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SIAM REVIEW
Vol. 21, No. 4, October 1979

0036-1445/79/2104-0004$01.00/0

APPLICATIONS OF THE GORDAN–STIEMKE THEOREM IN COMBINATORIAL MATRIX THEORY*

Dedicated to Leon Mirsky on the occasion of his 60th birthday

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Abstract. By use of the Gordan–Stiemke Theorem of the alternative we demonstrate the similarity of four theorems in combinatorial matrix theory. Each theorem contains five equivalent conditions, one of which is the existence in a given pattern of a line-sum-symmetric or constant-line-sum matrix which is semi-positive or strictly positive for the pattern. A generalization of the Gordan–Stiemke Theorem is stated in terms of complementary faces of the positive orthant and combinatorial applications are given. Many of our results are classical, but some may be new.

Introduction. The fascinating interplay between the theory of linear inequalities and combinatorics has been much exploited in the past quarter century; see the surveys by A. J. Hoffman [25], [26]. Our paper is offered as another example of this interplay. However, our thrust is directed towards basic and classical results, not recent generalizations. Specifically, we use a geometric form of a theorem of the alternative in the theory of linear inequalities to derive results in combinatorial matrix theory.

Some of the combinatorial results obtained have a long history and many applications. For example, we derive the famous Frobenius–König Theorem [29], [18]. A slightly more general theorem is P. Hall's theorem on systems of distinct representatives [22], [36, p. 27]. This is a basic result in combinatorics, which Mirsky in his book on Transversal Theory has called "the master key which has unlocked many closed doors" [36, p. 38].

Theorems of the alternative play a fundamental role in linear programming and hence in such related areas as linear complementarity and nonlinear programming, see Dantzig [11, pp. 136–139], where a short historical survey may be found [11, p. 21]. See also Gale [19], Cottle–Dantzig [10], Mangasarian [32], [33].

Early theorems of the alternative are the theorems of Gordan [21] and Stiemke [45]. Expressed in terms of complementary subspaces the theorems coincide in what we call the *Gordan–Stiemke Theorem.* This is our chief tool and it permits us to prove combinatorial results on matrices. Though the theorems we obtain are largely known, they have not been considered previously in a unified manner. By means of our approach, we stress the underlying similarity of the structure of four theorems.

Our four principal theorems are labeled Theorems AA, AB, BA, BB and each contains five equivalent conditions numbered (i)–(v), where corresponding numbers refer to corresponding parts of the theorems. Thus condition (ii) in each theorem refers to the existence of a certain type of matrix for a given pattern P. (For precise definitions, see § 2 and § 3.) The types of matrices are line-sum-symmetric matrices (ith row sum equal to ith column sum) and constant-line-sum matrices (all row and column sums

* Received by the editors July 10, 1978, and in revised form December 29, 1978. This research was partly supported by the National Science Foundation under Grants MCS 76-06958 and MCS 76-06374.

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equal), and the four conditions numbered (ii) are:

	A	B
A	There exists a line-sum-symmetric matrix semi-positive for P	There exists a line-sum-symmetric matrix strictly positive for P
B	There exists a constant-line-sum matrix semi-positive for P	There exists a constant-line-sum matrix strictly positive for P

Conditions (i) and (ii) are easily seen to be equivalent, and from conditions (ii) we derive by means of the Gordan–Stiemke Theorem the equivalent conditions (iii), which involve the nonexistence in the given pattern of certain types of matrices which we have called difference and bidifference matrices. Conditions (iii), (iv) and (v) are then shown to be equivalent by elementary arguments. A direct graph theoretic proof of the equivalence of (i), (iii), (v) is easy and known in the case of Theorems AA and AB (in the case of, Theorem AA, cf. Harary–Norman–Cartwright [24, Thm. 10.1]). We have not found such a proof in the case of Theorems BA and BB, and here the equivalence of (iii) to the other conditions may be new.

There are two types of proof of the Gordan–Stiemke Theorem in the literature: those that depend on a separation theorem in real n-space, e.g. Nikaido [38, § 3.3], Ben-Israel [6], Levine and Shapiro [31], and inductive proofs that use inequalities and are valid over any (totally) ordered field, e.g. Gordan [21], Stiemke [45], Tucker [46], Gale [19, § 2.3]. We have chosen to state our result for matrices with elements in an ordered field and we use Gale [19] as our basic reference. The real numbers obviously form the most important example of an ordered field. But our results also hold over the rational numbers, and it is then an easy exercise to show that our Theorems AA, AB, BA, BB as well as Theorems 4.8 and 4.9 are also valid for matrices of *integers*.

We have stated our results in terms of patterns, i.e. (0, 1)-matrices. There is an obvious 1–1 correspondence between patterns and directed simple graphs, and in order to minimize definitions, we use graphs in a very restricted manner in the formal part of the paper. But in our comments we use some familiar concepts that have not been defined formally.

In § 1 we give the relevant geometric definitions and we state the Gordan–Stiemke Theorem without proof. In § 2 we apply the Gordan–Stiemke Theorem to nonnegative line-sum-symmetric matrices and nonnegative difference matrices. In § 3 the applications are to nonnegative constant-line-sum matrix and nonnegative bidifference matrices. A semi-positive constant-line-sum matrix is just a multiple of a doubly-stochastic matrix; we have used our own (new) terminology in the theorems in order to have a name for the class of matrices involved. Further, the pattern given by the 0-matrix would be exceptional in the results if they were stated in terms of doubly-stochastic matrices.

In § 4 we state and prove a generalization of the Gordan–Stiemke Theorem to complementary faces of the positive orthant in a form which is more general than required subsequently. Though several closely related theorems are known, e.g. Gale [19, p. 71, Ex. 24], Ben-Israel [5], we have not found in the literature our formulation which is natural for the combinatorial application in this section. In § 5 we summarize without proof additive analogs of applications of the Gordan–Stiemke Theorem to multiplicative minmax theorems for real matrices which are considered in detail in [42]. For these results the duality between line-sum-symmetric matrices and difference matrices, and between constant-line-sum matrices and bidifference matrices is crucial. It was the need to explore the dualities implicit in the above and similar situations (e.g. [12], [13], [40], [41]) that motivated the investigations in this paper.

For related theorems valid over arbitrary (unordered) fields see [41]. References to alternative proofs of many results will be found in our comments throughout.

