Matrices Diagonally Similar to a Symmetric Matrix

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

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ABSTRACT

Let $\mathbb{F}$ be field, and let A and B be $n\times n$ matrices with elements in $\mathbb{F}$. Suppose that A is completely reducible and that B is symmetric. If the principal minors of A determined by the 1- and 2-circuits of the graph of B and by the chordless circuits of the graph of A are equal to the corresponding principal minors of B, then A is diagonally similar to B; and conversely.

1. INTRODUCTION

Let A and B be completely reducible matrices with elements in a field. Then A and B are diagonally similar if and only if each circuit product of A equals the corresponding circuit product of B: see Bassett, Maybee, and Quirk [1, Proposition 2] and Fiedler and Pták [4, Theorem 3.1]; see also [2, Corollary 4.4], [5]. Diagonal similarity of A and B immediately implies that each principal minor of A equals the corresponding principal minor of B, but results in the converse direction are not yet well understood. A result of this type is essentially to be found in [3, Corollary 6.7]: If A and B are completely

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reducible matrices with elements in the complex field and B is an M-matrix, then A is diagonally similar to an M-matrix. In this note an example is given showing that the equality between corresponding principal minors of A and B does not guarantee that A is diagonally similar to B or B^t: see Remark and Example 3.7(ii). However, if we replace the requirement that B be an M-matrix by the requirement that B be symmetric, then equality between corresponding principal minors is equivalent to the diagonal similarity of A and B. In fact, a stronger result is proved, for the condition of equality need not be imposed on all principals minors, but only on minors determined by certain classes of circuits: see our main result, which is stated in the abstract and as Theorem 3.5.

Our theorem is an easy consequence of some lemmas. The techniques of proof of these lemmas are graph theoretic, and they employ the concept of a chordless circuit of a graph.

Our result suggests problems which we call *inverse minor problems*; see Problem 4.2.

2. NOTATION AND DEFINITIONS

Throughout this paper n will denote a fixed positive integer and $\langle n\rangle=\{1,\dots,n\}$.

DEFINITION 2.1.

(i) A (directed) graph G is a subset of $\langle n\rangle\times\langle n\rangle$. If $(i,j)\in G$, we call (i,j) an *arc* of G. [Note that G corresponds to the arcset of the graph $(\langle n\rangle,G)$ as defined by most authors.]

(ii) $G_n=\langle n\rangle\times\langle n\rangle$.

(iii) A graph G is *symmetric* if $(i,j)\in G$ implies that $(j,i)\in G$.

(iv) Let s be a positive integer. A *circuit of G of length s* (or s-circuit) is a sequence $\gamma=(i_1,\dots,i_s)$ of distinct integers in $\langle n\rangle$ such that for $k=1,\dots,s$, $(i_k,i_{k+1})\in G$, where $i_{s+1}=i_1$. We call (i_k,i_{k+1}) an *arc of* $v\gamma$, and i_k a *vertex of* $\gamma, k=1,\dots,s$. We put $|\gamma|=s$. Also $\bar{\gamma}=\{i_1,\dots,i_s\}$ is called the *support* of γ. We identify $(i_1,\dots,i_s)$ and $(i_k,\dots,i_s,i_1,\dots,i_{k-1})$, $k=2,\dots,s$.

(v) If $\gamma=(i_1,\dots,i_s)$ is a circuit of G, then γ^{-1} is the circuit $(i_s,i_{s-1},\dots,i_1)$ of G_n.

(vi) Let γ be a circuit of G. A *chord* of γ is an arc (i,j) of G such that i and j are distinct vertices of γ, but neither (i,j) nor (j,i) is an arc of γ.

(vii) A circuit γ of G is *chordless* if there is no chord of γ.

REMARK 2.2.

(i) If γ is a 1-, 2-, or 3-circuit of G, then γ is chordless. If $G=G_n$, then the converse holds.

(ii) If γ is a chordless circuit of G, and α is a circuit of G such that $\bar{\alpha} \subseteq \bar{\gamma}$, then $|\alpha| \leqslant 2$ or $\alpha = \gamma$ or $\alpha = \gamma^{-1}$.

(iii) If β is a circuit of G, there is a chordless circuit γ of G such that $\bar{\gamma} \subseteq \bar{\beta}$.

