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FLOWS ON GRAPHS APPLIED TO DIAGONAL SIMILARITY AND DIAGONAL EQUIVALENCE FOR MATRICES*

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Three equivalence relations are considered on the set of $n \times n$ matrices with elements in $\mathbf{F}_o$, an abelian group with absorbing zero adjoined. They are the relations of diagonal similarity, diagonal equivalence, and restricted diagonal equivalence. These relations are usually considered for matrices with elements in a field, but only multiplication is involved. Thus our formulation in terms of an abelian group with o is natural. Moreover, if $\mathbf{F}$ is chosen to be an additive group, diagonal similarity is characterized in terms of flows on the pattern graph of the matrices and diagonal equivalence in terms of flows on the bipartite graph of the matrices. For restricted diagonal equivalence a pseudo-diagonal of the graph must also be considered. When no pseudo-diagonal is present, the divisibility properties of the group $\mathbf{F}$ play a role. We show that the three relations are characterized by cyclic, polygonal, and pseudo-diagonal products for multiplicative $\mathbf{F}$. Thus, our method of reducing propositions concering the three equivalence relations to propositions concerning flows on graphs, provides a unified approach to problems previously considered independently, and yields some new or improved results. Our consideration of cycles rather than circuits eliminates certain restrictions (e.g., the complete reducibility of the matrices) which have previously been imposed. Our results extend theorems in Engel and Schneider [5], where however the group $\mathbf{F}$ is permitted to be non-commutative.

0. Introduction

In Section 1 we give preliminaries, which have been included for the sake of clarity because many intuitive graph theoretic concepts have been formalized in a variety of ways by different authors. In the main, we have followed Berge [2] or Pearl [11]; no attempt is made to trace graph theoretic results to their origins. Certain graph theoretic definitions will also be found in Sections 3 and 4. In Section 5 we make comments concerning alternative versions of our theorems and we relate our results to the published literature. Applications to the scaling problem for real or complex matrices will appear in a forthcoming paper.

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1. Preliminaries

Definition 1.1. A set $\mathbf{F}_o$, furnished with a binary operation $*$, is an *abelian group with* o, if

(i) $\mathbf{F}_o = \mathbf{F} \cup \{o\}$, where $o \notin \mathbf{F}$,

(ii) $\mathbf{F}$ is an abelian group under $*$,

(iii) $a * o = o * a = o$ for all $a \in \mathbf{F}_o$.

Except where specifically indicated otherwise, the composition $*$ will be taken to be multiplication (juxtaposition). As usual, we write S^m for the set of all (column) vectors with entries in the set S and S^{mn} for the set of all $(m \times n)$ matrices with entries in S.

The three equivalence relations mentioned in the abstract are the following:

Definition 1.2. Let $\mathbf{F}_o$ be an abelian group with o and let $A, B \in \mathbf{F}_o^{nr}$.

(i) A is *diagonally similar* to B if $n = r$ and there exists an invertible diagonal matrix $X \in \mathbf{F}_o^{nn}$ such that $XAX^{-1} = B$.

(ii) A is *diagonally equivalent* to B if there exist invertible diagonal matrices $X \in \mathbf{F}_o^{nn}$, $Y \in \mathbf{F}_o^{rr}$ such that $XAY^{-1} = B$.

(iii) A is *restricted diagonally equivalent* to B if there exist invertible diagonal matrices $X \in \mathbf{F}_o^{nn}$, $Y \in \mathbf{F}_o^{rr}$ such that $XAY^{-1} = B$ and $\det X \cdot \det Y^{-1} = 1$, where 1 is the identity of $\mathbf{F}$.

For the sake of completeness, we observe that a matrix is diagonal if the off-diagonal entries are o, that $X = \operatorname{diag}(x_1, \ldots, x_n)$ is invertible if $x_i \in \mathbf{F}$, for $i = 1, \ldots, n$, that $X^{-1} = \operatorname{diag}(x_1^{-1}, \ldots, x_n^{-1})$, and that $\det X = \prod_{i=1}^{n} x_i$.

Notations 1.3. (i) The symbol o denotes the absorbing element of an abelian group with o. The symbol 0 denotes the additive identity of the additive group $\mathbf{Z}$ of integers or of $\mathbf{Z}^m$. When $\mathbf{F}$ is an additive group, we also use 0 for its group identity.

(ii) Elements of $\mathbf{Z}$ or $\mathbf{Z}^m$ (and occasionally of $\mathbf{Q}^m$, where $\mathbf{Q}$ is the rational field) will be denoted by lower case Greek letters. Matrices in $\mathbf{Z}^{mn}$ will be denoted by upper case Greek letters. Graphs, their elements (arcs), vertex sets, vertices, and positive integers used for counting purposes will be denoted by upper or lower case Roman letters in the range G to V. The remainder of the Roman alphabet $(A-F, W-Z)$ will be reserved for elements of $\mathbf{F}_o$ and $\mathbf{F}_o^n$ in the lower case and for matrices in $\mathbf{F}_o^{nr}$ in the upper case.

Definition 1.4. (i) A (directed) *graph* G on the vertex set V is a subset of $V \times V$.

(ii) A (directed) *bipartite graph* H on the pair of vertex sets (V, V'), where $V \cap V' = \emptyset$, is a subset of $V \times V'$.

This definition of bipartite graph excludes graphs which have arcs directed from V' to V. Such graphs are not used in this paper. Normally, for our purposes, $V=\{1,2,\ldots,n\}=\langle n\rangle$ and $V'=\{n+1,n+2,\ldots,n+r\}=\langle r\rangle'$. For the sake of convenience, we assume that a graph G has been ordered lexicographically. Thus $G=\{g_1,\ldots,g_m\}$ where $m=|G|$, the number of elements of G, and if $g_q=(i,j)$, $g_s=(k,l)$ then $q<s$ if and only if either $i<k$ or $i=k$ and $j<l$.