1. The Gordan–Stiemke Theorem. In this paper $\mathbb{F}$ will denote an ordered field. For $m = 0, 1, 2, \cdots$, we call $x \in \mathbb{F}^m$ *nonnegative* $(x \geqq 0)$ if $x_i \geqq 0$, $i = 1, \cdots, m$, *semi-positive* $(x \geq 0)$ if $x_i \geqq 0$, $i = 1, \cdots, m$, and for some k, $1 \leqq k \leqq m$, $x_k > 0$; and *strictly positive* $(x > 0)$ if $x_i > 0$, $i = 1, \cdots, m$. It is advantageous to include the apparently trivial case $m = 0$; here $\mathbb{F}^0 = \{0\}$, and 0 is nonnegative, strictly positive but *not* semi-positive. We put

$$\mathbb{F}^m_+ = \{x \in \mathbb{F}^m : x \geqq 0\}.$$

If W is a subspace of $\mathbb{F}^m$ we write $W_+ = W \cap \mathbb{F}^m_+$. A (convex) *cone* K is a nonempty subset of $\mathbb{F}^m$ such that $x, y \in K$, $\alpha \geqq 0$, $\alpha \in \mathbb{F}$ imply that $x + y \in K$ and $\alpha x \in K$. The (positive) *dual* in $\mathbb{F}^m$ of a cone K is the set K^D defined by

$$K^D = \{y \in \mathbb{F}^m : y'x \geqq 0, \forall x \in K\}.$$

It is easy to check that K^D is again a cone. For a subspace W of $\mathbb{F}^m$, $W^D = W^\perp$, the orthogonal complement. If $K, L \subseteq \mathbb{F}^m$, then $K + L = \{x + y : x \in K, y \in L\}$.

We are now ready to state our principal lemmas. Our first lemma follows immediately by combining the Corollary on p. 58 of Gale [19], with parts of his Theorem (2.14):

LEMMA 1.1. *Let W be a subspace of $\mathbb{F}^m$. Then*

(i) $(W \cap \mathbb{F}^m_+)^D = W^D + \mathbb{F}^m_+$,

(ii) $(W + \mathbb{F}^m_+)^D = W^D \cap \mathbb{F}^m_+$.

LEMMA 1.2. (Gordan–Stiemke theorem, Gale [19, p. 48]). *Let W be a subspace of $\mathbb{F}^m$. Then the following are equivalent*:

(i) *W contains a semi-positive vector.*

(ii) *$W^\perp$ contains* no *strictly positive vector.*

Comments. (i) Observe the result is valid in the case $m = 0$, when clearly $W = W^\perp = \{0\}$, and (i), (ii) are both false.

(ii) Let $\Gamma \in \mathbb{F}^{mn}$, and let

$$W = \{y \in \mathbb{F}^m : y'\Gamma = 0\}.$$

Then Lemma (1.2) becomes the result of Gordan [21], rediscovered by Stiemke [45, Thm. II]: Either there is a semi-positive $y \in \mathbb{F}^m$ such that $y'\Gamma = 0$, or there is an $x \in \mathbb{F}^n$ such that Γx is strictly positive, but not both. Observe that

$$W^\perp = \{\Gamma x : x \in \mathbb{F}^n\}.$$

If we interchange the roles of W and $W^\perp$ in Lemma (1.2) we obtain Stiemke [45, Thm. I]: Either there is a strictly positive $y \in \mathbb{F}^m$ such that $y'\Gamma = 0$, or there is an $x \in \mathbb{F}^n$ such that Γx is semi-positive, but not both. Conversely, each of the results quoted implies our Lemma 1.2, since a subspace W of $\mathbb{F}^m$ may be considered as either the left-hand kernel or the right-hand image of matrix $\Gamma \in \mathbb{F}^{mn}$.

2. Line-sum-symmetric matrices and difference matrices.

DEFINITION 2.1. (i) A *pattern* P is a square $\{0, 1\}$-matrix, viz. $p_{ij} = 0$ or $p_{ij} = 1$, for $i, j = 1, \cdots, n$. Throughout, $n = n(P)$ will denote the order of P and $m = m(P)$ the number of entries of P that equal 1. (Thus $n > 0$, and $0 \leqq m \leqq n^2$).

(ii) Let P be an $(n\times n)$ pattern. Then

$$G(P)=\{\{i,j\}\in\langle n\rangle\times\langle n\rangle : p_{ij}=1\}$$

where $\langle n\rangle=\{1,\cdots,n\}$. We order $G(P)$ lexicographically and we thus obtain $G(P)=\{g_1,\cdots,g_m\}$. (The set $G(P)$ is of course the arc set of a directed graph on the vertex set $\langle n\rangle$.)

(iii) The *incidence matrix* $\Gamma=\Gamma(P)$ of P is the $(m\times n)$ matrix defined thus:

$$\gamma_{qi}=1,\quad \text{if } g_q=(i,j) \text{ and } i\neq j$$

$$\gamma_{qi}=-1,\quad \text{if } g_q=(j,i) \text{ and } i\neq j$$

$$\gamma_{qi}=0,\quad \text{otherwise.}$$

(Observe $\gamma_{qi}=0$ if $g_q=(i,i)$.)

(iv) *The set of matrices in the pattern P* is defined as

$$M(P)=\{A\in\mathbb{F}^{nn}: a_{ij}=0 \text{ whenever } p_{ij}=0\}.$$

(v) Let $A\in M(P)$. We say that A is a *line-sum-symmetric matrix for P* (or, in $M(P)$) if $\sum_{j=1}^n a_{ij}=\sum_{i=1}^n a_{ji}$, $j=1,\cdots,n$. The *line-sum-symmetric space $W_s=W_s(P)$ for* P consists of all line-sum-symmetric matrices for P.

(vi) Let $A\in M(P)$. We call A a *circuit matrix for P* (or, in $M(P)$) if there exists a sequence $(i_1,\cdots,i_k)$ of distinct integers in $\langle n\rangle$ such that $a_{ij}=1$ if $i=i_p$, $j=i_{p+1}$, when $1\leqq p\leqq k$ and $i_{k+1}=i_1$; but $a_{ij}=0$ otherwise. (A circuit matrix is line-sum-symmetric.)

(vii) Let $A\in M(P)$. Then A is called a *difference matrix* for P if there is a diagonal matrix X in $\mathbb{F}^{nn}$ such that $A=XP-PX$, (viz. $a_{ij}=x_i-x_j$ whenever $(i,j)\in G(P)$, and $a_{ij}=0$ otherwise).

(viii) For $\phi\subseteq\alpha\subseteq\langle n\rangle$, let $\alpha'=\langle n\rangle\backslash\alpha$. Put $Z_\alpha=\operatorname{diag}(z_1,\cdots,z_n)$, where $z_i=1$ if $i\in\alpha$, $z_i=0$ if $i\in\alpha'$. The matrix $A\in M(P)$ is called a *cocircuit matrix* for P if A is semi-positive and $A=Z_\alpha P-PZ_\alpha$, for some α, $\phi\subset\alpha\subseteq\langle n\rangle$. (We use $\subset$ for strict inclusion.) (A cocircuit matrix is a difference matrix.)