(iv) If G is a symmetric graph, then every arc of G lies on a chordless circuit.

Henceforth, $\mathbb{F}$ will denote a field, and $\mathbb{F}^{nn}$ the set of all $n \times n$ matrices with elements in $\mathbb{F}$. If $\varnothing \subset \omega \subseteq \langle n \rangle$, then $A[\omega]$ is the principal submatrix of A lying in the intersection of the rows and columns indexed by ω in their natural orders. A *principal minor* is the determinant of a principal submatrix. The matrices A and B in $\mathbb{F}^{nn}$ are *diagonally similar* if there exists a nonsingular diagonal matrix X in $\mathbb{F}^{nn}$ for which $XAX^{-1} = B$.

DEFINITION 2.3. Let $A \in \mathbb{F}^{nn}$.

(i) The *graph* $G(A)$ *of* A is defined by

$$G(A) = \{(i,j) \in \langle n \rangle \times \langle n \rangle : a_{ij} \neq 0\}.$$

(ii) The matrix A is said to be *combinatorially symmetric* if $G(A)$ is a symmetric graph.

(iii) The matrix $A \in \mathbb{F}^{nn}$ is called *completely reducible* if every arc of $G(A)$ is the arc of a circuit of $G(A)$.

(iv) If $\gamma = (i_1, \ldots, i_s)$ is a circuit of G_n, then the *circuit product* $\Pi_\gamma(A)$ is defined by

$$\Pi_\gamma(A) = a_{i_1 i_2} \cdots a_{i_s i_{s+1}},$$

where $i_{s+1} = i_1$.

REMARK 2.4.

(i) Evidently $\Pi_\gamma(A) \neq 0$ if and only if γ is a circuit of $G(A)$.

(ii) It is well known (e.g. [2]) that A is completely reducible if and only if after simultaneous permutation of rows and columns A is the direct sum of irreducible matrices.

3. MAIN RESULTS

LEMMA 3.1. *Let* $A, B \in \mathbb{F}^{nn}$, *where* B *is combinatorially symmetric. Let* (i) $\det A[\bar{\gamma}] = \det B[\bar{\gamma}]$ *if* γ *is a* 1- *or* 2-*circuit of* $G(A)$ *or* $G(B)$.

Then

(ii) $\Pi_\gamma(A)=\Pi_\gamma(B)$ *if* γ *is a* 1- *or* 2-*circuit of* G_n, *and*
(iii) $G(B)\subseteq G(A)$.

Proof. (ii): It is easy to see that (i) implies that $\det A[\bar{\gamma}]=\det B[\bar{\gamma}]$ for all 1- and 2-circuits γ of G_n. Hence $a_{ii}=b_{ii}$ for all $i\in\langle n\rangle$. Thus $a_{ii}a_{jj}-a_{ij}a_{ji}=b_{ii}b_{jj}-b_{ij}b_{ji}$ now implies that $a_{ij}a_{ji}=b_{ij}b_{ji}$, $i\neq j$, $i,j\in\langle n\rangle$.

(iii): Suppose that $b_{ij}\neq 0$. If $i=j$, then $a_{ij}=b_{ij}\neq 0$ by (ii). Suppose that $i\neq j$. Then, by the combinatorial symmetry of $b_{ji}=0$. But then, by (ii), $a_{ij}a_{ji}=b_{ij}b_{ji}\neq 0$ and so $a_{ij}\neq 0$. ■

LEMMA 3.2. *Let* $A,B\in\mathbb{F}^{nn}$, *where* B *is combinatorially symmetric. Suppose that*

(i) $\det A[\bar{\gamma}]=\det B[\bar{\gamma}]$ *if* γ *is a* 1- *or* 2-*circuit of* $G(B)$ *or* γ *is a chordless circuit of* $G(A)$.

Then

(ii) $\Pi_\gamma(A)+\Pi_{\gamma^{-1}}(A)=\Pi_\gamma(B)+\Pi_{\gamma^{-1}}(B)$ *if* γ *is a chordless circuit of* $G(A)$.