Definition 1.5. Let G be a graph on $\langle n\rangle$ and suppose that $|G|=m$. Then the *incidence matrix* $\Gamma=\Gamma(G)\in\mathbf{Z}^{mn}$ is defined thus: $\Gamma=(\gamma_{qi})$ where

$$\gamma_{qi}=1 \quad \text{if } g_q=(i,j) \text{ and } i\neq j,$$
$$\gamma_{qi}=-1 \quad \text{if } g_q=(j,i) \text{ and } i\neq j,$$
$$\gamma_{qi}=0 \quad \text{otherwise.}$$

(Observe that $\gamma_{qi}=0$ if $g_q=(i,i)$.)

Certain notions which have intuitive meanings for a graph G are most easily defined in terms of vectors in $\mathbf{Z}^m$.

Definition 1.6. Let G be a graph on $\langle n\rangle$ with

$$\Gamma=\Gamma(G)\in\mathbf{Z}^{mn}.$$

(i) An (integer) *flow* on G is a vector $\gamma\in\mathbf{Z}^m$ such that $\gamma^t\Gamma=0$.

(ii) Let $i,j\in\langle n\rangle$, where $i\neq j$. A *chain from i to j* in G is a vector $\alpha\in\mathbf{Z}^m$ such that $(\alpha^t\Gamma)_i=1$, $(\alpha^t\Gamma)_j=-1$ and $(\alpha^t\Gamma)_k=0$, otherwise. The 0 vector in $\mathbf{Z}^m$ is a chain from i to i, for any $i\in\langle n\rangle$.

(iii) The *vertex components* of G are the subsets $V_1,\ldots,V_p$ of the vertex set $V=\langle n\rangle$ which form the equivalence classes under the relation "there is a chain from i to j in G". Thus if $(i,j)\in G$, then i and j belong to the same component V_s. We define the *arc component* (index set) G_s of G by

$$G_s=\{q\in\langle m\rangle : g_q=(i,j) \text{ and } i,j\in V_s\}.$$

Note that $\gamma_{qi}\neq 0$ implies that $q\in G_s$ and $i\in V_s$, for some s. Then Γ is a direct sum of matrices Γ_s where

$$\Gamma_s=(\gamma_{qi}), \quad q\in G_s, i\in V_s,$$

and each Γ_s is not decomposable as a direct sum after independent permutation of rows and columns. If V itself is a vertex component then G is called *connected*.

We simply refer to "components", when we count such components and it does not matter whether arc or vertex components are intended. For technical reasons, arc components are defined as index sets, but they correspond to the usual topological notion.

We now make the association between graphs and matrices which is crucial for our development. We begin by defining two graphs commonly associated with $A \in \mathbf{F}_o^{nn}$.

Definitions 1.7. (i) Let $A \in \mathbf{F}_o^{nn}$. The *pattern graph* $G(A)$ on $\langle n \rangle$ for A consists of the set of pairs (i, j) such that $a_{ij} \neq$ o.

(ii) Let $A \in \mathbf{F}_o^{nr}$. The *bipartite pattern graph* $H(A)$ on $(\langle n \rangle, \langle r \rangle')$ for A consists of the set of pairs $(i, j+n)$ such that $a_{ij} \neq$ o.

Then for the incidence matrices we write $\Gamma(G(A)) = \Gamma(A) = \Gamma$, but $\Gamma(H(A)) = \Delta(A) = \Delta$.

1.8. Let H be a fixed bipartite graph on $(\langle n \rangle, \langle r \rangle')$ and let $m = |H|$, the number of arcs in H. Consider the set of all matrices $A \in \mathbf{F}_o^{nr}$ with $H(A) = H$. There is a natural bijection from this set onto $\mathbf{F}^m$ given by $A \rightarrow a$, where

$$a_q = a_{ij}, \quad \text{if } h_q = (i, j+n).$$

The point of the mapping is this: If $\mathbf{F}$ is a multiplicative group, our equivalence relations will be characterized by certain products. If $A \in \mathbf{F}_o^{nn}$, we may use $G(A)$ to define the same bijection.

Definition 1.9. Let $\beta \in \mathbf{Z}^m$. Then

$$\Pi_\beta(A) = \prod_{q=1}^{m} a_q^{\beta_q}.$$

If β is a flow for $G(A)$ or $H(A)$, such a product will be called a *flow product*. Similar terminology will be used for *chain products*, etc.

Thus if $\mathbf{F}$ is additive, the product $\Pi_\beta(A)$ corresponds to the sum $\beta^t a$. It is precisely for this reason that in some proofs (where explicitly stated) we assume that $\mathbf{F}$ is an additive group. This allows us to use the standard notations of $\mathbf{Z}$-modules and matrices. For example, the multiplicative $XAX^{-1} = B$ corresponds to

$$b = a + \Gamma x,$$

where $X = \text{diag}(x_1, \ldots, x_n)$ and $x = (x_1, \ldots, x_n)^t$, cf. the proof of Theorem 2.1.