(ix) We define the *canonical isomorphism* η of $M(P)$ onto $\mathbb{F}^m$ thus: Let $A\in M(P)$. Then $a=\eta(A)$ is the vector in $\mathbb{F}^m$ given by $a_q=a_{ij}$, if $g_q=(i,j)\in G(P)$.

For some additional remarks on some of the concepts defined above see [34].

The mapping η is an isomorphism of $M(P)$ onto $\mathbb{F}^m$ considered as an inner product space where we use the usual inner product $\langle x,y\rangle=y^tx$ in $\mathbb{F}^m$ and $\langle A,B\rangle=\operatorname{trace}(A^tB)$ in $M(P)$. The isomorphism η will allow us to apply the results of § 1 to $M(P)$. Let $A\in M(P)$. It is easy to see that $A\in W_s$ if and only if $a=\eta(A)\in\mathbb{R}^m$ satisfies $a^t\Gamma=0$, cf. [41]. Further, if $b\in\mathbb{F}^m$, then $a^tb=0$ for all a satisfying $a^t\Gamma=0$ if and only if $b=\Gamma x$, for some $x\in\mathbb{R}^n$. But this is equivalent to $b=\eta(B)$, where B is a difference matrix for B. Hence, on applying Lemma 1.1 to $W=W_s$ we obtain:

THEOREM 2.2. *The following cones are dual to each other in $M(P)$:*

(i) *The cone of all nonnegative line-sum-symmetric matrices in $M(P)$.*

(ii) *The cone of all matrices A of form $A=B+C$, where B is a difference matrix for P, and C is nonnegative in $M(P)$.*

For $A\in M(P)$ we use the terms strictly positive and semi-positive relative to $M(P)$: A is *strictly positive for P* if $a_{ij}>0$ whenever $(i,j)\in G(P)$ and $a_{ij}=0$ otherwise.

We now come to the first of our applications of the Gordan–Stiemke Theorem.

THEOREM 2.3 (AA). *Let P be a $(n\times n)$ pattern and let $A\in M(P)$. Then the following are equivalent:*

(i) *There is a circuit matrix in $M(P)$.*

(ii) *There is a semi-positive line-sum-symmetric matrix in $M(P)$.*

(iii) *There is* no *difference matrix strictly positive for P.*

(iv) *There is* $(i, j) \in G(P)$ *such that* $a_{ij} = 0$ *for every cocircuit matrix A for P.*

(v) *There is* no *permutation matrix Q such that* QPQ^{-1} *is strictly upper triangular.*

Proof. (i)⇒(ii): The proof is trivial.

(ii)⇒(i): Let A be a semi-positive line-sum-symmetric matrix in $M(P)$. Since A contains at least one positive element, we have $m(P) > 0$. But A is line-sum-symmetric; hence if the ith row (column) of A contains a positive element, so does the ith column (row). Thus there exists an infinite sequence $(k_1, k_2, k_3, \cdots)$ of elements of $\langle n \rangle$ such that $a_{k_q k_{q+1}} > 0$, $q = 1, 2, 3, \cdots$. Since $\langle n \rangle$ is finite there exists a subsequence of distinct integers $(i_1, \cdots . i_p)$ such that $a_{i_q i_{q+1}} > 0$, $q = 1, \cdots, p$, where $i_{p+1} = i_1$. If C is the corresponding circuit matrix, then $C \in M(P)$.

(ii)⇔(iii): This is the Gordan–Stiemke Theorem applied to $W = W_s$, the space of line-sum-symmetric matrices.

Not (iv)⇒*Not* (iii): Suppose that for every $(i, j) \in G(P)$, $X^{(i,j)}P - PX^{(i,j)}$ is a cocircuit matrix whose (i, j)-entry is 1. Let $X = \sum_{(i,j)\in G(P)} X^{(i,j)}$. Then $XP - PX$ is a difference matrix strictly positive for P.

Not (v)⇒*Not* (iv): If $A = XP - PX$ is a difference matrix for P, and Q is a permutation matrix, then $QAQ^{-1} = (QXQ^{-1})(QPQ^{-1}) - (QPQ^{-1})(QXQ^{-1})$ is difference matrix for the pattern QPQ^{-1}. Hence we may assume that P itself is strictly upper triangular. Let $(i, j) \in G(P)$, and let $\alpha = \langle i \rangle$. Observe that $i < j$ and hence $\phi \subset \alpha \subset \langle n \rangle$. Then $A = Z_\alpha P - PZ_\alpha$ is nonnegative, since $p_{kl} = 0$ if $k \in \alpha'$, $l \in \alpha$. But $a_{ij} = 1$. Hence A is a cocircuit matrix.

Not (iii)⇒*Not* (v): Let $XP - PX$ be strictly positive for P. After simultaneous permutation of rows and columns we may assume that $x_1 \geqq \cdots \geqq x_n$. So if $1 \leqq j \leqq i \leqq n$, then $x_i - x_j \leqq 0$, and hence $(i, j) \notin G(P)$. It follows that P is strictly upper triangular. □

Comments. The implication (ii)⇒(i) of Theorem AA and Theorem AB below is closely related to Afriat [3, Thm. 1], which in turn is an analog of a theorem to be found in Berge [7, p. 91, Thm. 4] concerning circulations on graphs. The essence of (ii)⇒(i) is that the extremals of the cone of nonnegative line-sum-symmetric matrices are nonnegative multiples of circuit matrices. Our proof is an application of the following obvious graph theoretic observation: A finite (directed) graph with no sink vertex must contain a circuit, e.g. [24, p. 64, Thm. 3.8]. When considering alternative proofs of Theorem AA, note that the implication (i)⇒(v) is trivial, and the converse (v)⇒(i) is a consequence of the result that the transitive closure of a graph without a circuit is a partial order. For a proof along these lines see Harary–Norman–Cartwright [24, p. 268, Thm. 10.1], where one may also find the equivalence of (i), (iii) (in a restricted form) and (v).

DEFINITION 2.4. Let P be an $(n \times n)$ pattern.

(i) If $\phi \subset \alpha, \beta \subseteq \langle n \rangle$, then $P[\alpha|\beta]$ is the $|\alpha| \times |\beta|$ submatrix of P indexed by the rows of α and columns of β in their natural orders, where $|\alpha|, |\beta|$ denote the cardinality of α, β resp.

