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THE GROWTH OF POWERS OF A NONNEGATIVE MATRIX*

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Abstract. Let A be a nonnegative $n \times n$ matrix. In this paper we study the growth of the powers A^m , $m = 1, 2, 3, \dots$ when $\rho(A) = 1$. These powers occur naturally in the iteration process

$$x^{(m-1)} = Ax^{(m)}, \quad x^{(0)} \geq 0,$$

which is important in applications and numerical techniques. Roughly speaking, we analyze the asymptotic behavior of each entry of A^m . We apply our main result to determine necessary and sufficient conditions for the convergence to the spectral radius of A of certain ratios naturally associated with the iteration above.

1. Introduction. Let A be a nonnegative $n \times n$ matrix. In the iteration process

$$(1.1) \quad x^{(m+1)} = Ax^{(m)}, \quad x^{(0)} \geq 0,$$

which is important in applications and numerical techniques, the powers A^m , $m = 1, 2, \dots$ occur naturally. In this paper, we study the growth of these powers. In the literature there are several studies of the growth of A^m when the elementary divisors belonging to the spectral radius $\rho(A)$ of A are linear. For example, see Gantmacher [7, Chap. 13, § 5-7] Varga [19, pp. 32-34] when A is irreducible, and Meyer-Plemmons [10] when $\lim_{m \rightarrow \infty} A^m$ exists. We deal here with the general nonnegative case, when the elementary divisors belonging to $\rho(A)$ may have degrees greater than 1. At the cost of ignoring nilpotent A , where the problem is trivial, we assume that $\rho(A) > 0$.

For a complex $n \times n$ matrix A , with $\rho(A) = 1$, there is a least integer k for which $m^{-k}A^m$ is bounded, $m = 1, 2, \dots$. However, even in the simple case of an imprimitive, irreducible nonnegative A , $\lim_{m \rightarrow \infty} \|m^{-k}A^m\|$ and, a fortiori $\lim_{m \rightarrow \infty} m^{-k}A^m$, do not in general exist. To obtain precise results for general nonnegative A with $\rho(A) = 1$, it is thus necessary to introduce some smoothing. For example, in [14] Rothblum considered Cesaro means of powers of A . In this paper we study the growth of

$$(1.2) \quad B^{(m)} = A^m(I + \dots + A^{q-1}), \quad m = 1, 2, \dots,$$

where q is a certain positive integer.

After some preliminaries in § 2, we use elementary analytic methods in § 3 to prove a theorem on the growth of $B^{(m)}$. As corollary, we obtain a known theorem on the index of the eigenvalue 1 of A , cf. Schaefer [17, Chap. 1, Thm. 2.7]. We also give a local form of the theorem; that is, we show that for $1 \leq i, j \leq n$ there exist integers $k = k(i, j)$ and $q = q(i, j) > 0$ such that the element $b_{ij}^{(m)}$ of the matrix given by (1.2) satisfies

$$(1.3) \quad \lim_{m \rightarrow \infty} m^{-k} b_{ij}^{(m)} > 0.$$

The analytic results of § 3 motivate the investigations in the rest of the paper.

The main thrust of the paper is the use of the graph structure of the matrix A to decrease the integer $q(i, j)$ and to determine the integer $k(i, j)$ in (1.3). The requisite graph theoretic concepts are developed in § 4, and in § 5 we state our main result, Theorem (5.10). As a corollary, we obtain a striking theorem on the index of 1 due to

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Rothblum [13]. Our results are related to those of U. G. Rothblum [14], [15], and in some instances, would also follow from his. But where Rothblum considers A^{qm} , $m = 1, 2, \dots$, we consider $B^{(m)}$ and this allows us to choose a smaller integer q . Our definitions of $q(i, j)$ involves the greatest common divisor (g.c.d.) of certain periods where one might expect the least common multiple (l.c.m.). Consider the example

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then, by direct computation, for $1 \leq i, j \leq 2$, $\lim_{m \rightarrow \infty} b_{ij}^{(m)} = 1$, where $B^{(m)} = A^m(I + A)$. Thus $k(i, j) = 0$, and we may choose $q(i, j) = 2$ if $1 \leq i, j \leq 2$. Similarly $k(i, j) = 0$, $q(i, j) = 3$ if $3 \leq i, j \leq 5$. Yet $\lim_{m \rightarrow \infty} m^{-1} a_{ij}^{(m)} = \frac{1}{6}$ if $1 \leq i \leq 2, 3 \leq j \leq 6$, and so we have $k(i, j) = 1, q(i, j) = 1$. We might add that it may be possible that our choice of $q(i, j)$ can be improved in the general case where we use an l.c.m. of certain g.c.d.'s.

In § 6, we apply our results to the iteration process (1.1) for any nonnegative matrix A satisfying $\rho(A) > 0$. For $x \geq 0, x \neq 0$ denote

(1.4i) $r(x) = \sup \{ \mu : \mu x \leq Ax \},$

(1.4ii) $R(x) = \inf \{ \mu : \mu x \geq Ax \}.$

In Theorem 6.8, we find necessary and sufficient conditions for $r(A^m x)$ and $R(A^m x)$ to converge to the spectral radius of A . We show that whether or not this happens depends only on what is in general a small part of the vector x . In § 7, we show that a theorem due to D. H. Carlson [3] on the existence of nonnegative solutions y for $(I - A)y = x, x \geq 0, \rho(A) = 1$ is a consequence of our main results and we extend the theorem.

2. Preliminaries.

Notations. Let $\varphi(1), \varphi(2), \dots$, be a sequence of nonnegative numbers and $k \geq 0$ be an integer.

- (2.1) (i) $\varphi(m) = O(m^k)$ will denote that $\varphi(m)/m^k, m = 1, 2, \dots$, is bounded.
- (ii) $\varphi(m) = o(m^k)$ will denote that $\lim_{m \rightarrow \infty} \varphi(m)/m^k = 0$.
- (iii) $\varphi(m) \approx m^k$ will denote that $\lim_{m \rightarrow \infty} \varphi(m)/m^k$ exists and is positive.
- (iv) The above notations will also be used for $k = -1, -\infty$. In case that $k = -1$ $\varphi(m) = O(m^k), \varphi(m) = o(m^k), \varphi(m) \approx m^k$ will each indicate that there exists $\rho, 0 < \rho < 1$, such that $\varphi(m)\rho^{-m} = O(1)$. In case that $k = -\infty$ the above notations will mean that $\varphi(m) = 0$ for all sufficiently large m . (Thus $\varphi(m) \approx m^{-\infty}$ implies $\varphi(m) \approx m^{-1}$.)
- (v) The notation $A(m) \approx m^k$ will be used for a sequence of nonnegative matrices $A(1), A(2), \dots$ to indicate the relation holds for each element.

Combinatorial result. Let $r \geq 0$ and $t > 0$ be integers. Then

(2.2)
$$\Gamma_r^t = \sum_{p_1 + \dots + p_t = r} 1^{p_1} 1^{p_2} \dots 1^{p_t},$$

where the summation is taken over all nonnegative integers p_1, \dots, p_t whose sum is r . That is, Γ_r^t is the number of collections of r objects chosen from t distinct objects, with

repetitions allowed. It is well-known that

$$(2.3) \quad \Gamma'_t = \binom{r+t-1}{r}.$$

A simple way to prove this equality is by considering the coefficient of x^r of both sides of the identity

$$\sum_{r=0}^{\infty} \binom{r+t-1}{r} x^r = \left(\sum_{r=0}^{\infty} x^r \right)^{-t}$$

which is derived from $(1-x)^{-t} = (1-x)^{-1} \cdots (1-x)^{-1}$. For a purely combinatorial proof see for example Brualdi [2, p. 37]. For $t = 0$ the above formula implies $\Gamma'_0 = 1$ for all $r \geq 0$.

We shall also need some results on the convergence of series.