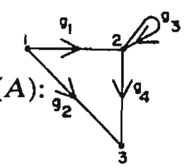
1.10. We give an example and indicate the intuitive background of some of the terms defined. For this purpose we need some further standard graph theoretic notions. An (elementary) *cycle* γ for a graph G is a flow for G whose entries are $-1, 0$, or 1 (in brief, a $\{-1, 0, 1\}$-vector) and which has minimal support, where support is defined as $\{q \in \langle m \rangle : \gamma_q \neq 0\}$. An (elementary) *circuit* is a $\{0, 1\}$-cycle. If $A \in \mathbf{F}_o^{nn}$, a cycle for $G(A)$ corresponds to a (family of) sequence(s) $(i_1, \ldots, i_k)$, $k \geq 2$ such that $i_1, \ldots, i_{k-1}$ are pairwise distinct, $i_k = i_1$ and either $(i_q, i_{q+1}) \in G(A)$

or $(i_{q+1}, i_q) \in G(A)$ for $q = 1, \ldots, k-1$. A cycle for $H(A)$ corresponds to a (family of) sequence(s) $(i_1, j_1, i_2, j_2, \ldots, i_k, j_k)$, $k \geq 2$, where $i_1, \ldots, i_{k-1}$ are pairwise distinct, $j_1, \ldots, j_{k-1}$ are pairwise distinct, $i_1 = i_k$, $j_1 = j_k$, and both $(i_q, j_q) \in H(A)$ and $(i_{q+1}, j_q) \in H(A)$. Because of the location of the corresponding entries in the matrix, we will call a cyclic product for $H(A)$ a *polygonial product for* A, and generally reserve the term *cyclic product for* A to mean cyclic product for $G(A)$.

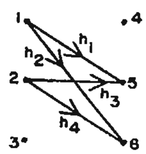
Example. Let

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

where $a_{12}, a_{13}, a_{22}, a_{23}$ are in $\mathbf{F}$. Then we have:

$G(A)$: 

$$\Gamma = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$H(A)$: 

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$a^t = (a_{12}\, a_{13}\, a_{22}\, a_{23})$.

Let $\gamma = (1, -1, 0, i)^t$, $\gamma' = (0, 0, 1, 0)^t$, $\delta = (1, -1, -1, 1)^t$. Then $\pm\gamma$, $\pm\gamma'$ are the cycles for $G(A), \pm\delta$ are the only cycles for $H(A)$, and the corresponding cyclic and polygonal products for A are

$$\Pi_\gamma(A) = \frac{a_{12}\, a_{23}}{a_{13}}, \qquad \Pi_{\gamma'}(A) = a_{22}, \qquad \Pi_\delta(A) = \frac{a_{12}\, a_{23}}{a_{13}\, a_{22}}.$$

2. A theorem on diagonal similarity

We are ready to state our first theorem. Theorems related to the results of this section are mentioned in Comments 5.1 and 5.2.

Theorem 2.1. *Let* $A, B \in \mathbf{F}_o^{nn}$, *where* $\mathbf{F}_o$ *is a abelian group with* o. *Then the following are equivalent*:

(i) A *is diagonally similar to* B,

(ii) $G(A) = G(B)$ *and for all flows* γ *of* $G(A)$,

$$\Pi_\gamma(A) = \Pi_\gamma(B).$$

Proof. In this proof, $\mathbf{F}$ is taken to be an additive group.

(i)$\Rightarrow$(ii): By definition, there exist $x_i \in \mathbf{F}$, $i=1,\ldots,n$, such that $b_{ij}=x_i+a_{ij}-x_j$, for $i,j=1,\ldots,n$. Since o is the absorbing element of $\mathbf{F}_o$, we have $b_{ij}=\mathrm{o}$ if and only if $a_{ij}=\mathrm{o}$, and hence $G(B)=G(A)$. If the matrices are represented by m-vectors as explained in 1.8, then the hypothesis is $b=a+\Gamma x$. Thus if γ is a flow for $G(A)$,

$$\gamma^{t}b=\gamma^{t}(a+\Gamma x)=\gamma^{t}a,$$

since $\gamma^{t}\Gamma=0$.

(ii)$\Rightarrow$(i): Let $V_1,\ldots,V_p$ be the vertex components of $G(A)$, cf. Definition 1.6. Thus for $i,j\in\langle n\rangle$, there exists a chain from i to j in $G(A)$ if and only if i and j belong to the same V_s. Suppose (without loss of generality) that $s\in V_s$. Let $i\in V_s$ and let δ be a fixed chain from i to s. We define $x_i=\delta^{t}(b-a)$ and we shall show that $b_{ij}=x_i+a_{ij}-x_j$ for $i,j=1,\ldots,n$.

If $a_{ij}=\mathrm{o}$, then $b_{ij}=\mathrm{o}$ and there is nothing to prove, so suppose that $a_{ij}\neq\mathrm{o}$ and q is such that $g_q=(i,j)$. Let β be any chain from s to i. Then $\beta+\delta$ is a flow, whence by assumption, $(\beta+\delta)^{t}a=(\beta+\delta)^{t}b$. Hence

$$\beta^{t}(b-a)=\delta^{t}(a-b)=-x_i.$$

Let σ be the qth unit vector in $\mathbf{Z}^m$. Then $\beta+\sigma$ is a chain from s to j. Hence, by a similar argument, $(\beta+\sigma)^{t}(b-a)=-x_j$. But $\sigma^{t}(b-a)=b_q-a_q=b_{ij}-a_{ij}$. Thus $-x_i+(b_{ij}-a_{ij})=-x_j$, and the result follows.

In order to check the next proposition concerning uniqueness in Theorem 2.1 we make a definition:

Definition 2.2. We call a matrix $A\in\mathbf{F}_o^{nn}$ *pattern connected* if the pattern graph $G(A)$ is connected.

Thus A is pattern connected if and only if PAP^{t} is not a direct sum of two matrices, for any permutation P, i.e. either A is irreducible or A is not completely reducible. Observe that any $A\in\mathbf{F}_o^{nn}$ is the direct sum of p pattern connected principal submatrices, where p is the number of components of $G(A)$.

Proposition 2.3. *Let $A\in\mathbf{F}_o^{nn}$. If $\mathbf{F}$ has at least two distinct elements, then the following are equivalent:*

(i) *A is pattern connected,*

(ii) *if $XAX^{-1}=X'A(X')^{-1}$, then $X'=fX$ for some $f\in\mathbf{F}$.*

Proof. If $\mathbf{F}$ is taken to be an additive group, we shall show that each of (i) and (ii) is equivalent to

(iii) if $u\in\mathbf{F}^n$ and $\Gamma(A)u=0$, then $u=f\varepsilon$, for some $f\in\mathbf{F}$, where $\varepsilon\in\mathbf{Z}^n$ is the vector all of whose entries are 1.