(ii) We call *P completely reducible* if $P[\alpha|\alpha'] = 0$ implies that $P[\alpha'|\alpha] = 0$, for all α, $\phi \subset \alpha \subset \langle n \rangle$, where as before $\alpha' = \langle n \rangle \backslash \alpha$. Thus an irreducible P is completely reducible. Note that some other authors have used the term completely reducible in a slightly different sense.

THEOREM 2.5 (AB). *Let P be an* $(n \times n)$ *pattern. Then the following are equivalent.*

(i) *For every* $(i, j) \in G(P)$, *there is a circuit matrix* $C \in M(P)$ *with* $c_{ij} = 1$.

(ii) *There is a line-sum-symmetric matrix strictly positive for P.*

(iii) *There is* no *semi-positive difference matrix for P.*

(iv) *There is* no *cocircuit matrix for P.*

(v) *P is completely reducible.*

Proof. (i)$\Rightarrow$(ii): By assumption, for every $(i, j) \in G(P)$ there is a circuit matrix whose (i, j)-th entry is 1. The sum of all such matrices is a line-sum-symmetric matrix positive for P.

(ii)$\Rightarrow$(i): The proof is by induction on $m(P)$, the number of positive elements in P. If $P = 0$, the result is true. If $m(P) > 0$, by Theorem (AA), (ii)$\Rightarrow$(i): we can find a circuit matrix $C \in M(P)$. Let A be a line-sum-symmetric matrix strictly positive for P. If

$$\mu = \min \{a_{kl} : c_{kl} = 1\}$$

then $A - \mu C$ is a line-sum-symmetric matrix strictly positive for a pattern P' with $m(P') < m(P)$. For $(i, j) \in G(P)$, either $c_{ij} = 1$ or else $(i, j) \in G(P')$ and the conclusion follows from the inductive hypothesis.

(ii)$\Leftrightarrow$(iii): This is the Gordan–Stiemke Theorem with $W^\perp = W_s$.

Not (iv)$\Rightarrow$*Not* (iii): This is trivial.

Not (v)$\Rightarrow$*Not* (iv): There exists α, $\phi \subset \alpha \subset \langle n \rangle$ such that $P[\alpha|\alpha'] = 0$ but $P[\alpha'|\alpha] \neq 0$. Then $A = Z_{\alpha'}P - PZ_{\alpha'}$ is a cocircuit matrix, since $a_{ij} = 1$ for some $(i, j) \in \alpha' \times \alpha$.

Not (iii)$\Rightarrow$*Not* (v): Let $XP - PX$ be a semi-positive difference matrix. Without loss of generality assume $x_1 \leqq \cdots \leqq x_n$. There exist $(k, l) \in G(P)$ such that $x_k - x_l > 0$. Hence there is a r, $l \leqq r < k$, such that $x_k \leqq \cdots \leqq x_r < x_{r+1} \leqq \cdots \leqq x_n$. Then for $\alpha = \langle r \rangle$, $P[\alpha|\alpha'] = 0$, but $P[\alpha'|\alpha] \neq 0$, since $k \in \alpha'$, $l \in \alpha$. □

Comments. The equivalence of (i) and (v) is of course well known, see, for example, [13, Remark (2.15)]. In graph theoretic language, it asserts that the connected components of a directed graph are strongly connected if and only if the vertex-adjacency matrix (pattern) of the graph is completely reducible. A direct proof of (ii)$\Rightarrow$(v) is easy.

3. Constant-line-sum matrics and bidifference matrices.

DEFINITION 3.1. Let P be an $(n \times n)$ pattern.

(i) Let $\Delta = \Delta(P)$ be the incidence matrix of

$$P^+ = \begin{bmatrix} 0 & P \\ 0 & 0 \end{bmatrix}$$

where all blocks are $(n \times n)$. Note Δ is an $(m \times 2n)$ matrix.

(ii) Let $A \in M(P)$. Then A is a *constant-line-sum matrix for P* if $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{jk}$, $i, k = 1, \cdots, n$. The *constant-line-sum space W_c for P* consists of all such matrices.

(iii) A matrix $L \in M(P)$ is called a *polygon matrix* for P, if there exist sequences $(i_1, \cdots, i_k)$, $(j_1, \cdots, j_k)$ in $\langle n \rangle$ with the following properties:

(a) Each sequence consists of distinct integers;

(b) $(i_q, j_q) \in G(P)$, $(i_{q+1}, j_q) \in G(P)$, $q = 1, \cdots, k$, where $i_{k+1} = i_1$;

(c) $l_{ij} = 1$ if $(i, j) = (i_q, j_q)$, for some q, $l_{ij} = -1$ if $(i, j) = (i_{q+1}, j_q)$, for some q, $l_{ij} = 0$ otherwise.

(A polygon matrix is a constant-line-sum matrix.)

(iv) A matrix $A \in M(P)$ is called a *bidifference matrix* for P, if there exist diagonal matrices X, Y with trace X = trace Y such that $A = XP - PY$; viz. there exist $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ such that $a_{ij} = x_i - y_j$ if $(i, j) \in G(P)$ and $a_{ij} = 0$ otherwise.

(v) The matrix $A \in M(P)$ is called a *copermutation matrix for P* if A is semi-positive for P and $A = Z_\alpha P - PZ_\beta$, for some α, β with $|\alpha| = |\beta|$, $\phi \subset \alpha$, $\beta \subseteq \langle n \rangle\}$, where Z_α, Z_β are defined as in Definition 2.1(viii). (A copermutation matrix is a bidifference matrix.) If we apply the canonical isomorphism η (see Definition 2.1(ix)) to W_c we see that

$$W_c \cong \{a \in \mathbb{F}^m : \exists k \in \mathbb{F}, a'\Delta = k\varphi'\}$$

where $\varphi \in \mathbb{R}^{2n}$, $\varphi_i = 1$, $i = 1, \cdots, n$, $\varphi_i = -1$, $i = n+1, \cdots, 2n$. It follows that

$$(W_c)^\perp \cong \{\Delta z : z \in \mathbb{F}^{2n} : \varphi' z = 0\}.$$

But this latter space is the isomorphic image under η of the space of bidifference matrices.