LEMMA 2.4. *Given integers $k \geq 1, q > 0$, and let $b_p \geq 0, p = 0, 1, 2, \dots$ be a sequence such that*

$$(2.5) \quad \lim_{p \rightarrow \infty} p^{-(k-1)}(b_p + \dots + b_{p+q-1}) = v,$$

where $q > 0$. Then

$$(2.6) \quad \lim_{m \rightarrow \infty} m^{-k} \sum_{p=1}^m b_p = \frac{v}{kq}.$$

Proof. Elementary. Alternatively, check that $c_{m,p} = m^{-k}kp^{k-1}$ satisfies the assumptions of Hardy [8, Thm. 2, p. 43]. \square

LEMMA 2.7. *Suppose (2.5) holds. If $\lim_{m \rightarrow \infty} a_m = u$ then*

$$(2.8) \quad \lim_{m \rightarrow \infty} m^{-k} \sum_{p=1}^m a_p b_{m-p} = \frac{uv}{kq}.$$

Proof. According to Hardy [8, Thm. 16, p. 64]

$$(2.9) \quad \lim_{m \rightarrow \infty} \frac{\sum_{p=1}^m a_p b_{m-p}}{\sum_{p=1}^m b_p} = u$$

since

$$0 \leq \frac{b_m}{\sum_{p=1}^m b_p} \leq \frac{b_m + \dots + b_{m+q-1}}{\sum_{p=1}^m b_p} \leq \frac{2vm^{(k-1)}}{v(2kq)^{-1}m^k},$$

and the last expression tends to 0. If we apply (2.6) to (2.9) we obtain (2.8). \square

3. Analytic approach. By \mathbb{R} , resp. \mathbb{C} , we denote the real, resp. complex field, and by \mathbb{R}_+ the nonnegative numbers. The set of real, resp. complex, nonnegative $r \times n$ matrices will be denoted by \mathbb{R}^n , resp. $\mathbb{C}^n, \mathbb{R}_+^n$. We also write $A \geq 0$ for $A \in \mathbb{R}_+^n$ (A is nonnegative) and $A > 0$ when A is positive ($a_{ij} > 0, i = 1, \dots, r, j = 1, \dots, n$).

Let $A \in \mathbb{C}^{nn}$. By $\text{spec } A$ we denote the set of eigenvalues of A . Suppose that $\text{spec } A = \{\lambda_1, \dots, \lambda_r\}$, where the λ_α are pairwise distinct. It is known (cf. Gantmacher [7, Chap. 5, § 3]) that there exist nonnegative integers p_1, \dots, p_r and unique matrices $Z^{(\alpha\beta)} \in \mathbb{C}^{nn}, \beta = 0, \dots, p_\alpha, \alpha = 1, \dots, r$ which are linearly independent such that for each polynomial $f(\tau)$,

$$(3.1) \quad f(A) = \sum_{\alpha=1}^r \sum_{\beta=0}^{p_\alpha} f^{(\beta)}(\lambda_\alpha) Z^{(\alpha\beta)}.$$

The $Z^{(\alpha\beta)}$ are polynomials in A , $p_\alpha + 1$ is the size of a largest Jordan-block belonging to λ_α . The columns of $Z^{(\alpha p_\alpha)}$ are eigenvectors of A corresponding to the eigenvalue λ_α , the rank of $Z^{\alpha p_\alpha}$ is equal to the number of Jordan blocks of size $p_\alpha + 1$ corresponding to λ_α . (The simplest way to obtain (3.1) is by assuming that A is in Jordan form.) As usual we define

$$\text{index}(\lambda_\alpha) = p_\alpha + 1.$$

That is, $p_\alpha + 1$ is the multiplicity of λ_α in the minimal polynomial of A . We shall also use a localized index. For $1 \leq i, j \leq n$ we put

$$\text{index}_{ij}(\lambda_\alpha) = 1 + \max\{\beta : z_{ij}^{(\alpha\beta)} \neq 0, \beta = 0, \dots, p_\alpha\},$$

where $\text{index}_{ij}(\lambda_\alpha) = 0$ if $z_{ij}^{(\alpha\beta)} = 0, \beta = 0, \dots, p_\alpha$. If $A \in \mathbb{C}^{n \times n}$ and m is any integer we shall denote the elements of A^m by $a_{ij}^{(m)}, 1 \leq i, j \leq m$.

Let $A \in \mathbb{R}_+^{n \times n}$. We assume throughout the normalization $\rho(A) = 1$. It is well-known (see Frobenius [6], Gantmacher [7, Chap. 13], Berman–Plemmons [1, Chap. 2]) that if λ is an eigenvalue of A and $|\lambda| = 1$, then λ is a root of 1. Hence, there is a positive integer q such that $\lambda^q = 1$, for all $\lambda \in \text{spec } A, |\lambda| = 1$. The smallest such integer q will be called the *period* of A . If $q = 1$, A will be called *aperiodic*. For an irreducible and aperiodic matrix $A \geq 0$, the Frobenius theorem and the formula (3.1) imply

$$\lim_{m \rightarrow \infty} A^m = Z^{(10)} > 0,$$

where $\lambda_1 = 1$, see for example Berman–Plemmons [1, Chap. 2, Thm. 4.1]. Theorem 3.4 extends the above equality in a local way. Part (i) of the theorem is an extension of the known inequality apparently due to Schaefer [16, Thm. 2.4, p. 264],

$$(3.2) \quad \text{index}(\lambda) \leq \text{index}(1) \quad \text{if } |\lambda| = 1,$$

for nonnegative matrices; see also Schaefer [17, Chap. 1, Thm. 2.7], Berman–Plemmons [1, Chap. 1, Thm. 3.2]. This result and part (i) of Theorem 3.4 could easily be deduced from the classical Pringsheim theorem on analytic functions; e.g., see Titchmarsh [18, p. 214]. The use of the Pringsheim theorem in analyzing the spectral properties of nonnegative matrices can be traced back to Ostrowski [11] (see also Karlin [9] and Schaefer [16, Appendix] for the infinite dimensional case). See Friedland [5] for a detailed analysis of the Pringsheim theorem for rational functions which has certain analogs to the Frobenius theorem. For sake of completeness we bring a short and elementary independent proof of Theorem 3.4. To do so we need an easy lemma which probably is known.

LEMMA 3.3. *Let $\lambda_\alpha, z_\alpha, \alpha = 1, \dots, r$ be complex numbers, where the λ_α are pairwise distinct. If $\lim_{m \rightarrow \infty} (\sum_{\alpha=1}^r \lambda_\alpha^m z_\alpha)$ exists, then $z_\alpha = 0$ if $|\lambda_\alpha| \geq 1, \lambda_\alpha \neq 1$.*

Proof. Since $\lim_{m \rightarrow \infty} \lambda_\alpha^m$ exists for $|\lambda_\alpha| < 1$ or $\lambda_\alpha = 1$, without loss of generality we may assume that $|\lambda_\alpha| \geq 1, \lambda_\alpha \neq 1, \alpha = 1, \dots, r$. Put $z = (z_1, \dots, z_r)^t \in \mathbb{C}^r$ and $u^{(m)} = (u_m, \dots, u_{m+r-1})^t$, where $u_m = \sum_{\alpha=1}^r \lambda_\alpha^m z_\alpha$. Let $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_r\} \in \mathbb{C}^r$ and let $V = (v_{\alpha\beta})_1^r \in \mathbb{C}^r$ be the Vandermond matrix given by $v_{\alpha\beta} = \lambda_\beta^{\alpha-1}, \alpha, \beta = 1, \dots, r$. Then

$$u^{(m)} = V \Lambda^m z.$$

The assumption of the lemma implies that $\lim_{m \rightarrow \infty} u^{(m)}$ exists. Since V is nonsingular, $\lim_{m \rightarrow \infty} \Lambda^m z = \lim_{m \rightarrow \infty} V^{-1} u^{(m)}$ and so $z = 0$. \square

THEOREM 3.4. *Let $A \in \mathbb{R}_+^{n \times n}$ where $\rho(A) = 1$. Let $1 \leq i, j \leq n$.*

- (i) *If $\lambda \in \text{spec } A, |\lambda| = 1$, then $\text{index}_{ij}(\lambda) \leq \text{index}_{ij}(1)$.*
- (ii) *Let q be a positive integer such that $\lambda^q = 1$ if $\lambda \in \text{spec } A, |\lambda| = 1$ and $\text{index}_{ij}(\lambda) =$*

$\text{index}_{ij}(1)$. Put $k + 1 = \text{index}_{ij}(1)$ and let

$$B^{(m)} = A^m(I + \dots + A^{q-1}).$$

Then $b_{ij}^{(m)} \approx m^k$. In particular, $a_{ij}^{(m)} \neq o(m^k)$ if $k \geq 0$.