First observe that (i) holds if and only if $G(A)$ is connected, and it is known that $G(A)$ is connected if and only if (iii) is satisfied (cf. Pearl [11, p. 390] for a proof that can be adapted to our situation). Next, (ii) corresponds to: if $a+\Gamma(A)x = a+\Gamma(A)x'$, then $x' = x+f\varepsilon$, where $f\in\mathbf{F}$. But this is clearly also equivalent to (iii).

It is not necessary to check all flow products for A and B, as in Theorem 2.1, to prove diagonal similarity. Suppose K is a spanning forest (tree, if $G(A)$ is connected). Complete each arc of $G(A)$ which does not belong to K to the unique cycle all the other arcs of which are in K. In this way we obtain cycles $\gamma_1, \ldots, \gamma_s$ where $s = m-n+p$, n is the order of A (or the number of vertices of $G(A)$), m is the number of non-zero entries of A (or the number of arcs of $G(A)$), and p is the number of components of $G(A)$. Then each flow γ of $G(A)$ is linear combination with *integral* coefficients of $\gamma_1, \ldots, \gamma_s$, see Berge [2, pp. 26–27], Chen [3, p.43] and Pearl [11, pp. 373, 397] for related results. The cycles we have constructed have the property required in the following corollary.

Corollary 2.4. *Let $A, B\in\mathbf{F}_o^{nn}$. Let $\gamma_1, \ldots, \gamma_s$ be flows for $G(A)$ such that every flow for $G(A)$ is a linear combination with integral coefficients of $\gamma_1, \ldots, \gamma_s$. If $G(A) = G(B)$ and $\Pi_{\gamma_i}(A) = \Pi_{\gamma_i}(B)$, $i = 1, \ldots, s$, then A is diagonally similar to B.*

Corollary 2.5. *Let $A\in\mathbf{F}_o^{nn}$. If $\mathbf{F}$ has at least two elements then the following are equivalent:*

(i) *The only flow for $G(A)$ is 0 (i.e. $G(A)$ is a forest),*

(ii) *For all $B\in\mathbf{F}_o^{nn}$ such that $G(B) = G(A)$, B is diagonally similar to A.*

Proof. (i)$\Rightarrow$(ii): Direct application of Theorem 2.1.

(ii)$\Rightarrow$(i): By Corollary 2.4, it is enough to prove that there is no cycle for $G(A)$. So, for the sake of contradiction, suppose that γ is a cycle for $G(A)$. If $\mathbf{F}$ is chosen to be additive, then, as in the proof of Theorem 2.1, $\gamma^t(b-a) = 0$. Now let $q\in\langle m\rangle$ such that $\gamma_q\neq 0$, and let $f\in\mathbf{F}$, where $f\neq 0$. Then we may choose $b\in\mathbf{F}^m$ such that $b_q - a_q = f$ and $b_r - a_r = 0$, for all $r\neq q$. From $\gamma^t(b-a) = 0$ we obtain that $0 = b_q - a_q = \gamma_q f$. Hence $\gamma_q \neq \pm 1$. But γ is a cycle, and so we have a contradiction.

3. A theorem on diagonal equivalence

Comments 5.3, 5.4, and 5.5 relate to this section.

We could prove our next theorem in the same manner as Theorem 2.1. Instead, we prefer to derive it from that theorem. For $A\in\mathbf{F}_o^{nr}$, we put

$$A^+ = \begin{bmatrix} \text{o} & A \\ \text{o} & \text{o} \end{bmatrix} \in \mathbf{F}_o^{n+r,n+r}.$$

where o now stands for the $(n\times n)$, $(r\times n)$, or $(r\times r)$ matrix with all entries equal to o. Then $H(A) = G(A^+)$. Also, if $X\in\mathbf{F}_o^{nn}$, $Y\in\mathbf{F}_o^{rr}$ are invertible diagonal matrices and $Z=X\oplus Y$, then $XAY^{-1}=B$ if and only if $ZA^{+}Z^{-1}=B^{+}$. Hence as an immediate corollary to Theorem 2.1 we have:

Theorem 3.1. *Let* $\mathbf{F}_o$ *be an abelian group with* o. *Let* $A, B \in \mathbf{F}_o^{nr}$. *Then the following are equivalent*:

(i) *A is diagonally equivalent to B,*

(ii) $H(A)=H(B)$, *and for all flows* γ *of* $H(A)$,

$$\Pi_\gamma(A)=\Pi_\gamma(B).$$

In the rest of this section, we omit proofs, since item (2.1) corresponds to item (3.i) for $i=2, 3, 4, 5$.

Definition 3.2. Let $A \in \mathbf{F}_o^{nr}$. Then A is called *chainable* if the bipartite pattern graph $H(A)$ is connected.

See Sinkhorn and Knopp [13] and Engel and Schneider [5] for equivalent definitions.

Proposition 3.3. *Let* $A \in \mathbf{F}_o^{nr}$. *If* $\mathbf{F}$ *has at least two distinct elements, then the following are equivalent*:

(i) *A is chainable*

(ii) *If* $XAY^{-1}=X'A(Y')^{-1}$, *then* $X'=fX$ *and* $Y'=fY$, *for some* $f\in\mathbf{F}$.

It is not necessary to check all flow products for $H(A)$ as in Theorem 3.1, to prove diagonal equivalence. By a construction similar to that preceding Corollary 2.4, we obtain cycles $\gamma_1, \ldots, \gamma_s$ for $H(A)$ such that each flow for $H(A)$ is a linear combination with integral coefficients of $\gamma_1, \ldots, \gamma_s$. In this case $s=m-(n+r)+p'$ where m and n are as before and p' is the number of components of $H(A)$. (Note that when $n=r$, $p'\geq p$, where p is the number of components of $G(A)$.)