We now immediately obtain from Lemma 1.1:

THEOREM 3.2. *Let P be an $(n \times n)$ pattern. The following cones are dual to each other in $M(P)$.*

(i) *The cone of nonnegative constant-line-sum matrices.*

(ii) *The cone of matrices A of form $A = B + C$, where B is a bidifference matrix for P, and C is nonnegative in $M(P)$.*

THEOREM 3.3 (BA). *Let P be an $(n \times n)$ pattern. The following are equivalent*:

(i) *There is a permutation matrix in $M(P)$.*

(ii) *There is a semi-positive constant-line-sum matrix in $M(P)$.*

(iii) *There is* no *bidifference matrix strictly positive for P.*

(iv) *There is $(i, j) \in G(P)$ such that $a_{ij} = 0$ for every copermutation matrix A for P.*

(v) *P has* no *0-submatrix of order $r \times s$, where $r + s = n + 1$.*

Comment. (ii) is equivalent to: There is a doubly-stochastic matrix in $M(P)$.

Proof. (i)$\Rightarrow$(ii): This is trivial.

(ii)$\Rightarrow$(i): The proof is by induction on the number $m(P)$ of nonzero elements in P. Let A be a semi-positive constant-line-sum matrix in $M(P)$. Since A contains at least one positive element in each line (i.e. row or column), $m(P) \geqq n$. If A is a multiple of a permutation matrix (in particular if $m(P) = n$) then the implications holds. So suppose inductively that $m(P) > n$ and that A is not a multiple of a permutation matrix. Let the pattern P^* be given by

$$p^*_{ij} = 1 \quad \text{if and only if} \quad 0 < a_{ij} < s,$$

where s is the (positive) sum of each line of A. Since A has at least two nonzero entries in some line, $P^* \neq 0$. But, again since A is a constant-line-sum matrix, every line of P^* either is 0 or has at least two positive entries. It follows that there is a polygon matrix L in $M(P^*)$. Let

$$\mu = \{\min a_{ij} : l_{ij} > 0\}.$$

Then $A - \mu L$ is a semi-positive constant-line-sum matrix and $A - \mu L \in M(P')$, where $m(P') < m(P)$. Hence by inductive hypothesis there is a permutation matrix $Q \in M(P') \subseteq M(P)$.

(ii)$\Leftrightarrow$(iii): This is the Gordan–Stiemke Theorem applied to $W = W_c$.

Not (iv)$\Rightarrow$*Not* (iii): For each $(i, j) \in G(P)$, let $X^{(i,j)}P - PY^{(i,j)}$ be a copermutation matrix whose (i, j)th entry is 1. Let $X = \sum_{(i,j)\in G(P)} X^{(i,j)}$ and $Y = \sum_{(i,j)\in G(P)} Y^{(i,j)}$. Then $XP - PY$ is a bidifference matrix strictly positive for P.

Not (v)$\Rightarrow$*Not* (iv): Let $P[\alpha|\beta] = 0$, where $|\alpha| + |\beta| = n + 1$. Let $(i, j) \in G(P)$. Then $(i, j) \notin \alpha \times \beta$ and three cases arise.

(a) $(i, j) \in \alpha' \times \beta$. Let $\bar{\alpha} = \alpha'$ and $\bar{\beta} = \beta \backslash \{j\}$,

(b) $(i, j) \in \alpha' \times \beta'$. Let $\bar{\alpha} = \alpha' \cup \{k\}$, $\bar{\beta} = \beta$, where k is any element of α.

(c) $(i, j) \in \alpha \times \beta'$. Let $\bar{\alpha} = \alpha' \cup \{i\}$, $\bar{\beta} = \beta$.

In each case, $|\bar{\alpha}| = |\bar{\beta}|$, and $A = Z_{\bar{\alpha}}P - PZ_{\bar{\beta}}$ is a copermutation matrix with $a_{ij} = 1$.

Not (iii)$\Rightarrow$*Not* (v): There exist diagonal X, Y with trace X = trace Y such that $XP-PY$ is strictly positive for P. After replacing P by Q_1PQ_2, X by $Q_1PQ_1^{-1}$, and Y by $Q_2^{-1}YQ_2$ we may suppose that $x_1 \leqq \cdots \leqq x_n$, and $y_1 \leqq \cdots \leqq y_n$. Since trace X = trace Y, there is a $k \in \langle n \rangle$ such that $x_k \leqq y_k$. Let $\alpha = \{1, \cdots, k\}$, $\beta = \{k, \cdots, n\}$. Since $x_i - y_j \leqq 0$ for $(i, j) \in \alpha \times \beta$, it follows that $P[\alpha|\beta] = 0$. Clearly $|\alpha| + |\beta| = n + 1$. □

Comments. The result (ii)$\Rightarrow$(i) is due to König [29], who deduced it from a more general theorem in graph theory; see also [30, p. 238, A]. Subsequently, Frobenius [18], proved (ii)$\Rightarrow$(i) by observing the rather easy implication (ii)$\Rightarrow$(v) and giving a combinatorial proof of (v)$\Rightarrow$(i) (which is now known as the Frobenius–König Theorem). Birkhoff [9] and most recent authors, e.g. Marcus–Minc [34, pp. 238–9], Mirsky [36, pp. 184–5], have followed Frobenius' sequence (ii)$\Rightarrow$(v)$\Rightarrow$(i). In Theorem BA, on the other hand, we obtain the combinatorial result (v)$\Rightarrow$(i) by the sequence (v)$\Rightarrow$(iii) (an easy computation with inequalities), (iii)$\Rightarrow$(ii) (Gordan–Stiemke Theorem) and (ii)$\Rightarrow$(i), where our proof of this last implication rests on the following obvious graph-theoretic result: A finite undirected graph with no vertex of degree 1 and a nonempty arc sets contains a cycle [23, p. 34, Thm. 4.1(4)], for a polygon matrix corresponds to a cycle of $G(P^+)$. Our proof owes something to von Neumann's [37] proof of Birkhoff's theorem (see comment following Theorem BB), but it is a little different, for von Neumann also uses the theorem that every element in a convex cone is a positive linear combination of extremals. Also, with minor changes in the definitions, our inductive proof applies to matrices over ordered Abelian groups, cf. [44], while von Neumann's proof avoids induction at the cost of requiring a field and division by 2. For proofs by means of the duality theorem of linear programming of results closely related to the Frobenius–König theorem, see Hoffman–Kuhn [27] and Hoffman [25]. More information concerning the origins of Theorems BA and BB may be found in Biggs–Lloyd–Wilson [8, pp. 203–4] and [43].

DEFINITION 3.4. Let P be an $(n \times n)$ pattern. Then P is called *completely decomposable* if for every α, β, $\phi \subset \alpha, \beta \subset \langle n \rangle$ with $|\alpha| + |\beta| = n$, $P[\alpha|\beta] = 0$ implies $P[\alpha'|\beta'] = 0$.