Proof. (i) Let $\{\lambda_1, \dots, \lambda_r\}$ be the eigenvalues with $|\lambda_\alpha| = 1, \alpha = 1, \dots, r$, where the λ_α are pairwise distinct. Let

$$d + 1 = \max \{ \text{index}_{ij} \{ \lambda_\alpha \} : \alpha = 1, \dots, r \}.$$

If $d = -1$ then there is nothing to prove. So assume that $d \geq 0$. Suppose that $z_\alpha \equiv z_{ij}^{(\alpha d)} \neq 0$ for $\alpha = 1, \dots, s$ where $1 \leq s \leq r$ and $z_{ij}^{(\alpha d)} = 0$ for $\alpha = s + 1, \dots, r$. It follows immediately from (3.1) that

$$a_{ij}^{(m)} = m^d \left(\sum_{\alpha=1}^s \lambda_\alpha^{m-d} z_\alpha \right) + o(m^d).$$

Hence, by Lemma (3.3), $a_{ij}^{(m)} \neq o(m^d)$.

Let q be a positive integer such that $\lambda_\alpha^q = 1, \alpha = 1, \dots, s$. Define

$$\varphi_m(\tau) = \tau^m(1 + \tau + \dots + \tau^{q-1}).$$

If we take the d th derivative of $\varphi_m(\tau)$, we obtain

$$\varphi_m^{(d)}(\tau) = m^d \varphi_{m-d}(\tau) + o(m^d)$$

for any fixed $\tau, |\tau| \leq 1$, and also $\varphi_{m-d}(\lambda_\alpha) = 0$ for $|\lambda_\alpha| = 1, \lambda_\alpha \neq 1, 1 \leq \alpha \leq s$. Put $B^{(m)} = \varphi_m(A)$. By (3.1) and the equality above we have

$$(3.5) \quad b_{ij}^{(m)} = m^d \left(\sum_{\alpha=1}^r \varphi_{m-d}(\lambda_\alpha) z_\alpha \right) + o(m^d).$$

Now suppose that $\text{index}_{ij}(1) < d + 1$. Then (3.5) implies that $b_{ij}^{(m)} = o(m^d)$. But $b_{ij}^{(m)} = a_{ij}^{(m)} + \dots + a_{ij}^{(m+q-1)} \geq a_{ij}^{(m)} \geq 0$, and this is a contradiction. Thus $d = k$ and this proves (i).

(ii) Suppose that $\lambda_1 = 1$. If $k = -1$, by an argument like that above, $a_{ij}^{(m)} = b_{ij}^{(m)} \approx m^k$. Let $k \geq 0$. By (3.5) and the preceding argument we obtain

$$b_{ij}^{(m)} = m^k q z_1 + o(m^k),$$

where $z_1 = z_{ij}^k > 0$. This proves (ii). \square

We now state a global version of Theorem 3.4 (ii) which follows immediately from Theorem 3.4.

THEOREM 3.6. Let $A \in \mathbb{R}_+^{n \times n}$ where $\rho(A) = 1$. Let q be a positive integer such that $\lambda^q = 1$ if $\lambda \in \text{spec } A, |\lambda| = 1$ and $\text{index}(\lambda) = \text{index}(1) = k + 1$. Let

$$B^{(m)} = A^m(I + \dots + A^{q-1}).$$

Then

$$(3.7) \quad \lim_{m \rightarrow \infty} m^{-k} B^{(m)} = F,$$

where $F \geq 0$ and F is not identically zero.

It should be noted that the assumption that A is nonnegative was used crucially in the proof of Theorems 3.4 and 3.6. For example, let $A = -I$; then there are no k, q for which the limit of (3.7) exists and is nonzero. Also, the assumption that $\rho(A) = 1$ is used

in an essential way. Let

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}.$$

Then $\lim_{m \rightarrow \infty} \rho(A)^{-2m} A^{2m} (I + A)$ and $\lim_{m \rightarrow \infty} \rho(A)^{-(2m+1)} A^{2m+1} (I + A)$ exist, but are distinct. It follows that no k, q exist for which $\lim_{m \rightarrow \infty} \rho(A)^{-m} m^{-k} B^{(m)}$ exists and is nonzero.

Our subsequent work discusses the nature of k, q and F .

4. Graph theoretical concepts. Let $A \in \mathbb{R}_+^{nn}$ and let $\rho(A) > 0$. We may assume, without loss of generality, that after simultaneous permutations of rows and columns, A is in the Frobenius [6] normal form which can be found in many references, e.g., Gantmacher [7, Vol. II, p. 75]. Thus

$$(4.1) \quad A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1\nu} \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{\nu\nu} \end{bmatrix},$$

where the diagonal blocks $A_{\alpha\alpha}, \alpha = 1, \dots, \nu$ are irreducible and all subdiagonal blocks are 0. (The 1×1 matrix 0 is considered to be irreducible.)

Let A be in Frobenius normal form (4.1). Then the (reduced) graph $G(A)$ of A is a subset of $\langle \nu \rangle \times \langle \nu \rangle$, where $\langle \nu \rangle = \{1, \dots, \nu\}$ and $G(A) = \{(\alpha, \beta) \in \langle \nu \rangle \times \langle \nu \rangle : A_{\alpha\beta} \neq 0\}$. (Observe that many authors would call $G(A)$ the arcset of the graph $(\langle \nu \rangle, G(A))$, but we have no need to mention the vertex set $\langle \nu \rangle$ explicitly.)

If $(\alpha, \beta) \in G(A)$, we call (α, β) an arc of $G(A)$. If (α, β) is an arc of $G(A)$, then $\alpha \preceq \beta$; also $(\alpha, \alpha) \in G(A), 1 \preceq \alpha \preceq \nu$, unless $A_{\alpha\alpha}$ is the 1×1 matrix 0. Thus we define a (simple) path from α to β in $G(A)$ to be a sequence $\pi = (\alpha_0, \dots, \alpha_s)$, where either $s \geq 1, 1 \preceq \alpha = \alpha_0 < \dots < \alpha_s = \beta \preceq \nu$ and $(\alpha_{i-1}, \alpha_i) \in G(A), i = 1, \dots, s$, or $s = 0$ and $\alpha = \alpha_0 = \beta$ and $(\alpha, \alpha) \in G(A)$. The support of π is the set $\text{supp } \pi = \{\alpha_0, \dots, \alpha_s\} \subseteq \{1, \dots, \nu\}$. We always assume that the $\alpha_i, i = 0, \dots, s$, have been listed in strictly ascending order.