Corollary 3.4. *Let* $A, B \in \mathbf{F}_o^{nr}$. *Let* $\gamma_1, \ldots, \gamma_s$ *be flows for* $H(A)$ *such that every flow for* $H(A)$ *is a linear combination with integral coefficients of* $\gamma_1, \ldots, \gamma_s$. *If* $H(A)=H(B)$ *and* $\Pi_{\gamma_i}(A)=\Pi_{\gamma_i}(B)$, $i=1, \ldots, s$, *then A is diagonally equivalent to B.*

Corollary 3.5. *Let* $\mathbf{F}$ *be an abelian group with more than one element and let* $A \in \mathbf{F}_o^{nr}$. *Then the following are equivalent*:

(i) *The only flow for* $H(A)$ *is* 0,

(ii) *For all* $B \in \mathbf{F}_o^{nr}$ *such that* $H(B)=H(A)$, *B is diagonally equivalent to A.*

4. Theorems on restricted diagonal equivalence

Necessary and sufficient conditions for two matrices to be restricted diagonally equivalent involve the divisibility properties of the group $\mathbf{F}$ and some graph theoretic concepts which will be defined as the need arises.

Notations 4.1. (i) By ε we shall denote (as before) the vector in $\mathbf{Z}^n$ or $\mathbf{Z}^{n+r}$ with all entries equal to 1.

(ii) By φ we shall denote the vector in $\mathbf{Z}^{n+r}$ with first n entries equal to 1, and last r entries equal to -1.

Lemma 4.2. *Let G be a connected graph on $\langle n\rangle$, let $\eta\in\mathbf{Z}^n$ and let $\Gamma=\Gamma(G)$. The following are equivalent:*

(i) $\eta^t\varepsilon=0$.

(ii) *There exists an $\alpha\in\mathbf{Z}^m$ such that $\alpha^t\Gamma=\eta^t$.*

Proof. (ii)$\Rightarrow$(i): Since $\Gamma\varepsilon=0$, $\alpha^t\Gamma=\eta^t$ implies that $\eta^t\varepsilon=\alpha^t\Gamma\varepsilon=0$.

(i)$\Rightarrow$(ii): Since G is connected, the rank of Γ is $n-1$ and a basis for the right null-space of Γ in $\mathbf{Q}^n$ is $\{\varepsilon\}$, cf. Pearl [11, pp. 390–391]. Suppose that $\eta^t\varepsilon=0$. Since the orthogonal complement in $\mathbf{Q}^n$ of the right null-space of Γ is the left hand range of Γ, there exists an $\alpha\in\mathbf{Q}^n$ such that $\alpha^t\Gamma=\eta^t$. We may suppose that the top left hand $(n-1)\times(n-1)$ submatrix Γ_1 of Γ is non-singular. Then the first $(n-1)$ rows of Γ form a basis for the left hand range, and so there exists

$$\alpha=\begin{bmatrix}\alpha'\\0\end{bmatrix}$$

where $\alpha'\in\mathbf{Q}^{n-1}$, $0\in\mathbf{Q}^{m-n+1}$ such that $(\alpha')^t\Gamma=\eta^t$, and so also, $\alpha^t\Gamma=\eta^t$. It follows that $(\alpha')^t\Gamma_1=(\eta')^t$ where

$$\eta=\begin{bmatrix}\eta'\\\eta''\end{bmatrix}$$

and $\eta'\in\mathbf{Z}^{n-1}$, $\eta''\in\mathbf{Z}^{m-n+1}$. But all minors of Γ_1 are $1, 0$ or -1 (cf. Chen [3, p. 80]), whence $\Gamma_1^{-1}\in\mathbf{Z}^{n-1,n-1}$. It follows that $\alpha'=\Gamma_1^{-1}\eta'\in\mathbf{Z}^{n-1}$ and so $\alpha\in\mathbf{Z}^m$.

Definitions 4.3. Let H be a bipartite graph on (V, V').

(i) If K is a vertex component of H then the *excess* of K is the absolute value of $|K\cap V|-|K\cap V'|$.

(ii) The graph H is *vertex-balanced* if the excess of all of its vertex components is 0. Otherwise H is *vertex-unbalanced.*

Let $A\in\mathbf{F}_o^{nr}$. The graph $H(A)$ is vertex-balanced if and only if the chainable direct summands of A are square. Hence, if $H(A)$ is vertex-balanced, then A is square.

Definitions 4.4. Let H be a bipartite graph on $(\langle n\rangle,\langle r\rangle')$.

(i) A *pseudo-diagonal* of H is a vector $\delta\in\mathbf{Z}^m$ such that $\delta^t\Delta=\varphi^t$ (here $\Delta=\Gamma(H)$).

(ii) A *diagonal* (or one-factor) of H is a $\{0, 1\}$-pseudo-diagonal.

Note that if $n\neq r$, then H will have no pseudo-diagonals. An example of a graph that has a pseudo-diagonal but no diagonal is furnished by $H(A)$ where

$$A=\begin{bmatrix}1&1&1\\1&o&o\\1&o&o\end{bmatrix},\quad \Delta=\begin{pmatrix}1&0&0&-1&0&0\\1&0&0&0&-1&0\\1&0&0&0&0&-1\\0&1&0&-1&0&0\\0&0&1&-1&0&0\end{pmatrix}$$

and the pseudo-diagonal is $(-1,1,1,1,1)$.

Intuitively, the diagonals of $H(A)$ have the usual meaning in terms of location in the matrix A, viz., they correspond to permutation matrices. A pseudo-diagonal corresponds to a matrix in $\mathbf{Z}^{nn}$ with all row and column sums equal to 1.