THEOREM 3.4 (BB). *Let P be an $(n \times n)$ pattern. Then the following are equivalent*:

(i) *For every $(i, j) \in G(P)$ there is a permutation matrix Q in $M(P)$ such that $q_{ij} = 1$.*

(ii) *There is a constant-line-sum matrix strictly positive for P.*

(iii) *There is* no *semi-positive bidifference matrix for P.*

(iv) *There is* no *copermutation matrix for P.*

(v) *P is completely decomposable.*

Comment. If $P \neq 0$, condition (ii) is obviously equivalent to (ii)′: There is a doubly-stochastic matrix in $M(P)$ strictly positive for P. But if $P = 0$, then (ii) and (ii)′ are *not* equivalent, see § 1.

Proof. (i)$\Rightarrow$(ii): For each $(i, j) \in G(P)$, choose a permutation matrix whose (i, j)th element is positive. Their sum is a constant-line-sum matrix strictly positive for P.

(ii)$\Rightarrow$(i): If $P = 0$, there is nothing to prove. If $P \neq 0$, the proof is by induction on $m(P)$. Let A be a constant-line-sum matrix strictly positive for P. By (ii)$\Rightarrow$(i) of Theorem (BA) there is a permutation matrix Q in $M(P)$. Let

$$\mu = \min \{a_{ij} : q_{ij} = 1\}.$$

Then $A - \mu Q$ is a constant-line-sum matrix strictly positive for a pattern P' with $m(P') < m(P)$. For $(i, j) \in G(P)$ either $q_{ij} = 1$ or $(i, j) \in G(P')$ and the conclusion follows from the inductive hypothesis.

(ii)$\Leftrightarrow$(iii): This is the Gordan–Stiemke Theorem with $W^\perp = W_c$.

Not (iv)$\Rightarrow$*Not* (iii): This is trivial.

Not (v)$\Rightarrow$*Not* (iv): Let $P[\alpha|\beta]=0$ and $P[\alpha'|\beta']\neq 0$, where $|\alpha|+|\beta|=n$. Then $B=Z_{\alpha'}P-PZ_\beta$ is a copermutation matrix for P.

Not (iii)$\Rightarrow$*Not* (v): Let $XP-PY$ be a semi-positive bidifference matrix. Let $X=\operatorname{diag}(x_1,\cdots,x_n)$, $Y=\operatorname{diag}(y_1,\cdots,y_n)$. Without loss of generality we may assume that $x_1\leqq\cdots\leqq x_n$, $y_1\leqq\cdots\leqq y_n$. There are two cases.

Case 1. Suppose $X=Y$. In this case P is not completely reducible by Theorem (AB), *Not* (iii)$\Rightarrow$*Not* (v), and hence P is completely decomposable.

Case 2. Suppose $X\neq Y$. Since tr X = tr Y, there is a k, $1\leqq k\leqq n$, such that $x_k<y_k$. Let $\alpha=\{1,\cdots,k\}$, $\beta=\{k,\cdots,n\}$. Since $x_i-y_j<0$ for $(i,j)\in\alpha\times\beta$ it follows that $P[\alpha|\beta]=0$. If P is also completely decomposable, the following argument shows that $P=0$, which however is impossible, since we have assumed the existence of a semi-positive matrix in $M(P)$. Let $(r,s)\in\langle n\rangle\times\langle n\rangle$. If $(r,s)\in\alpha\times\beta$, then $p_{rs}=0$. So suppose that $(r,s)\notin\alpha\times\beta$, say $r\notin\alpha$. Then $|\alpha|<n$ and $|\beta|\geqq 2$. If $s\in\beta$, choose $t=s$. If $s\notin\beta$, let t be any element of β. Put $\beta_t=\beta\backslash\{t\}$. Then $\beta_t\neq\phi$, $P[\alpha|\beta_t]=0$ and $|\alpha|+|\beta_t|=n$. Hence also $P[a'|\beta_t']=0$ and so $p_{rs}=0$. If $s\notin\beta$, a similar argument again yields $p_{rs}=0$. Thus $P=0$, and the result follows. □

Comments. An easy consequence of (ii)$\Rightarrow$(i) is the Birkhoff–König Theorem: A constant-line-sum matrix is a linear combination with positive coefficients of permutation matrices. This result was proved for integral matrices by König [29], [30, p. 239, Thm. B] and for real matrices by Birkhoff [9]; see Mirsky [35] for much information and see [44] for a previous unification. We refer to Birkhoff's Theorem or König's Theorem when we wish to distinguish between the real and integer case. A proof of Birkhoff's Theorem using inequalities and induction has been given by Hoffman–Wielandt [28]. The equivalence (v)$\Leftrightarrow$(ii) is due to Perfect–Mirsky [39], cf. [36, p. 199, Thm. 11.4.1].

4. A generalization of the Gordan–Stiemke Theorem with applications. In this section we apply a generalization of the Gordan–Stiemke Theorem to obtain further results. Theorem 4.6 below is easily derived from Gale's "key theorem" [19, p. 44, Thm. 2.6]. Here we give a simple proof based on standard geometric results which are part of [19, Thm. 2.14]. (As remarked by Gale [19, p. 59], his Theorem 2.6 may be proved in a similar manner.) We first define the relevant concepts.

DEFINITION 4.1. (i) A cone is *polyhedral* (called finite in [19, p. 55]) if it is the set of all linear combinations with nonnegative coefficients of a finite set of vectors in $\mathbb{F}^m$.

(ii) A *face* F of $\mathbb{F}^m$ is a subset of $\mathbb{F}^m_+$ such that for some α, $\phi\subseteq\alpha\subseteq\langle n\rangle$

$$F=F_\alpha=\{x\in\mathbb{F}^m_+ : x_i>0\Rightarrow i\in\alpha\}.$$

(iii) If F is a face of $\mathbb{F}^m_+$, the *complementary face* F' is the face of $\mathbb{F}^m_+$ satisfying $\{0\}=F\cap F'$ and $\mathbb{F}^m_+=F+F'=\{x+y : x\in F,\ y\in F'\}$. Indeed, if $F=F_\alpha$, $\phi\subseteq\alpha\subseteq\langle n\rangle$, then $F'=F_{\alpha'}$.

(iv) If K is any subset of $\mathbb{F}^m_+$ then the *face* $\varphi(K)$ *of* $\mathbb{F}^m_+$ *generated by* K is the smallest face of $\mathbb{F}^m_+$ which contains K.

We collect properties of $\varphi(K)$ in the following lemma.

LEMMA 4.2. (i) *If* $K\subseteq\mathbb{F}^m_+$ *then* $\varphi(K)=F_\alpha$, *where*

$$\alpha=\{i\in\langle n\rangle : \exists x\in K,\ x_i>0\}.$$

(ii) *If K is a subcone of* $\mathbb{F}^m_+$, *then there is an* $x\in\varphi(K)=F_\alpha$ *such that* $x_i>0$, *for all* $i\in\alpha$.