If $1 \preceq \alpha \preceq \nu$, then we call α a singular vertex (of $G(A)$) if $\rho(A_{\alpha\alpha}) = \rho(A)$. (This terminology is consistent with that of Richman-Schneider [12].) Let $1 \preceq \alpha \preceq \beta \preceq \nu$. For any path π from α to β in $G(A)$, let $k(\pi) + 1$ be the number of singular γ in the support of π . (Thus note each distinct γ is counted only once in $k(\pi) + 1$.) Let $\alpha_{j_0} < \alpha_{j_1} < \dots < \alpha_{j_k}$, where $k = k(\pi)$, be all singular vertices in $\text{supp } \pi$. We set

$$(4.2) \quad k(\alpha, \beta) = \max \{k(\pi) : \pi \text{ is a path from } \alpha \text{ to } \beta \text{ in } G(A)\}.$$

If there is no path from α to β in $G(A)$ we put $k(\alpha, \beta) = -\infty$. We shall call $k(\alpha, \beta)$ the singular distance from α to β . If (i, i) is a position in $A_{\alpha\alpha}$ and (j, j) a position in $A_{\beta\beta}$ then we shall also call $k[i, j] = k(\alpha, \beta)$ the singular distance from i to j (note our use of square brackets).

A path π from α to β will be called a maximal path if the number of singular vertices in the support of π is $k(\alpha, \beta) + 1$. Let $1 \preceq \alpha, \beta \preceq \nu$. Let $\mathcal{P}(\alpha, \beta)$ be the set of maximal paths from α to β . For each $\pi \in \mathcal{P}(\alpha, \beta)$ let $q(\pi)$ be the g.c.d. of periods of $A_{\gamma\gamma}$ with $\gamma \in \text{supp } \pi$ and singular (viz. $\rho(A_{\gamma\gamma}) = \rho(A)$).

Then we define

$$(4.3) \quad q(\alpha, \beta) = \text{l.c.m. } \{q(\pi) : \pi \in \mathcal{P}(\alpha, \beta)\}.$$

We shall call $q(\alpha, \beta)$ the local period of (α, β) . If $k(\alpha, \beta) < 0$ then $q(\alpha, \beta) = 1$. Also if

(i, i) is a position in $A_{\alpha\alpha}$ and (j, j) is a position in $A_{\beta\beta}$ then we shall put $q(\alpha, \beta) = q[i, j]$, the local period of (i, j) .

5. The main results. Let $A \in \mathbb{R}_+^{nn}$, where $\rho(A) = 1$, be in Frobenius normal form (4.1). It follows from the Perron-Frobenius theory for nonnegative matrices, e.g., Gantmacher [7, Chap. 13] that there is a diagonal matrix X with positive diagonal elements so that, upon replacing A by $X^{-1}AX$,

$$(5.1) \quad A_{\alpha\alpha} = \rho(A_{\alpha\alpha})A'_{\alpha\alpha},$$

where $A'_{\alpha\alpha}$ is a stochastic matrix,

$$(5.2) \quad \|A_{\alpha\beta}\|_{\infty} \leq \sigma, \quad 1 \leq \alpha < \beta \leq \nu,$$

where $1 > \sigma$ and $\sigma > \max \{ \rho(A_{\alpha\alpha}) : \rho(A_{\alpha\alpha}) < 1, \alpha = 1, \dots, \nu \}$ if such α exist. Here $\| \cdot \|_{\infty}$ is the l_{∞} -operator norm,

$$\|Z\|_{\infty} = \max \left\{ \sum_{j=1}^n |z_{ij}| : i = 1, \dots, r \right\} \quad \text{for } Z \in \mathbb{R}^{rn}.$$

The diagonal matrix X can be constructed as follows. Let $u^{(\alpha)}$ be a positive vector satisfying $A_{\alpha\alpha}u^{(\alpha)} = \rho(A_{\alpha\alpha})u^{(\alpha)}$. Denote by X_{α} a diagonal matrix, whose diagonal entries are the elements of $u^{(\alpha)}$. Then X is of the form $\text{diag} \{ X_1, \varepsilon X_2, \dots, \varepsilon^{\nu-1} X_{\nu} \}$ for some small enough positive ε . In our subsequent proofs we may assume that A has been normalized as above.

Let π be a path in $G(A)$. Denote by $s + 1$ the cardinality of $\text{supp } \pi$. That is

$$(5.3i) \quad \text{supp } \pi = \{ \beta_0, \dots, \beta_s \}, \quad 1 \leq \beta_0 < \beta_2 < \dots < \beta_s \leq \nu.$$

We define the *path matrix* $A(\pi)$ by

$$(5.3ii) \quad \begin{aligned} A_{ii}(\pi) &= A_{\beta_i\beta_i}, & i = 0, \dots, s, \\ A_{i,i+1}(\pi) &= A_{\beta_i\beta_{i+1}}, & i = 0, \dots, s-1, \\ A_{ij}(\pi) &= 0, & i, j = 0, \dots, s \text{ otherwise,} \end{aligned}$$

$$(5.3iii) \quad A(\pi) = (A_{ij}(\pi))_0^s.$$

Thus $A(\pi)$ is in Frobenius normal form and has $s + 1$ irreducible diagonal blocks $A_{ii}(\pi) = A_{\beta_i\beta_i}$, $i = 0, \dots, s$. To avoid ambiguity, we write $A(\pi)_{ij}^{(m)}$ for the (i, j) block component of $A(\pi)^m$, $i, j = 0, \dots, s$.

We now prove a sequence of lemmas for the path matrix $A(\pi)$ of a given path.

LEMMA 5.4. Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Let $1 \leq \alpha, \beta \leq \nu$ and π be a path in $G(A)$ from α to β . Put $k = k(\pi)$, where $k(\pi) + 1$ is the number of singular vertices in $\text{supp } \pi$. If $A(\pi)$ is the path matrix given by (5.3), then $\|A(\pi)_{0s}^{(m)}\|_{\infty} = O(m^k)$.

Proof. We note that

$$(5.5) \quad A(\pi)_{0s}^{(m)} = \sum_{p_0 + \dots + p_s = m-s} A_{00}^{p_0}(\pi) A_{01}(\pi) A_{11}^{p_1}(\pi) \cdots A_{(s-1)s}(\pi) A_{ss}^{p_s}(\pi).$$

So

$$\|A(\pi)_{0s}^{(m)}\|_{\infty} \leq \sigma^s \sum_{p_0 + \dots + p_s = m-s} \|A_{00}(\pi)\|_{\infty}^{p_0} \cdots \|A_{ss}(\pi)\|_{\infty}^{p_s}.$$

Suppose first that π does not contain singular vertices, i.e., $k = -1$. Then

$$\|A(\pi)_{0s}^{(m)}\|_{\infty} \leq \sigma^m \sum_{p_0 + \dots + p_s = m-s} 1^{p_0} \cdots 1^{p_s} = \sigma^m \Gamma_{s+1}^{m-s},$$

where Γ'_i is given by (2.3). As $\Gamma_s^{m-s} \cong m^s$ we immediately deduce

$$\lim_{m \rightarrow \infty} \tau^{-m} A(\pi)_{0_s}^{(m)} = 0 \quad \text{for any } \tau, \sigma < \tau < 1.$$

Suppose now that $k \geq 0$. Then

$$\begin{aligned} \|A(\pi)_{0_s}^{(m)}\|_\infty &\leq \sigma^s \sum_{q_0 + \dots + q_s = m-s} 1^{q_0} \dots 1^{q_k} \sigma^{q_{k+1}} \dots \sigma^{q_s} \\ &= \sigma^s \sum_{u=0}^{m-s} \left(\sum_{q_0 + \dots + q_k = u} 1^{q_0} \dots 1^{q_k} \right) \left(\sum_{q_{k+1} + \dots + q_s = m-s-u} \sigma^{q_{k+1}} \dots \sigma^{q_s} \right) \\ &= \sigma^s \sum_{u=0}^{m-s} \Gamma_{k+1}^u \Gamma_{s-k}^{m-s-u} \sigma^{m-s-u}. \end{aligned}$$

Hence

$$\|A(\pi)_{0_s}^{(m)}\|_\infty \leq \Gamma_{k+1}^{m-s} \left(\sum_{\nu=0}^\infty \Gamma_{s-k}^\nu \sigma^{\nu+s} \right).$$

The last series converges by the ratio test and $\Gamma_{k+1}^{m-s} \leq m^k$. This establishes the lemma. \square

LEMMA 5.6. *Let the assumptions of Lemma 5.4 hold. Assume furthermore that $k \geq 0$, i.e., the support of π contains singular vertices. Then, for sufficiently large m ,*

$$(5.7) \quad \sum_{j=0}^{2(s+1)(n-1)} A(\pi)_{0_s}^{(m+j)} \geq Gm^k,$$

where G is a positive matrix.