Lemma 4.5. *Let H be a bipartite graph on $(\langle n\rangle, \langle r\rangle')$. Then the following are equivalent*:

(i) *H is vertex-balanced,*

(ii) *H has a pseudo-diagonal.*

Proof. (i)$\Rightarrow$(ii): For any $\eta\in\mathbf{Z}^{n+r}$ we define $\eta^{(s)}$ to be the vector indexed by V_s with $\eta_i^{(s)}=\eta_i$, $i\in V_s$, where V_s is a vertex component of H. For $\alpha\in\mathbf{Z}^m$, $\alpha^{(s)}$ will be a vector indexed by the arc component H_s, where $\alpha_q^{(s)}=\alpha_q$, $q\in H_s$. Further $\Delta^{(s)}$ is a submatrix of the incidence matrix Δ of H with columns indexed by V_s, and rows by H_s. Suppose H is vertex-balanced. If $\varepsilon\in\mathbf{Z}^m$ and $\varphi\in\mathbf{Z}^{n+r}$ are defined as above, then we have $(\varphi^{(s)})^t\varepsilon^{(s)}=0$. Hence by Lemma 4.2 applied to H_s there exists a vector $\beta^{(s)}$ indexed by H_s such that $(\beta^{(s)})^t\Delta^s=(\varphi^{(s)})^t$. Since (after row and column permutation) Δ is a direct sum of $\Delta^{(1)},\dots,\Delta^{(p)}$ it follows that $\beta^t\Delta=\varphi^t$, where $\beta\in\mathbf{Z}^m$ is defined by $\beta_i=\beta_i^{(s)}$ for $i\in H_s$, $s=1,\dots,p$.

(ii)$\Rightarrow$(i). Let δ be a pseudo-diagonal of G. Then $\delta^t\Delta=\varphi^t$ and so for each s, $(\delta^{(s)})^t\Delta^{(s)}=\varphi^{(s)}$. Thus by Lemma 4.2, $(\varphi^{(s)})^t\varepsilon^{(s)}=0$. It follows that $\varphi^{(s)}$ has as many 1's as -1's, or, in other words, $|V_s\cap\langle n\rangle|=|V_s\cap\langle r\rangle'|$.

Theorem 4.6. *Let $\mathbf{F}_o$ be an abelian group with o. Let $A,B\in\mathbf{F}_o^{nn}$. If $H(A)$ is a vertex-balanced graph and δ is a pseudo-diagonal of $H(A)$, then the following are equivalent*:

(i) *A is restricted diagonally equivalent to B,*

(ii) *A is diagonally equivalent to B and*

$$\Pi_\delta(A)=\Pi_\delta(B).$$

Proof. It is enough to show that $XAY^{-1}=B$ implies that $\det XY^{-1}=\Pi_\delta(A)\Pi_\delta(B)^{-1}$. In the rest of this proof, $\mathbf{F}$ will be an additive group. If $x=(x_1,\dots,x_n)^t$, $y=(y_1,\dots,y_n)^t$ and $u=\binom{x}{y}$, then the diagonal equivalence relation becomes $b-a=\Delta u$, $\det XY^{-1}$ corresponds to $\varphi^t u$, and $\Pi_\delta(A)\Pi_\delta(B)^{-1}$ corresponds to $\delta^t(b-a)$. Since $\delta^t\Delta=\varphi^t$, we obtain

$$\delta^t(b-a)=\delta^t\Delta u=\varphi^t u.$$

Definition 4.7. Let $\mathbf{F}$ be an abelian (multiplicative) group.

(i) If k is a positive integer and $a \in \mathbf{F}$, then a is *divisible by* k if there exists $b \in \mathbf{F}$ such that $b^k = a$.

(ii) Let k be a positive integer. The group $\mathbf{F}$ is *divisible by* k if, for every $a \in \mathbf{F}$, a is divisible by k,

(iii) The group $\mathbf{F}$ is *divisible* if, for every positive integer k, $\mathbf{F}$ is divisible by k.

Theorem 4.8. *Let $\mathbf{F}_o$ be an abelian group with o. Let $A, B \in \mathbf{F}_o^{nr}$. Let $H(A)$ have a component with excess k, $k \geqslant 1$ and suppose that $\mathbf{F}$ is divisible by k. Then the following are equivalent.*

(i) *A is restricted diagonally equivalent to B,*

(ii) *A is diagonally equivalent to B.*

Proof. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(i). Suppose V_s is a vertex component with excess k. We assume that $\mathbf{F}$ is an additive group and use the notation of Lemma 4.5. By hypothesis, there is a $u \in \mathbf{F}^{n+r}$ such that $b - a = \Delta u$. We want to find a $u' \in \mathbf{F}^{n+r}$ such that $b - a = \Delta^t u'$ and $\varphi^t u' = 0$. Note that $\Delta^{(s)}\varepsilon^{(s)} = 0$. Let $\eta \in \mathbf{F}^{n+r}$ be defined by $\eta_i = \varepsilon_i^{(s)}$ if $i \in V_s$, and $\eta_i = 0$ otherwise. Then $\Delta\eta = 0$ and $\varphi^t\eta = k'$, where $|k'| = k$. Then for $u' = u - f\eta$, where $f \in \mathbf{F}$ satisfies $k'f = \varphi^t u$, we have $b - a = \Delta u'$ and $\varphi^t u' = \varphi^t u - k'f = 0$.

Corollary 4.9. *Let $A, B \in \mathbf{F}_o^{nr}$. If $H(A)$ is vertex-unbalanced and $\mathbf{F}$ is divisible, then* (i) *and* (ii) *of Theorem* 4.8 *are equivalent.*

Observe that the multiplicative group of non-zero complex numbers and the multiplicative group of positive real numbers are divisible.

