Linear and Multilinear Algebra, 1976, Vol. 4, pp. 155–176

Printed in Great Britain

The Hadamard-Fischer Inequality for a Class of Matrices Defined by Eigenvalue Monotonicity

GERNOT M. ENGEL†

IBM, 101 BM 54, Owego, NY 13827

and

HANS SCHNEIDER‡

Mathematics Department, University of Wisconsin, Madison, WI 53706

(*Received July 23, 1976*)

For $A \in \mathbb{C}^{nn}$ and $\phi \subset \mu \subseteq \langle n \rangle = \{1, \ldots, n\}$, let $A[\mu] = (a_{ij})$, $i, j \in \mu$ and $A(\mu) = (a_{ij})$, $i, j \in \langle n \rangle \backslash \mu$. We define a subset $\omega_{\langle n \rangle}$ of $\mathbb{C}^{nn}$ by $A \in \omega_{\langle n \rangle}$ if

1) $\operatorname{Spec} A[\mu] \cap \mathbb{R} \neq \phi$, for $\phi \subset \mu \subseteq \langle n \rangle$,

2) $l(A[\mu]) \leqslant l(A[\nu])$, if $\phi \subset \nu \subseteq \mu \subseteq \langle n \rangle$,

where $l(A[\mu]) = \min(\operatorname{Spec} A[\mu] \cap \mathbb{R})$. For $A, B \in \omega_{\langle n \rangle}$, define $A \leqslant_\tau B$ by

$$l(A[\mu]) \leqslant l(B[\mu]), \quad \text{for} \quad \phi \subset \mu \subseteq \langle n \rangle.$$

By definition, $A \in \tau_{\langle n \rangle}$ if $A \in \omega_{\langle n \rangle}$ and $0 \leqslant_\tau A$.

For $0 \leqslant_\tau A \leqslant_\tau B$ (where $A, B \in \omega_{\langle n \rangle}$) it is shown that

3) $0 \leqslant \det A \leqslant \det B - \det(B - l(A)I) \leqslant \det B$.

For $A \in \tau_{\langle n \rangle}$, $A \leqslant_\tau A[\mu] \oplus A(\mu)$, and hence we obtain the Hadamard–Fischer inequality

4) $0 \leqslant \det A \leqslant \det A[\mu] \det A(\mu)$

for the class $\tau_{\langle n \rangle}$ which includes the positive semi-definite, totally nonnegative and M-matrices. Cases of equality in (3) are treated in detail and are related to the cyclic structure of A and B.

† The research of this author was made possible through the cooperation of IBM and the Department of Mathematics, State University of New York at Binghamton.

‡ The research of this author was supported in part by NSF Grant GP–3798X.

INTRODUCTION

It has been observed [e.g. Taussky (1958), Fan (1960) etc.] that the following three classes of matrices share many common properties: (a) the positive semi-definite matrices, (b) the M-matrices [for definitions, see Ostrowski (1937), Fan (1960), (1966), called class K by Fiedler and Ptak (1962), and (c) the totally nonnegative matrices [Gantmacher (1959, Vol. 2, p. 98)], Gantmacher and Krein (1937) or (1960, p. 85). For example, the three classes share the property given by (0.1) and (0.2) below, which we refer to as *eigenvalue monotonicity*.

Let† $A \in \mathbb{C}^{nn}$ and define $A[\mu] = (a_{ij}), i, j \in \mu, \phi \subset \mu \subseteq \langle n \rangle$. Then

0.1) Spec $A[\mu] \cap \mathbb{R} \neq \phi$, for $\phi \subset \mu \subseteq \langle n \rangle = \{1, \ldots, n\}$

(i.e. each principal submatrix has a real eigenvalue) and

0.2) $$l(A[\mu]) \leqslant l(A[\nu]), \quad \text{if} \quad \phi \subset \nu \subseteq \mu \subseteq \langle n \rangle$$

where

0.3) $$l(A[\mu]) = \min(\operatorname{Spec} A[\mu] \cap \mathbb{R}).$$

In addition, if A is in one of the three classes (a), (b), (c), then

0.4) $$l(A) \geqslant 0.$$

For A positive semi-definite, eigenvalue monotonicity is due to Cauchy (1829), cf. Beckenback and Bellman (1961), for M-matrices it is an immediate consequence of a theorem of Frobenius (1908, p. 471) cf. Gantmacher (1959, Vol. II, p. 67), and for totally nonnegative matrices the result is due to S. Friedland (unpublished).