Proof. Let

$$B_{ii}(\pi) = I + A_{ii}(\pi) + \dots + A_{ii}(\pi)^{(n-1)}, \quad i = 1, \dots, s.$$

Since $A_{ii}(\pi)$ is irreducible, and its dimension does not exceed n , we have $B_{ii}(\pi) > 0$, Wielandt [20], Berman–Plemmons [1, Chap. 2, Thm. 1.3]. Clearly (5.5) implies, for $t = 2(s+1)(n-1)$,

$$\begin{aligned} \sum_{j=0}^t A(\pi)_{0_s}^{(m+j)} &\geq n^{-(s+1)} \sum_{p_0 + \dots + p_s = m-s} B_{00}(\pi) A_{00}(\pi)^{p_0} B_{00}(\pi) A_{01}(\pi) B_{11}(\pi) A_{11}^{p_1} B_{11}(\pi) \dots \\ &\quad \cdot A_{s-1,s}(\pi) B_{ss}(\pi) A_{ss}^{p_s}(\pi) B_{ss}(\pi). \end{aligned}$$

For $i, j = 0, \dots, s$, let E_{ij} be the matrix all of whose entries equal 1 and whose dimension is that of $A_{ij}(\pi)$. Clearly $B_{00}(\pi) \geq c'_0 E_{00}$, $B_{ss}(\pi) \geq c'_s E_{ss}$ where $c'_0, c'_s > 0$. Since $A_{i,i+1}(\pi) \neq 0$, we have

$$B_{ii}(\pi) A_{i,i+1}(\pi) B_{i+1,i+1}(\pi) \geq c_i E_{i,i+1},$$

where $c_i > 0$, $i = 1, \dots, s-1$, and hence, for some $c > 0$,

$$(5.8) \quad \sum_{j=1}^t A(\pi)_{0_s}^{(m+j)} \geq c \sum_{p_0 + \dots + p_s = m-s} E_{00} A_{00}(\pi)^{p_0} E_0 \dots E_{s-1,s} A_{ss}(\pi)^{p_s} E_{ss}.$$

In the inequality (5.8) we may restrict the sum on the right-hand side by letting $p_j = 0$ if $\rho(A_{jj}(\pi)) < 1$. So let $\gamma_0 < \dots < \gamma_k$ be the subscripts of A_{ii} which are singular vertices

and put $\bar{A}_{ii} = A_{\gamma_i \gamma_i}(\pi)$. Since $E_{ij}E_{jk} \geq E_{ik}$, it follows that

$$\sum_{j=0}^t A(\pi)_{0s}^{(m+j)} \geq c' \sum_{p_0+\dots+p_k=m-t} \bar{E}_{-1,0} \bar{A}_{00}(\pi)^{p_0} \bar{E}_{01} \cdots \bar{A}_{kk}^{p_k} \bar{E}_{k,k+1},$$

where $c' > 0$ and the $\bar{E}_{i,i+1}$, $i = -1, \dots, k$ are matrices all of whose entries are 1. But $\bar{A}_{ii}(\pi)$ is a stochastic matrix, $i = 0, \dots, k$, whence $\bar{A}_{ii}(\pi)^{p_0} \bar{E}_{i,i+1} = \bar{E}_{i,i+1}$, $i = 0, \dots, k$. It follows that

$$\sum_{j=0}^t A(\pi)_{0s}^{(m+j)} \geq 2\Gamma_{k+1}^{m-t} G,$$

where $G > 0$. The lemma now follows from (2.3) since $\Gamma_k^{m-t} \geq \frac{1}{2}m^k$ for sufficiently large m . \square

LEMMA 5.9. *Let the assumptions of Lemma 5.4 hold, and suppose that $k = k(\pi) \geq 0$. Let $q = q(\pi)$ be the g.c.d. of periods of $A_{\gamma\gamma}$ for singular $\gamma \in \text{supp } \pi$. Let*

$$B(\pi)^{(m)} = A(\pi)^m (I + A(\pi) + \cdots + A(\pi)^{q-1}).$$

- (i) *If (i, j) is any position in $A(\pi)_{0s}$, then, in $A(\pi)$, $\text{index}_{ij}(1) = k + 1$.*
- (ii) *$b(\pi)_{ij}^{(m)} \approx m^k$.*

Proof. (i) Let $k^* + 1 = \text{index}_{ij}(1)$ in $A(\pi)$. By Theorem 3.4 there is a positive integer q^* such that for

$$B^*(\pi)^{(m)} = A(\pi)^m (I + A(\pi) + \cdots + A(\pi)^{q^*-1}),$$

we have $b^*(\pi)_{ij}^{(m)} \approx m^{k^*}$. But $k^* > k$ contradicts Lemma 5.4. Since the sum in (5.7) can be majorized by a sum of terms of the form $B^*(\pi)_{0s}^{(m+j)}$, $j = 0, \dots, 2(s+1)(n-1)$, it follows that $k^* < k$ contradicts Lemma 5.6. Hence $k^* = k$.

(ii) Now suppose that $\lambda \in \text{spec } A(\pi)$, $|\lambda| = 1$ and $\text{index}_{ij}(\lambda) = \text{index}_{ij}(1) = k + 1$ in $A(\pi)$. Then

$$\text{index}_{ij}(\lambda) \leq \text{index}(\lambda) \leq \text{mult}(\lambda),$$

where $\text{mult}(\lambda)$ is the algebraic multiplicity of λ in $A(\pi)$. But, by the Perron-Frobenius theorem for irreducible matrices,

$$\text{mult}(\lambda) \leq \text{mult}(1) = k + 1.$$

Hence $\text{mult}(\lambda) = k + 1$ and, by Perron-Frobenius, λ is an eigenvalue of every $A_{\gamma\gamma}$ for which γ is singular. It follows that $\lambda^q = 1$, where $q = q(\pi)$. Hence the conditions of Theorem 3.4 (ii) are satisfied and the lemma follows.

We state our main result.

THEOREM 5.10. *Let A be nonzero $n \times n$ matrix normalized by the condition $\rho(A) = 1$. Assume $1 \leq i, j \leq n$. Let $k = k[i, j]$ be the singular distance from i to j and $q = q[i, j]$ be the local period of (i, j) . Put $B^{(m)} = A^m (I + A + \cdots + A^{q-1})$. Then $b_{ij}^{(m)} \approx m^k$.*

Proof. As usual, we assume that A is in the Frobenius form (4.1). Suppose that (i, j) is a position in $A_{\alpha\beta}$. Denote by $\Pi(\alpha, \beta)$ the set of all paths connecting α to β . Then we obviously have

$$A_{\alpha\beta}^{(m)} = \sum_{\pi \in \Pi(\alpha, \beta)} A(\pi)_{0s(\pi)}^{(m)}.$$

So

$$B_{\alpha\beta}^{(m)} = \sum_{\pi \in \Pi(\alpha, \beta)} B(\pi)_{0s(\pi)}^{(m)}.$$

Assume first that $k = k(\pi) = -\infty$; then, clearly, $B_{\alpha\beta}^{(m)} = A_{\alpha\beta}^{(m)} = 0$. If $k = -1 \cong k(\pi)$ then Lemma 5.4 implies that each $A(\pi)_{0s(\pi)}^{(m)} \approx m^{-1}$. So $A_{\alpha\beta}^{(m)} \approx m^{-1}$ and again $A_{\alpha\beta}^{(m)} = B_{\alpha\beta}^{(m)}$.