Corollary 4.10. *Let $A, B \in \mathbf{F}_o^{nr}$. If $\mathbf{F}_o = \mathbf{R}$, the reals under multiplication, and $H(A)$ has a component with odd excess, then* (i) *and* (ii) *of Theorem* 4.8 *are equivalent.*

Corollary 4.11. *Let $A, B \in \mathbf{F}_o^{nr}$. If $H(A)$ has a component with excess* 1, *then* (i) *and* (ii) *of Theorem* 4.8 *are equivalent.*

We now show that, conversely, the equivalence of (i) and (ii) of Theorem 4.8 implies the divisibility properties of $\mathbf{F}$. For $k = 1, 2, \ldots$, let $A^{(k)}$ be the matrix in $\mathbf{F}_o^{(k+2)(k+2)}$ defined by

$$a_{ij}^{(k)} = 1 \quad \text{if } i = 1,\ j \neq 1 \text{ or } i \neq 1,\ j = 1,$$
$$a_{ij}^{(k)} = \text{o} \quad \text{otherwise}.$$

Thus, for example,

$$A^{(2)} = \begin{bmatrix} \text{o} & 1 & 1 & 1 \\ 1 & \text{o} & \text{o} & \text{o} \\ 1 & \text{o} & \text{o} & \text{o} \\ 1 & \text{o} & \text{o} & \text{o} \end{bmatrix}$$

Observe that there are no polygonal products for $A^{(k)}$ (i.e. $H(A^{(k)})$ has no cycles), and thus, for $B \in \mathbf{F}_o^{(k+2)(k+2)}$ with $H(B)=H(A^{(k)})$, Corollary 3.5 implies that B is diagonally equivalent to $A^{(k)}$. For $f \in \mathbf{F}$, we define a matrix $B^{(k)}(f) \in \mathbf{F}^{(k+2)(k+2)}$ which has the same entries as $A^{(k)}$ except that the entry in position $(1,2)$ is f.

Lemma 4.12. *Let $\mathbf{F}_o$ be an abelian group with* o *and let k be a positive integer. Let $f \in \mathbf{F}$. If the matrices $A^{(k)}$ and $B^{(k)}(f)$ are restricted diagonally equivalent, then f is divisible by k.*

Proof. In this proof $\mathbf{F}$ will be an additive group. By assumption there exists $x, y \in \mathbf{F}^{k+2}$ such that $b-a=\Delta u$ and $\varphi^t u=0$, where $u=\binom{x}{y}$, and a, b are the vectors in $\mathbf{F}^{2k+2}$ corresponding to $A^{(k)}, B^{(k)}(f)$, respectively. The equations for $b-a=\Delta u$ are

$$f=x_1-y_2,$$
$$0=x_1-y_j, \quad j=3,\dots,k+2,$$
$$0=x_i-y_1, \quad i=2,\dots,k+2.$$

On adding these $(2k+2)$ equations we obtain

$$f=k(x_1-y_1)+\sum_{i=1}^{k+2}(x_i-y_i)=k(x_1-y_1),$$

since

$$0=\varphi^t u=\sum_{i=1}^{k+2}(x_i-y_i).$$

Theorem 4.13. *Let $\mathbf{F}_0$ be an abelian group with* o. *Then the following are equivalent*:

(i) $\mathbf{F}$ *is a divisible group,*

(ii) *For all n and all $A \in \mathbf{F}_o^{nn}$* such that *$H(A)$ is vertex-unbalanced, the following implication holds: If $B \in \mathbf{F}_o^{nn}$ is diagonally equivalent to A, then B is restricted diagonally equivalent to A.*

Proof. (i)$\Rightarrow$(ii): By Corollary 4.9.

(ii)$\Rightarrow$(i): Since $H(A^{(k)})$ is vertex-unbalanced for $k=1,2,\dots$ and $B^{(k)}(f)$ is diagonally equivalent to $A^{(k)}$ for all $f \in \mathbf{F}$, this follows immediately by Lemma 4.12.

If is easy to prove an analogous theorem in which, in (ii) $\mathbf{F}_o^{nn}$ is replaced by $\mathbf{F}_o^{1n}$, or other classes of matrices.

5. Comments

5.1. If the matrix A is completely reducible (i.e. for some permutation matrix P, PAP^t is the direct sum of irreducible matrices, c.f. Pearl [11, p. 292]) then every arc of $G(A)$ lies on a circuit. In this case, a proof similar to the one we have for Theorem 2.1, shows that in that theorem, "flow" may be replaced by "circuit". The result, with the proof we have indicated, was published by Fiedler and Ptak [6], see also Engel and Schneider [4, Corollary 4.4], and for non-commutative $\mathbf{F}$ see Engel and Schneider [5]. The result was published independently without proof by Bassett *et al.* [1]. As far as we know, the simple observation has not previously been made in print that if cycles are considered in place of circuits, the restriction to completely reducible matrices may be eliminated.

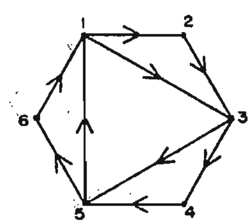
5.2. We define the *flow space* (over $\mathbf{Q}$) for a graph G to be the subspace of $\mathbf{Q}^m$ generated by all (integer) flows for G. Our remarks preceding Corollary 2.4 indicate the well-known result that the dimension of the flow space is $s = m - n + p$.

The cycles of Corollary 2.4 form a basis for the flow space. However, the examples below shows that some condition beyond this is necessary in Corollary 2.4. The condition in that corollary is ensured by the following: The $(s \times m)$ matrix whose rows are $\gamma_1^t, \ldots, \gamma_s^t$ has an $(s \times s)$ submatrix with determinant equal to ± 1. We also note that if A is completely reducible, it is possible to adapt the proof of Berge [2, Theorem 9, p. 29] to show that s *circuits* may always be found with the above property. Similar remarks may be made about Corollary 3.4.