Proof. In view of Lemma 3.1, we may suppose that γ is a chordless circuit of $G(A)$, where $|\gamma|\geq 3$. Then for $C=A$ or $C=B$,

$$\det C[\bar{\gamma}]=(-1)^{|\gamma|}(\Pi_\gamma(C)+\Pi_{\gamma^{-1}}(C))+\varphi(C), \tag{iii}$$

where $\varphi(C)$ is a sum of products of form $\pm\Pi_{\alpha_1}(C)\cdots\Pi_{\alpha_k}(C)$, and the α_i are 1- and 2-circuits. Hence, by Lemma 3.1, $\varphi(A)=\varphi(B)$. We now obtain $\Pi_\gamma(A)+\Pi_{\gamma^{-1}}(A)=\Pi_\gamma(B)+\Pi_{\gamma^{-1}}(B)$ from (iii) and our assumption (i). ■

The next lemma is our chief graph theoretic result.

LEMMA 3.3. *Let* $A,B\in F^{nn}$, *where* A *is completely reducible and* B *is combinatorially symmetric. Suppose that*

(i) $\det A[\gamma]=\det B[\gamma]$ *if* γ *is either a* 1- *or* 2-*circuit of* $G(B)$ *or* γ *is a chordless circuit of* $G(A)$.

Then

(ii) $G(A)=G(B)$.

Proof. By Lemma 3.1, $G(B) \subseteq G(A)$. To prove $G(A) \subseteq G(B)$ it is enough to show that $G(A)$ is symmetric, for then the roles of A and B may be interchanged in Lemma 3.1.

To prove $G(A)$ is symmetric, we first show that if β is a chordless circuit of $G(A)$, then β^{-1} is also a circuit of $G(A)$. Suppose this result is false for the chordless circuit β of $G(A)$. Clearly $|\beta| \geqslant 3$, and by Lemma 3.2,

$$0 \neq \Pi_\beta(A) = \Pi_\beta(A) + \Pi_{\beta^{-1}}(A) = \Pi_\beta(B) + \Pi_{\beta^{-1}}(B).$$

Hence either $\Pi_\beta(B) \neq 0$ or $\Pi_{\beta^{-1}}(B) \neq 0$. But, by the combinatorial symmetry of B, $\Pi_\beta(B) \neq 0$ implies that $\Pi_{\beta^{-1}}(B) \neq 0$. It follows that $\Pi_{\beta^{-1}}(B) \neq 0$. Since $G(B) \subseteq G(A)$, we deduce that $\Pi_{\beta^{-1}}(A) \neq 0$, and hence β^{-1} is a circuit of $G(A)$.

Now let $a_{ij} \neq 0$, $i \neq j$. We shall prove that $a_{ji} \neq 0$. Since A is completely reducible, there is a circuit γ of $G(A)$ of which (i,j) is an arc. Let γ be such a circuit of shortest length. If γ has a chord (k,l), then there is a chordless circuit β of $G(A)$ such that $\bar{\beta} \subseteq \bar{\gamma}$, $|\beta| < |\gamma|$, and (k,l) is an arc of β. By the previous paragraph, β^{-1} is also a circuit of $G(A)$, and so (l,k) is also a chord of γ. Since both (k,l) and (l,k) are arcs of $G(A)$, there is a circuit α of $G(A)$ of which (i,j) is an arc and $|\alpha| < |\gamma|$. This is contrary to our choice of γ. Hence γ is chordless and γ^{-1} is therefore also a circuit of $G(A)$. It follows that $a_{ji} \neq 0$. Thus A is combinatorially symmetric and the lemma is proved. ∎

REMARK 3.4.

(i) The assumption that A is completely reducible cannot be omitted in the hypothesis of Lemma 3.3. As an example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(ii) In Lemma 3.3 the assumption that B is combinatorially symmetric cannot be replaced by the weaker assumption that B is completely reducible. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then $G(A) \neq G(B)$ and $G(A) \neq G(B^t)$.

THEOREM 3.5. *Let $A \in \mathbb{F}^{nn}$, $B \in \mathbb{F}^{nn}$, where A is completely reducible and B is symmetric. Then the following are equivalent:*

(i) $\det A[\bar{\gamma}] = \det B[\bar{\gamma}]$ *if γ is a 1- or 2-circuit of $G(B)$ or γ is a chordless circuit of $G(A)$.*

(ii) *The matrices A and B are diagonally similar.*

Proof. (ii)⇒(i): Trivial.