Assume now that $k \geq 0$. If $k > k(\pi)$, Lemma 5.4 implies that $B(\pi)_{0s(\pi)}^{(m)} = O(m^k)$. However, if $k = k(\pi)$, then according to Lemma 5.9 $\lim_{m \rightarrow \infty} m^{-k} B(\pi)_{0s(\pi)}^{(m)} = F_{0s}(\pi) > 0$ as $q(\pi)$ divides $q(\alpha, \beta) = q[i, j]$. By the definition of $k(\alpha, \beta)$ there exists $\pi \in \Pi(\alpha, \beta)$ such that $k(\pi) = k(\alpha, \beta)$. So $\lim_{k \rightarrow \infty} m^{-k} B_{\alpha\beta}^{(m)} = F_{\alpha\beta} > 0$. \square

COROLLARY 5.11. *Under the conditions of Theorem 5.10,*

$$\sum_{p=1}^m a_{ij}^{(p)} \approx m^{(k+1)}.$$

Proof. For $k \geq 0$, the result is immediate by Lemma 2.4. If $k = -1$, then by Theorem 5.10 the nonnegative series above converges. The assumption $k = -1$ implies that at least one term is positive. Finally if $k = -\infty$, $a_{ij}^{(p)} = 0$, $p = 1, 2, \dots$, and the result follows. \square

Comparing Theorems 3.4 and 5.10 we first deduce a local version of Rothblum's equality and then the equality itself.

THEOREM 5.12. *Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Assume that $1 \leq i, j \leq n$; then*

$$\text{index}_{ij}(1) = k[i, j] + 1.$$

COROLLARY 5.13 (Rothblum [13]). *Let $A \in \mathbb{R}_+^{nn}$ where $\rho(A) = 1$. Then $\text{index}(1) = \max_{1 \leq i, j \leq n} \text{index}_{ij}(1) = \max_{1 \leq i, j \leq n} k[i, j] + 1$.*

6. Convergent iterative methods for the spectral radius of a nonnegative matrix.

Let $A \in \mathbb{R}_+^{nn}$ and assume that $\rho(A) > 0$. Let $r(x)$ and $R(x)$ be defined as in (1.4). Clearly $0 \leq r(x) \leq R(x) \leq +\infty$. It is obvious that

$$r(x) \leq r(Ax) \leq R(Ax) \leq R(x).$$

So the sequence $r(A^m x)$, $m = 0, 1, \dots$ is an increasing sequence bounded above by $R(x)$, and the sequence $R(A^m x)$, $m = 0, 1, \dots$ is a decreasing sequence bounded below by $r(x)$.

In [4], Collatz observed that, for $A \in \mathbb{R}_+^{nn}$ and $x > 0$,

$$(6.1) \quad r(x) \leq \rho(A) \leq R(x),$$

and when A is irreducible, this inequality is valid for all $x \geq 0$, $x \neq 0$; see Wielandt [20], Varga [19, p. 32]. Thus the question arises when, for $A \geq 0$ and $x \geq 0$, $x \neq 0$,

$$(6.2) \quad \lim_{m \rightarrow \infty} r(A^m x) = \rho(A) = \lim_{m \rightarrow \infty} R(A^m x).$$

Wielandt's [20] characterization of $\rho(A)$ for irreducible A easily implies that (6.2) holds for primitive A and all $x \in \mathbb{R}_+^n$, $x \geq 0$, $x \neq 0$ (cf. Varga [19, p. 34]). This result follows from the fact that

$$\lim_{m \rightarrow \infty} \rho(A)^{-m} A^m = Z > 0$$

when A is primitive, where $Z = uv'$, $v > 0$, $Au = \rho(A)u$, $v > 0$, $v'A = \rho(A)v'$, $v'u = 1$. If A is irreducible but imprimitive then (6.2) does not hold unless x is orthogonal on all eigenvectors of A' corresponding to λ such that $|\lambda| = \rho(A)$ and $\lambda \neq \rho(A)$. We shall show

that this condition can be put in equivalent forms. If A is irreducible and of period q , then by simultaneous permutations of rows and columns we now put A into the form

$$(6.3) \quad \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_{q-1,q} \\ A_{q1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where the diagonal blocks 0 are square (see Frobenius [6], Gantmacher [7, Vol II, p. 62], Berman–Plemmons [1, Chap. 2, Thm. 2.20]).

LEMMA 6.4. *Let A be an irreducible nonnegative matrix of period q in form (6.3), and suppose that $\rho(A) = 1$. Let $v^t A = v^t$, $Au = u$, where $v > 0$, $u > 0$, $v^t u = 1$, $A^j y^j = \omega^j y^j$, $j = 1, \dots, q-1$, $\omega = e^{2\pi i/q}$. Let $0 \neq x \in \mathbb{R}_+^n$ be partitioned conformally with A , $x^t = (x_{(1)}^t, \dots, x_{(q)}^t)$. Then the following are equivalent*

- (i) $\lim_{m \rightarrow \infty} A^m x = (v^t x) u$,
- (ii) $\lim_{m \rightarrow \infty} A^m x$ exists,
- (iii) $x^t y^j = 0$, $j = 1, \dots, q-1$,
- (iv) $v_{(1)}^t x_{(1)} = \dots = v_{(q)}^t x_{(q)}$,
- (v) $\lim_{m \rightarrow \infty} R(A^m x) = \lim_{m \rightarrow \infty} r(A^m x) = 1$,

where $v^t = (v_{(1)}^t, \dots, v_{(q)}^t)$ has been partitioned conformally with A .

Proof. We first derive a formula for $A^m x$, $m = 1, 2, \dots$. Let ω be a primitive q th root of unity. It is well-known that the eigenvalues of A on the unit circle are $\lambda_\alpha = \omega^{\alpha-1}$, $\alpha = 1, \dots, q$ and that each λ_α is a simple zero of the characteristic polynomial. It follows, in the notation of § 3, that $p_\alpha = 0$, $\alpha = 1, \dots, q$ and that

$$\begin{aligned} Z^{(\alpha 0)} &= D^{\alpha-1} u v^t D^{1-\alpha}, \quad \alpha = 1, \dots, q, \\ y^\alpha &= D^{(1-\alpha)} v, \quad \alpha = 1, \dots, q, \end{aligned}$$

where

$$D = \begin{bmatrix} I_{11} & & & \\ & \omega I_{22} & & 0 \\ & 0 & \ddots & \\ & & & \omega^{q-1} I_{qq} \end{bmatrix},$$

and $I_{\alpha\alpha}$ is an identity matrix of the same order of $A_{\alpha\alpha}$, $\alpha = 1, \dots, q$.

Hence by (3.1),

$$A^m = \sum_{\alpha=0}^{q-1} \omega^{m\alpha} D^\alpha u v^t D^{-\alpha} + o(1);$$

and so

$$(6.5) \quad A^m x = \sum_{\alpha=0}^{q-1} \omega^{m\alpha} a_\alpha (D^\alpha u) + o(1),$$

where

$$(6.6) \quad a_\alpha = v^t D^{-\alpha} x = x^t y^\alpha, \quad \alpha = 0, \dots, q-1.$$

Let

$$c_\beta = v_{(\beta+1)}^t x_{(\beta+1)}, \quad \beta = 0, \dots, q-1.$$

Then it follows immediately from (6.6) that

$$(6.7) \quad a_\alpha = \sum_{\beta=0}^{q-1} \omega^{-\alpha\beta} c_\beta, \quad \alpha = 0, \dots, q-1.$$

We now prove the equivalence of our five conditions. We show (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i) and (i) \Rightarrow (v) \Rightarrow (iv).