Proof. The proof is easy, for (ii); just add a finite number of vectors in K. □

Indeed, φ is a closure operator in the usual sense. In particular

$$K\subseteq K'\subseteq\mathbb{F}^m_+\Rightarrow\varphi(K)\subseteq\varphi(K')\subseteq\mathbb{F}^m_+.$$

For a corresponding definition of face for a general cone in $\mathbb{F}^m$ see Barker [4].

The following results are needed in our proof of Theorem 4.6. Let K and L be polyhedral cones. Then

$$K^D \text{ is a polyhedral cone,} \tag{4.3}$$

$$K^{DD} = K, \tag{4.4}$$

$$(K \cap L)^D = K^D + L^D. \tag{4.5}$$

Result (4.3) is easily derived from [19, p. 54, Thm. 2.14]. Result (4.4) is a form of Farkas' Lemma [14], (4.4) and (4.5) are parts of [19, p. 54, Thm. 2.14]. For the real field, proofs based on a separation theorem are to be found in Ben-Israel [6].

THEOREM 4.6 (generalization of Gordan–Stiemke). *Let K be a polyhedral cone in $\mathbb{F}^m$. Then $\varphi(K \cap \mathbb{F}^m_+)$ and $\varphi((-K^D) \cap \mathbb{F}^m_+)$ are complementary faces of $\mathbb{F}^m_+$.*

Proof. Let $\alpha \subseteq \langle n \rangle$. We first show that it is enough to prove the equivalence of (i) and (ii):

(i) $K \cap \mathbb{F}^m_+ \subseteq F_\alpha$,

(ii) $\varphi((-K^D) \cap \mathbb{F}^m_+) \supseteq F_{\alpha'}$.

For, by (4.3) and (4.4), the roles of K and $-K^D$ may be interchanged in (i) and (ii). Thus, if we also interchange α and α' we obtain the equivalence of

(iii) $(-K^D) \cap \mathbb{F}^m_+ \subseteq F_{\alpha'}$,

(iv) $\varphi(K \cap \mathbb{F}^m_+) \supseteq F_\alpha$.

The conclusion of the theorem then follows. It remains to show that (i) and (ii) are equivalent.

(i)$\Rightarrow$(ii): Let $K \cap \mathbb{F}^m_+ \subseteq F_\alpha$. Then $(K \cap \mathbb{F}^m_+)^D \supseteq -F_{\alpha'}$. Hence by (4.5) and since $(\mathbb{F}^m_+)^D = \mathbb{F}^m_+$, we have

$$K^D + \mathbb{F}^m_+ = (K \cap \mathbb{F}^m_+)^D \supseteq -F_{\alpha'}.$$

Let $x \in \mathbb{F}^m_+$, where $x_i > 0$ if $i \in \alpha'$, $x_i = 0$, if $i \in \alpha$. Since $x \in F_{\alpha'}$, there exist $u \in K^D$, $v \in \mathbb{F}^m_+$ such that $-x = u + v$. Hence $-u = x + v \in \mathbb{F}^m_+ \cap (-K^D)$ and $u_i > 0$ if $i \in \alpha'$. Thus $F_{\alpha'} \subseteq \varphi(-u) \subseteq \varphi((-K^D) \cap \mathbb{F}^m_+)$.

(ii)$\Rightarrow$(i): Let $i \in \alpha'$. By assumption, there exist $x \in (-K^D) \cap F^m_+$ such that $x_i > 0$. Let $y \in K \cap \mathbb{F}^m_+$. On the one hand, $y'x \geqq 0$, since $x, y \in \mathbb{F}^m_+$. On the other hand $y'x \leqq 0$ since $x \in -K^D$ and $y \in K$. Hence $y'x = 0$. It follows that $y_i = 0$ and so $y \in F_\alpha$. □

Comments. (i) In our applications below we use Theorem 4.6 when $K = W$, a subspace of $\mathbb{F}^m_+$. For this purpose we do not need the general form of (4.3) and (4.4), but instead we may use standard results on subspaces; also (4.5) reduces to Lemma 1.1.

(ii) Let $x \in \mathbb{F}^m_+$. Then $\varphi(x) = \mathbb{F}^m_+$ if and only if x is strictly positive. Hence the Gordan–Stiemke Theorem is the special case of Theorem 4.6 with $K = W$, a subspace, and $W \cap \mathbb{F}^m_+ = \{0\}$.

(iii) For $K = W$, a subspace, results close to Theorem 4.6 are Levin–Shapiro [31, Thm. 3.4] and the following theorem, Ben-Israel [5, Cor. 6], Gale [19, p. 71, Ex. 24]: Let W be a subspace of $\mathbb{F}^m$. Then there exists a strictly positive $x \in \mathbb{F}^m$ such that $x = y + z$, where $y \in W \cap \mathbb{F}^m_+$, $z \in W^\perp \cap \mathbb{F}^m_+$. By putting

$$W = \{y \in \mathbb{F}^m : y'\Gamma = 0\}$$

where $\Gamma \in \mathbb{F}^{mn}$, we obtain the following result due to Tucker [46, Thm. 1], cf. Ben-Israel [5], Nikaido [38, p. 36]: There exists $y \in \mathbb{F}^m$ and $x \in \mathbb{F}^n$, $x \geqq 0$, such that $y'\Gamma \geqq 0$, $\Gamma x = 0$ and $y'\Gamma + x' > 0$.

(iv) Theorem 4.6 also holds when $\mathbb{F}$ is the real field, and K is an arbitrary (not necessarily polyhedral) closed cone satisfying $K \cap \mathbb{F}_+^m = \{0\}$; see Nikaido [38, p. 33, Thm. 3.6]. Our proof essentially goes through in this situation with (4.5) replaced by the weaker result $(K \cap L)^D = \mathrm{cl}\,(K^D + L^D)$, which can be found in [5]. However, Theorem 4.6 is false for closed cones K without a further hypothesis. We give an example which is based on Ben-Israel [6, Ex. 1.4].

Example 4.7. Let $\mathbb{F}$ be the real field. Let K be the cone of all vectors in $\mathbb{F}^3$ which make an angle of 45° or less with $(1, -1, 0)'$. Thus

$$K = \{(x_1, x_2, x_3)' \in \mathbb{F}^3 : x_1 \geqq 0,\ x_2 \leqq 0,\ -2x_1x_2 \geqq x_3^2\}.$$

Then $K \cap \mathbb{F}_+^3 = F_{\{1\}}$ and $(-K^D) \cap \mathbb{F}_+^3 = F_{\{2\}}$, since $K^D = K$, but the complement of $F_{\{1\}}$ is $F_{\{2,3\}}$.