(i)⇒(ii): Suppose (i) holds. By Lemma 3.3, $G(A) = G(B)$. Let γ be a chordless circuit of $G(A)$. By Lemma 3.1, $\Pi_\gamma(A) = \Pi_\gamma(B)$ if γ is a 1- or 2-circuit. We shall prove that the equality holds also if $|\gamma| \geqslant 3$. In this case, by Lemma 3.2,

$$\Pi_\gamma(A) + \Pi_{\gamma^{-1}}(A) = \Pi_\gamma(B) + \Pi_{\gamma^{-1}}(B) = 2\Pi_\gamma(B), \tag{iii}$$

where the last equality holds because B is symmetric. Further, for all $C = A$ or $C = B$,

$$\Pi_\gamma(C)\Pi_{\gamma^{-1}}(C) = \Pi_{\alpha_1}(C) \cdots \Pi_{\alpha_k}(C)$$

where $\alpha_1, \ldots, \alpha_k$ are 2-circuits of $G(A)$. Hence, by Lemma 3.1,

$$\Pi_\gamma(A)\Pi_{\gamma^{-1}}(A) = \Pi_\gamma(B)\Pi_{\gamma^{-1}}(B) = \Pi_\gamma(B)^2. \tag{iv}$$

By the elementary theory of quadratic equations, it follows from (iii) and (iv) that $\Pi_\gamma(A) = \Pi_\gamma(B)$.

Now let β be any circuit of $G(A)$. Suppose that $|\beta| \geqslant 3$. Since $G(A) = G(B)$ is symmetric, it is easy to show that for $C = A$ or $C = B$,

$$\Pi_\beta(C) = \frac{\Pi_{\gamma_1}(C) \cdots \Pi_{\gamma_k}(C)}{\Pi_{\alpha_1}(C) \cdots \Pi_{\alpha_{k-1}}(C)}, \tag{v}$$

where $\gamma_1, \ldots, \gamma_k$ are chordless circuits of $G(A)$ and $\alpha_1, \ldots, \alpha_{k-1}$ are 2-circuits of $G(A)$. By Lemma 3.1 and the previous paragraph, we now obtain that $\Pi_\beta(A) = \Pi_\beta(B)$ for all circuits β of $G(A)$.

The diagonal similarity of A and B now follows by [5, (5.1)]. ■

COROLLARY 3.6. *Let $\mathbb{F}$ be a totally ordered field, and let A and B in $\mathbb{F}^{nn}$ be symmetric matrices with nonnegative elements. Then the following are equivalent:*

(i) $\det A[\bar{\gamma}] = \det B[\bar{\gamma}]$ *if γ is a 1- or 2-circuit of $G(B)$ or γ is a chordless circuit of $G(A)$.*
(ii) $A = B$.

Proof. (ii)$\Rightarrow$(i): Trivial.
(i)$\Rightarrow$(ii): By Theorem 3.5, there is a nonsingular diagonal matrix $X \in \mathbb{F}^{nn}$ such that $XAX^{-1} = B$. Suppose $a_{ij} \neq 0$ for some $i, j \in \langle n \rangle$. Then $x_i a_{ij} x_j^{-1} = b_{ij} = b_{ji} = x_j a_{ji} x_i^{-1}$, whence $x_i^2 = x_j^2$. It follows that $b_{ij} = \pm a_{ij}$. But both $a_{ij} \geqslant 0$, $b_{ij} \geqslant 0$, and hence $b_{ij} = a_{ij}$. ■

REMARK AND EXAMPLE 3.7.

(i) The chordless circuits of G_n are precisely the 1-, 2-, and 3-circuits. Hence, in this case, there are $c_n = n + n(n-1)/2 + n(n-1)(n-2)/6 = n(n^2+5)/6$ principal minors determined by the end chordless circuits. For $G \subseteq \langle n \rangle \times \langle n \rangle$, the number of such minors is bounded above by c_n.