Example: Let $\mathbf{F} = \{1, b\}$ with $b^2 = 1$ and consider the matrices $A, B \in \mathbf{F}_0^{66}$, where

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ b & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $G(A) = G(B)$ is given by



The circuits corresponding to the outer hexagon and the three quadrilaterals (e.g. with vertices 1, 3, 5, 6) form a basis for the flow space over $\mathbf{Q}$, but the circuit corresponding to the inner triangle is not an integral linear combination of the basis elements. Further, for each basis element γ, $\Pi_\gamma(A)=1=\Pi_\gamma(B)$, but if δ is the circuit corresponding to the inner triangle, $\Pi_\delta(A)=1\neq b=\Pi_\delta(B)$. Hence, by Theorem 2.1, A is not diagonally similar to B. On the other hand, a basis with the required integrality property, is furnished by the circuits corresponding to the inner triangle and the three quadrilaterals.

5.3. Theorem 3.1 stated in terms of polygonal products is known and goes back to Lallèment and Petrich [8, 9], cf. also Engel and Schneider [5]. At first sight, the theorem as contained in those references might appear to lack the condition $H(A)=H(B)$; but this is not so, for polygonal product is there defined to include e.g. $a_{11}\ a_{11}^{-1}\ a_{11}\ a_{11}^{-1}$. What we have added is the observation that polygonal products are cyclic products for $H(A)$ and hence that Theorems 2.1 and 3.1 are of the same type and, indeed, that Theorem 3.1 is an application of Theorem 2.1. Also we believe that our proof is consequently a little simpler, particularly if the well-known machinery of graph theory contained in Section 1 is taken for granted.

5.4. Corollary 3.5 formulated in terms of the polygonal products mentioned in 1.10 above, was announced by G.M. Engel in a talk in December 1974 at Gatlinburg VI, Munich.

5.5. Suppose $H(A)$ has a pseudo-diagonal δ. Then it is easy to see that δ together with the flow space for $H(A)$, cf. 5.2, generate the same space over $\mathbf{Q}$ that is generated by all pseudo-diagonals for $H(A)$. If follows that this space has dimension $m-2n+p'+1$ where m and p' are defined as in Corollary 3.4. It is also easy to see that one may find pseudo-diagonals $\delta_1,\ldots,\delta_t$, $t=m-2n+p'+1$, such that every integral flow or pseudo-diagonal of $H(A)$ is a linear combination of $\delta_1,\ldots,\delta_t$ with integral coefficients. Thus it is enough to verify that $\Pi_{\delta_i}(A)=\Pi_{\delta_i}(B)$, $i=1,\ldots,t$, to prove the restricted diagonal equivalence of A and B.

5.6. We define a matrix A to be *totally supported* if for suitable permutation matrices P, Q, the matrix PAQ is the direct sum of fully indecomposable matrices (cf. Marcus and Minc [10, p. 123] for definition). Suppose that A is totally supported. It then follows from the Frobenius–König theorem, Marcus and Minc [10, p. 97], König [7, p. 240], that each arc of $H(A)$ lies on a diagonal of $H(A)$. Also it is known that if $\delta\in\mathbf{Z}^m$, where δ is non-negative and $\delta^t\Delta(A)=k\varphi$, $k\in\mathbf{Z}$, $k\geqslant 0$, then δ is a combination of diagonals of $H(A)$ with positive integral coefficients. (This result is essentially König [7, p. 239, Theorem XIV, B]: A non-negative integral matrix with all row and column sums equal is a linear combination of permutation matrices with positive integral coefficients.) One may use this result to prove that, when A is totally supported, each pseudo-diagonal is a linear combination with integral coefficients of diagonals. Hence in this case, "pseudo-diagonal" may be replaced by "diagonal" in the result stated at the end of 5.5. We take this opportunity to observe that König's Theorem XIV, B (quoted above) can be proved in a manner completely analogous to a result in Berge [2, p. 91]. If A is square, then each non-negative flow in $G(A)$ is a linear combination of circuits with positive integral coefficients. Moreover, this approach shows that we require at most $(m-2n+p'+1)$ diagonals and $(m-n+p)$ circuits respectively. It is also interesting that it was known to Polya [12] already in 1916 that (in our terminology) if A has no entries equal to o, the diagonals of $H(A)$ span a space over $\mathbf{Q}$ of dimension $n^2-2n+2(m=n^2, p'=1)$. With a little hind-sight, one can easily find circuit, diagonal, and polygonal products in his paper.

5.7. We give two examples to show that some mention of $G(A)$ (or $H(A)$) is essential to our theorems, and that they cannot be formulated in terms of the complete graph G_n on $\langle n\rangle$.

First, in $\mathbf{F}_o^{22}$, if

$$A=\begin{pmatrix} o & o \\ o & o \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} o & b_{12} \\ o & o \end{pmatrix}$$

where $b_{12}\in\mathbf{F}$, then all the cyclic products of A and B are equal. Here cyclic product is defined intuitively, viz. a_{11}, a_{22}, $a_{12}\,a_{21}$ are the cyclic products of A. But A and B are not diagonally similar (or equivalent). We observe, however, that if a matrix A is irreducible then indeed the condition $\Pi_\gamma(A)=\Pi_\gamma(B)$ for all (intuitive) cycles or even circuits of G_n guarantees diagonal similarity (cf. Engel and Schneider [4]).

Second, even if $G(A)=G(B)$, it is not enough to check a cycle basis for the complete graph. Let

$$A=\begin{pmatrix} o & 1 & o \\ o & o & 1 \\ 1 & o & o \end{pmatrix}, \qquad B=\begin{pmatrix} o & 2 & o \\ o & o & 1 \\ 1 & o & o \end{pmatrix}$$

be in $\mathbf{Q}^{33}$. Here it would not be adequate to compare cyclic products only for cycles in a basis for the flow space of G_n if the basis did not contain the unique cycle in $G(A)=G(B)$, for certainly A and B are not diagonally similar.

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