In order to apply Theorem 4.6 we require some terminology.

If P is an $(n \times n)$ pattern, then a subpattern P' of P is an $(n \times n)$ pattern such that $p'_{ij} = 1$ implies $p_{ij} = 1$. If $A \in M(P)$ and P' is a subpattern of P, we say that A *is strictly positive* (*zero*) *on* P' if $a_{ij} > 0$ ($a_{ij} = 0$) for all $(i, j) \in G(P')$. We identify the space $M(P)$ with its image $\eta(M(P)) = \mathbb{F}^m$ under the canonical isomorphism η. Similarly we identify the cone $M(P)_+$ of all nonnegative matrices in $M(P)$ with $\mathbb{F}_+^m$.

THEOREM 4.8. *Let P be an $(n \times n)$ pattern. Then there exists a subpattern P' such that, for $P'' = P - P'$, the following conditions hold*:

(i) (a) *Every line-sum-symmetric matrix for P is zero on P''* and

(b) *there exists a line-sum-symmetric matrix for P which is strictly positive on P'.*

(ii) (a) *Every difference matrix for P is zero on P'*, and

(b) *there exists a difference matrix for P strictly positive on P''.*

Further, P' is the unique subpattern for which either (i) *or* (ii) *holds.*

Proof. Let P' be the subpattern of P such that $M(P')_+ = \varphi(W_s \cap M(P)_+)$, where, as usual, W_s is the space of line-sum-symmetric matrices in $M(P)$. The complementary face of $M(P')_+$ in $M(P)_+$ is $M(P'')_+$ and hence by Theorem 4.6, $M(P'')_+ = \varphi(W_s^\perp \cap M(P)_+)$. Thus ((i)(a)) and ((ii)(a)) hold and since $W_s \cap M(P)_+$ and $W_s^\perp \cap M(P)_+$ are cones, so do ((i)(b)) and ((ii)(b)) by Lemma 4.2 (ii).

Let P_1' be a subpattern of P for which (i) holds. By ((i)(a)), applied to P'' and ((i)(b)) applied to P_1', P_1' is a subpattern of P'. By ((i)(a)) applied to P_1'' and ((i)(b)) applied to P', P' is a subpattern of P_1'. Hence $P_1' = P'$.

By a similar argument, if P_2' is a subpattern of P for which (ii) holds, then $P_2' = P'$. □

THEOREM 4.9. *Let P be an $(n \times n)$ pattern. Then there exists a subpattern P' such that for $P'' = P - P'$, the following conditions hold*:

(i) (a) *Every constant-line-sum matrix for P is zero on P''*, and

(b) *there exists a constant-line-sum matrix for P which is positive on P'.*

(ii) (a) *Every bidifference matrix for P is zero on P'*, and

(b) *there exists a bidifference matrix positive on P''.*

Further, P' is the unique subpattern of P for which either (i) *or* (ii) *holds.*

Proof. The proof is similar to the proof of Theorem 4.8 with W_s replaced by W_c. □

Comments. (i) Graph theoretic versions of Theorem 4.8 are familiar, even in a stronger form: Every arc of a directed graph either lies on a circuit or on a cocircuit, but not both, see, e.g., Berge [7, p. 15]. To obtain the matrix version of this result, we would need to show the following: If A is a nonnegative difference matrix for P such that $a_{ij} > 0$, then there exists a cocircuit matrix B for P with $b_{ij} = 1$. This may easily be done by arguments similar to some found in § 3.

(ii) We may strengthen Theorem 4.9 in a similar manner. For, if A is a nonnegative bidifference matrix for P and $a_{ij}>0$, then there exists a copermutation matrix B for P such that $b_{ij}=1$.

(iii) Finally we remark that Theorems 4.8 and 4.9 could also have been deduced without use of Theorem 4.6 from the theorems in § 2 and § 3 by considering the Frobenius normal form of the pattern P. This normal form is found in many places, e.g. [17], [20, Vol. II, p. 75] or [12, Remark 3.5]. The pattern P' in Theorem 4.8 corresponds to P^c as defined in [12, Def. 2.12], while in Theorem 4.9 P' corresponds to P^s, see [12, Def. 2.17].

5. Minmax applications of Gordan–Stiemke. Theorems 5.1 and 5.2 are additive analogs of theorems on optimal multiplicative scalings of matrices by means of diagonal transformations. For further details and for proofs using the Gordan–Stiemke Theorem see [42]. Though the results in [42] were proved for the field of reals, here the theorems are stated in a form valid over every ordered field $\mathbb{F}$.

Let P be an $(n\times n)$ pattern. For $A\in M(P)$ we put

$$\mu(A)=\max\{a_{ij}:(i,j)\in G(P)\}.$$

THEOREM 5.1. *Let P be an $(n\times n)$ pattern and let $A\in M(P)$. Let*

$$\mu_1=\min\{\mu(A+B): B \text{ is a difference matrix for } P\}.$$

(i) *If there is no circuit matrix for P, then*

$$\mu_1=-\infty.$$

(ii) *If there is a circuit matrix for P, then*

$$\mu_1=\max\left\{\frac{\operatorname{tr} C'A}{\operatorname{tr} C'C}: C \text{ is a circuit matrix for } P\right\}.$$

Comment. For the full pattern P, $p_{ij}=1$, $i,j\in\langle n\rangle$, Afriat [1], [2], [3] proved the following special case of Theorem 5.1: $\mu_1\leqq 0$ if and only if $\operatorname{tr} C'A\leqq 0$ for all circuit matrices in $M(P)$. Afriat [2], [3], used the Gordan–Stiemke Theorem in his proof. For a multiplicative analogue of the special case with a different type of proof, see Fiedler–Pták [15], [16] and Engel–Schneider [12, Lemma 6.3]. In [13, Thm. 4.23] the result is proved for a matrix with elements in a lattice ordered group. A multiplicative analogue for the general case was proved for a real matrix in [13, Thm. 7.2, Remark 7.3].

THEOREM 5.2. *Let P be an $(n\times n)$ pattern, and let $A\in M(P)$. Let*

$$\mu_2=\min\{\mu(A+B): B \text{ is a bidifference matrix for } P\}.$$

(i) *If there is no permutation matrix for P, then*

$$\mu_2=-\infty.$$

(ii) *If there is a permutation matrix for P, then*

$$\mu_2=\frac{1}{n}\max\{\operatorname{tr} C'A: C \text{ is a permutation matrix for } P\}.$$

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