(i) \Rightarrow (ii). Trivial.

(ii) \Rightarrow (iii). Since $\lim_{m \rightarrow \infty} A^m x$ exists, $\lim_{m \rightarrow \infty} v^t D^{-\alpha} A^m x$ also exists, $\alpha = 0, \dots, q-1$. But $v^t u > 0$, and hence $a_\alpha = x^t y^\alpha = 0$, $\alpha = 1, \dots, q-1$ by Lemma 3.3.

(iii) \Rightarrow (iv). Consider the identity (6.7). Since the Vandermonde matrix $q^{-1/2}(\omega^{-\alpha\beta})$, $\alpha, \beta = 0, \dots, q-1$ is unitary the assumption $a_\alpha = x^t y^\alpha = 0$, $\alpha = 1, \dots, q-1$ implies that $c_0 = c_1 = \dots = c_{q-1}$, which proves (iv).

(iv) \Rightarrow (i). If (iv) holds, then $c_0 = c_1 = \dots = c_{q-1}$ and (6.7) implies $a_1 = \dots = a_{q-1} = 0$. This establishes (i) in view of (6.5) and (6.6).

(i) \Rightarrow (v). Trivial, since $v^t x > 0$ and $u > 0$.

(v) \Rightarrow (i). Let $m = ql + r$, $0 \leq r \leq q-1$. Then (6.5) implies

$$\lim_{l \rightarrow \infty} A^{ql+r} x = \tilde{x}^{(r)}, \quad r = 0, \dots, q-1$$

for some $\tilde{x}^{(r)} \geq 0$, $\tilde{x}^{(r)} \neq 0$. Also

$$A^r \tilde{x}^{(0)} = \tilde{x}^{(r)}, \quad r = 0, \dots, q-1, \quad A^q \tilde{x}^{(0)} = \tilde{x}^{(0)}.$$

As A^q is a direct sum of q irreducible and primitive matrices the assumption $x \geq 0$, $x \neq 0$ implies that $\lim_{l \rightarrow \infty} (A^q)^l x = \tilde{x}^{(0)} \neq 0$. Obviously $x^{(0)} \geq 0$.

Now (v) implies that

$$x^{0 \leq r \leq 1} = Ax^{0 \leq r \leq 0},$$

whence $x^{(1)} = x^{(0)}$ and thus $x^{(r)} = x^{(0)}$ for $r = 1, \dots, q-1$. So $\lim_{m \rightarrow \infty} A^m x = x^{(0)}$ and (i) follows. \square

In what follows, we give necessary and sufficient conditions on a reducible matrix A to satisfy (6.2). To do so we need a few more graph theoretical concepts.

Let G be a graph on $\langle \nu \rangle = \{1, \dots, \nu\}$. Let J be a nonvoid subset of $\langle \nu \rangle$. Then $\alpha \in J$ is called a final state with respect to J if for any $\beta \neq \alpha$ and $(\alpha, \beta) \in G$, $\beta \notin J$. Denoting by $\mathcal{F}(J)$ the set of all final states with respect to J . If $J = \langle \nu \rangle$, then α is called a final state, i.e., $(\alpha, \beta) \in G$ implies that $\beta = \alpha$. Define

$$d(\beta, J) = \max \{k(\beta, \alpha) : \alpha \in \mathcal{F}(J)\}.$$

If $J = \langle \nu \rangle$, then write $d(\beta)$ instead of $d(\beta, \langle \nu \rangle)$. Let $A \geq 0$ be a reducible matrix. We assume that A is in the Frobenius form (4.1).

As in § 4, denote by $G(A)$ the (reduced) graph of A . Let $x \geq 0$, $x \neq 0$. Partition x conformably with A given by (4.1). That is $x^t = (x^t_{(1)}, \dots, x^t_{(\nu)})$. The support of x is the set $\text{supp } x = \{\alpha_1, \dots, \alpha_s\} \subseteq \{1, \dots, \nu\}$ such that $x_{(i)} \neq 0$ if and only if $i \in \text{supp } x$. We shall always assume that α_i , $i = 1, \dots, s$ have been listed in strictly ascending order.

THEOREM 6.8. *Let $A \in \mathbb{R}_+^{n \times n}$, $\rho(A) > 0$. Assume that A is in the Frobenius form (4.1). Moreover, if A_{ii} is imprimitive then A_{ii} is the Frobenius form (6.3). Let $x \geq 0$, $x \neq 0$. Then (6.2) holds if and only if any final state α with respect to the support of x satisfies*

- (i) α is a singular vertex (i.e., $\rho(A_{\alpha\alpha}) = \rho(A)$),
- (ii) either $A_{\alpha\alpha}$ is primitive, or $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition (iv) of Lemma 6.4.

Proof. Without loss of generality we may assume that $\rho(A) = 1$. Next we note that

$$(6.9) \quad (A^m x)_\alpha = \sum_{\beta \in \text{supp } x} A_{\alpha\beta}^{(m)} x_{(\beta)}.$$

Suppose that $\alpha \in \mathcal{F}(\text{supp } x)$. Then

$$(A^m x)_\alpha = A_{\alpha\alpha}^m x_{(\alpha)}.$$

By the definition of $R(x)$ and $r(x)$ we have

$$r(A^m x) A^m x \leq A^{m+1} x \leq R(A^m x) A^m x.$$

So

$$r(A^m x) A_{\alpha\alpha}^m x_{(\alpha)} \leq A_{\alpha\alpha}^{m+1} x_{(\alpha)} \leq R(A^m x) A_{\alpha\alpha}^m x_{(\alpha)}.$$

Hence, since $A_{\alpha\alpha}$ is irreducible, by (6.1),

$$r(A^m x) \leq r(A_{\alpha\alpha}^m x_{(\alpha)}) \leq \rho(A_{\alpha\alpha}) \leq R(A_{\alpha\alpha}^m x_{(\alpha)}) \leq R(A^m x).$$

Assume now that (6.2) holds. Then for any final state α with respect to $\text{supp } x$, we must have

$$\lim_{m \rightarrow \infty} r(A_{\alpha\alpha}^m x_{(\alpha)}) = \lim_{m \rightarrow \infty} R(A_{\alpha\alpha}^m x_{(\alpha)}) = \rho(A_{\alpha\alpha}) = 1.$$

So α is a singular vertex. If $A_{\alpha\alpha}$ is imprimitive, then the condition (v) of Lemma 6.4 holds. Hence, $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy (iv) of Lemma 6.4. This proves one direction of our theorem.

Assume now that if $\alpha \in \mathcal{F}(\text{supp } x)$ then $\rho(A_{\alpha\alpha}) = 1$; and if $A_{\alpha\alpha}$ is not primitive then $A_{\alpha\alpha}$ and $x_{(\alpha)}$ satisfy the condition (iv) of Lemma 6.4.