(ii) In Theorem 3.5, it is not possible to replace the assumption that B is symmetric by the weaker assumption that B is combinatorially symmetric, even if $\det A[\omega] = \det B[\omega]$, for *all* ω, $\varnothing \subset \omega \subseteq \langle n \rangle$. These equalities hold in the following example, in which $G(A) = G(B) = G_n$: Let

$$A = \begin{bmatrix} 10 & -1 & -2 & -1 \\ -1 & 10 & -1 & -1 \\ -3 & -1 & 10 & -1 \\ -1 & -4 & -1 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} 10 & -1 & -3 & -1 \\ -1 & 10 & -1 & -1 \\ -2 & -1 & 10 & -1 \\ -1 & -4 & -1 & 10 \end{bmatrix}.$$

Let $\alpha = (1,3,2,4)$. Then $\Pi_\alpha(A) = 2$, $\Pi_\alpha(B) = 3, \Pi_\alpha(B^t) = 8$, whence by (the trivial direction of the) theorem of [1] and [4] quoted, A is diagonally similar neither to B nor to B^t. The matrices A and B are M-matrices (for definition see [2]) and hence they furnish the example mentioned in the introduction.

4. EXAMPLE AND PROBLEMS

Let A and B be completely reducible matrices. The result of Maybee, Bassett, and Quirk [1] and of Fiedler and Pták [4] may easily be strengthened. Thus, if $G(A) \supseteq G(B)$, to guarantee the diagonal similarity of A and B it is sufficient to assume $\Pi_\gamma(A) = \Pi_\gamma(B)$ for γ in a subset Γ of the set of all circuits of $G(A)$. The set Γ can be chosen to have basis properties standard in algebraic graph theory; for details see [5], particularly Corollary (2.4) and Remark (5.2). If $G(A)=G_n$, then Γ can be chosen to have n^2-n+1 elements while the number of minors in condition (i) of Theorem (3.5) is of order n^3 [see Corollary 3.7(i)]. But it does not appear to be easy to improve Lemma 3.3 or Theorem 3.5 by considering subsets of the sets of principal minors in conditions (i) of these results. This is shown by the following striking example.

EXAMPLE 4.1. Let $n \geqslant 3$ and let $a \in \mathbb{F}$. Let $A, B \in \mathbb{F}^{nn}$ be given by $b_{ii}=0$, $i=1,\ldots,n$, and $b_{ij}=1$, $i \neq j$, $i,j=1,\ldots,n$. Let $a_{12}=a$, $a_{21}=2-a$, and $a_{ij}=b_{ij}$ otherwise. Observe that A is completely reducible and B is symmetric. Further, with the exception of $\gamma=(1,2)$, $\det A[\bar{\gamma}]=\det B[\bar{\gamma}]$ for all 1-, 2-, and 3-circuits γ of G_n. But, if $a \neq 1$, then A and B are not diagonally similar, and if $a=0$ or $a=2$, it is even false that $G(B) \subseteq G(A)$.

PROBLEM 4.2 (Inverse minor problems).

(i) In view of Theorem 3.5 it is interesting to pose the following problem: Let $\mathcal{P}_n$ be the set of ω, $\varnothing \subset \omega \subseteq \langle n \rangle$. Characterize the set of families $\{b_\omega : \omega \in \mathcal{P}_n\}$ such that there exists a symmetric $B \in \mathbb{F}^n$ for which $\det B[\omega]= b_\omega$, for $\omega \in \mathcal{P}_n$.

(ii) It is possible to vary the problem. Let G be a symmetric graph, and let $\mathcal{Q}$ be the set of $\bar{\gamma}$ such that γ is a chordless circuit of G (or, of course, some other subset of $\mathcal{P}_n$). Characterize the set of families $\{b_\omega : \omega \in \mathcal{Q})$ such that there exists a symmetric B with $G(B)=G$ and $\det B[\omega]=b_\omega$ for $\omega \in \mathcal{Q}$.

(iii) Characterize the minimal subsets $\mathcal{Q}$ of $\mathcal{P}_n$ such that for completely reducible $A \in \mathbb{F}^{nn}$ and symmetric $B \in \mathbb{F}^{nn}$, $\det A[\omega]=\det B[\omega]$ for all $\omega \in \mathcal{Q}$ implies that A and B are diagonally similar.

(iv) Finally, in view of [3, Corollary 6.7] it is interesting to consider similar problems for completely reducible M-matrices in place of symmetric matrices.

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