Let $1 \leq \beta \leq \nu$. Let $d = d(\beta, J)$. By our assumption, $d \neq -1$. If $d = -\infty$, then $(A^m x)_\beta = 0$, $m = 1, 2, \dots$. If $d \geq 0$, then

$$m^{-d} (A^m x)_\beta = m^{-d} \sum_{\alpha \in K} A_{\beta\alpha}^{(m)} x_\alpha + o(1),$$

where $K = \{\alpha : k(\beta, \alpha) = d\}$. Clearly $K \subseteq \mathcal{F}(\text{supp } x)$. Thus, to show

$$(6.10) \quad \lim_{m \rightarrow \infty} m^{-d} (A^m x)_\beta > 0,$$

it is enough to prove

$$(6.11) \quad m^{-d} A_{\beta\alpha}^{(m)} x_\alpha > 0$$

for $\alpha \in \mathcal{F}(\text{supp } x)$, $k(\beta, \alpha) = d$. To prove (6.11), let D be the matrix obtained from A by setting $D_{\alpha\alpha} = 0$ and $D_{\gamma\delta} = A_{\gamma\delta}$ in all other cases, $1 \leq \gamma, \delta \leq \nu$. We then have

$$m^{-d} A_{\beta\alpha}^{(m)} x_\alpha = m^{-d} \sum_{p=0}^m D_{\beta\alpha}^{(m-p)} A_{\alpha\alpha}^p x_\alpha.$$

Since in D , the singular distance from β to α is $d - 1$, we have, by Corollary 5.11,

$$\lim_{m \rightarrow \infty} m^{-d} \sum_{p=0}^m D_{\beta\alpha}^{(m-p)} = U_{\beta\alpha} > 0,$$

and by Lemma (6.4)

$$\lim_{p \rightarrow \infty} A_{\alpha\alpha}^p x_\alpha = v_\alpha > 0.$$

It easily follows from Lemma 2.7 that

$$\lim_{m \rightarrow \infty} m^{-d} A_{\beta\alpha}^{(m)} x_\alpha = \frac{1}{d} U_{\beta\alpha} v_\alpha > 0.$$

Thus, for each β , $1 \leq \beta \leq \nu$, either $(A^m x)_\beta = 0$, $m = 1, 2, \dots$ or (6.10) is satisfied. From this (6.2) follows immediately. \square

COROLLARY 6.12. *Let $A \in \mathbb{R}_+^{n \times n}$, $\rho(A) > 0$. Assume that A is the Frobenius form (4.1). Let J be a nonempty set of $\langle \nu \rangle$. Then for any $x \geq 0$ whose support is the set J , (6.2) holds if and only if for all final states α with respect to J , $\rho(A_{\alpha\alpha}) = \rho(A)$ and $A_{\alpha\alpha}$ is primitive.*

COROLLARY 6.13. *Let $A \in \mathbb{R}_+^{n \times n}$, $\rho(A) > 0$. Assume that A is in the Frobenius form (4.1). Then for any $x \geq 0$, $x \neq 0$, (6.2) holds if and only if for each α , $\alpha = 1, \dots, \nu$, $\rho(A_{\alpha\alpha}) = \rho(A)$ and $A_{\alpha\alpha}$ is primitive.*

7. Nonnegative solutions of $(I - A)y = x$. As an application of our results we give a simple proof of a theorem concerning nonnegative solutions y of $(I - A)y = x$ for given $x \geq 0$. For $1 \leq \alpha, \beta \leq \nu$ we shall say that β has access to α in $G(A)$ if there is a path from β to α in $G(A)$, viz., $k(\beta, \alpha) \geq -1$.

THEOREM 7.1. *Let $A \in \mathbb{R}_+^{n \times n}$ with $\rho(A) = 1$, and suppose that A is in the Frobenius normal form (4.1). Let $x \in \mathbb{R}_+^n$. Then the following are equivalent:*

- (i) *there is a $y \in \mathbb{R}_+^n$ such that $(I - A)y = x$;*
- (ii) *no singular vertex β has access in $G(A)$ to any $\alpha \in \text{supp } x$;*
- (iii) *$\lim_{m \rightarrow \infty} (I + \dots + A^m)x$ exists;*
- (iv) *$\lim_{m \rightarrow \infty} A^m x = 0$.*

Further, if (iii) holds and $y = \lim_{m \rightarrow \infty} (I + A + \dots + A^m)x$, then $(I - A)y = x$ and

$$(7.2) \quad y_\beta = 0 \quad \text{if } \beta \text{ does not have access to any } \alpha \in \text{supp } x,$$

$$(7.3) \quad y_\beta > 0 \quad \text{if } \beta \text{ has access to some } \alpha \in \text{supp } x.$$

Proof. Let $S^{(m)} = I + A + \dots + A^m$. If $1 \leq \beta \leq \nu$, then

$$(7.4) \quad (S^{(m)} x)_\beta = \sum_{\alpha \in \text{supp } x} S_{\beta\alpha}^{(m)} x_\alpha,$$

and, by Corollary 5.11, for $k = k(\beta, \alpha) \geq -1$,

$$(7.5i) \quad \lim_{m \rightarrow \infty} m^{-(k+1)} S_{\beta\alpha}^{(m)} = U_{\beta\alpha} > 0;$$

while for $k(\beta, \alpha) = -\infty$,

$$(7.5ii) \quad S_{\beta\alpha}^{(m)} = U_{\beta\alpha} = 0, \quad m = 1, 2, 3, \dots$$

We shall prove (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i), (iii) \Rightarrow (iv) \Rightarrow (ii).

(i) \Rightarrow (ii). Suppose that $(I - A)y = x$, where $y \geq 0$. Then

$$S^{(m)} x = (I - A^{m+1})y \leq y.$$

Let β be a singular vertex. If β has access to α , then $k = k(\beta, \alpha) \geq 0$ and, by (7.4) and (7.5),

$$y_\beta \geq (S^{(m)} x)_\beta \geq \frac{1}{2} m^{(k+1)} U_{\beta\alpha} x_\alpha$$

for large m . Hence $x_\alpha = 0$ and $\alpha \notin \text{supp } x$.

(ii) \Rightarrow (iii). Suppose (ii) holds and let $1 \leq \beta \leq \alpha$.

If $\alpha \in \text{supp } x$, then $k = k(\beta, \alpha) = -1$, or $k = -\infty$. Hence, $\lim_{m \rightarrow \infty} S_{\beta\alpha}^{(m)} x_\alpha = U_{\beta\alpha} x_\alpha$ exists for $\alpha \in \text{supp } x_\alpha$. So, by (7.5), $\lim_{m \rightarrow \infty} S^{(m)} x$ exists.

(iii) \Rightarrow (i). Let $y = \lim_{m \rightarrow \infty} S^{(m)} x$. Clearly $y \geq 0$. Since $AS^{(m)} x = S^{(m+1)} x - x$, y satisfies $(I - A)y = x$. This proves (i).

(iii) \Rightarrow (iv). Trivial.

(iv) \Rightarrow (ii). Suppose that (iv) holds but that (ii) is false. Then there exists a singular β and an $\alpha \in \text{supp } x$ such that $k(\beta, \alpha) \geq 0$. Let $q = q(\beta, \alpha)$ be the local period and let $B^{(m)} = A^m(I + \dots + A^{q-1})$. Then $\lim_{m \rightarrow \infty} B^{(m)} x = 0$. But by Theorem 5.10 for all sufficiently large m ,

$$(B^{(m)} x)_\beta \geq B_{\beta\alpha}^{(m)} x_\alpha \geq cm^k x_\alpha,$$

where $c > 0$, and $x_\alpha \neq 0$. This is a contradiction, and the implication is proved.

To complete the proof of the theorem observe that, for $y = \lim_{m \rightarrow \infty} S^{(m)} x$,

$$y_\beta = \sum_{\alpha \in \text{supp } x} U_{\beta\alpha} x_\alpha$$

in view of (ii) and (7.5). Since $U_{\beta\alpha} > 0$, if β has access to α and $U_{\beta\alpha} = 0$ otherwise, we immediately obtain (7.2) and (7.3). \square

The equivalence of conditions (i) and (ii) in Theorem 7.1 is due to D. H. Carlson [3]. We remark that Carlson also showed that if a nonnegative solution y of $(I - A)y = x$ exists, then the solution satisfying (7.2) and (7.3) is unique. It should be observed that the assumption that A is in Frobenius normal form is not needed for conditions (i), (iii) and (iv) of Theorem 7.1, which may easily be proved equivalent directly. Conditions (iii) and (iv) are equivalent for general $A \in \mathbb{R}^{nn}$ and $x \in \mathbb{R}^n$. We observe that for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

there is a $y \in \mathbb{R}^n$ such that $(I - A)y = x$; yet the equivalent conditions (ii), (iii) and (iv) do not hold. Clearly, no y satisfying $(I - A)y = x$ can be nonnegative.

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