The second common property we wish to emphasize is the Hadamard–Fischer inequality

0.5) $$0 \leqslant \det A \leqslant \det A[\mu] \det A(\mu), \quad \text{where} \quad A(\mu) = A[\langle n \rangle \backslash \mu].$$

This inequality is due to Hadamard (1893) and Fischer (1908) [cf. Beckenbach and Bellman (1961, p. 63) or Marcus and Minc (1964, p. 117)] for positive semi-definite matrices. For M-matrices, the inequality is an immediate consequence of Ostrowski (1937, Theorem 1). For totally nonnegative matrices, see Gantmacher (1959, Vol. II, p. 100), Gantmacher and Krein (1935, 1937, 1960, p. 108, Theorem 8) or Karlin (1968, p. 88, Lemma 9.2), for the special case $\mu = \{1, \ldots, k\}$ of the inequality. For general μ the result may be derived by Karlin (1958, Lemma 9.1, p. 88 and 10F, p. 95), see also Gantmacher and Krein (1960, Theorem 9, p. 111) and the remark following this theorem.

In this paper we show that the eigenvalue monotonicity condition (0.1), (0.2) together with the nonnegativity condition (0.4) implies the Hadamard–

† For our notations see Section 1.

Fischer inequality (0.5). More precisely, we define the subset $w_{\langle n\rangle}$ of $\mathbb{C}^{nn}$ consisting of all $A \in \mathbb{C}^{nn}$ which satisfy (0, 1) and (0.2). (Such A are called ω-matrices by us), and then we define the subset $\tau_{\langle n\rangle}$ of $\omega_{\langle n\rangle}$ of matrices $A \in \omega_{\langle n\rangle}$ which also satisfy (0.4), (such matrices are called τ-matrices). Evidently $\tau_{\langle n\rangle}$ contains the three classes (a), (b), (c) of matrices mentioned at the beginning of this introduction. We show that a τ-matrix A satisfies (0.5). In fact, the inequality (0.5) is a consequence of a better and more general inequality for τ-matrices. For A, $B \in \omega_{\langle n\rangle}$ we define a transitive, reflexive relation $A \leqslant_\tau B$ by

0.6) $$l(A[\mu]) \leqslant l(B[\mu]), \quad \text{if} \quad \phi \subset \mu \subseteq \langle n\rangle.$$

This relation has the property that if A, B and $B-A$ are positive semi-definite, or if A, B are M-matrices and $a_{ij} \leqslant b_{ij}$, $i, j = 1, 2, \ldots n$, then $A \leqslant {}_\tau B$. If A, $B \in \tau_{\langle n\rangle}$ and $A \leqslant_\tau B$, then

0.7) $$0 \leqslant \det A \leqslant \det B - \det(B - l(A)) \leqslant \det B,$$

cf. our main result, Theorem (3.8). Since, for $A \in \omega_{\langle n\rangle}$, $A \leqslant_\tau (A[\mu] \oplus A(\mu))$, the inequality (0.5) follows. The inequalities (0.7) are proved by a simple inductive argument using the *fundamental expansion*

0.8) $$\det(A + tI) = \sum_{\phi \subseteq \mu \subseteq \langle n\rangle} t^{|\mu|} \det A(\mu),$$

where $|\mu|$ denote the cardinality of μ.

Sections 4 and 5 are devoted to studying in detail the cases of equality in (0.7) when A is nonsingular. When the cyclic products of A and B (see (1.1)) satisfy an apparently artificial condition (see (4.1.1)) then the equalities in (0.8) are characterized in terms of the c-equivalence of matrices (see (1.11) and Theorem (4.1) and (5.1)). However this condition is clearly satisfied when $B = A[\mu_1] \oplus \ldots \oplus A[\mu_k]$, where $(\mu_1, \ldots, \mu_k)$ is a partition of $\langle n\rangle$ The concept of c-equivalence was examined by us in Engel and Schneider (1973b, 1975). Under the condition that $A[\mu_1], \ldots, A[\mu_k]$ are irreducible, it is shown (cf. Theorem (4.5)) that

$$0 < \det A = \det A[\mu_1] \ldots \det A[\mu_k]$$

if and only if A is (block) triangulable for $(\mu_1, \ldots, \mu_k)$ (Definition (4.3). We also characterize matrices satisfying

$$0 < \det A = \det A[\mu_1] \ldots \det A[\mu_k] - \det((A - l(A)\, I)[\mu_1]) \ldots \det((A - l(A)\, I)[\mu_k]),$$

see Theorem (5.4).

Thus we obtain well-known conditions for equality in Hadamard–Fischer for positive definite matrices [e.g. Marcus and Minc (1964)], and we also obtain as special cases the equality results for nonsingular M-matrices in our paper (1973a). Indeed, it was our observation at the end of (1973a) that the determinant expansion (0.8) could be used to prove the Hadamard inequality for both M-matrices and positive semi-definite matrices and could also be used to examine the cases for equality, which motivated our definition of τ-matrices.

In Section 6 of this paper we give some necessary conditions for all minors of A to satisfy Hadamard–Fischer, and some necessary conditions for A to be in $\omega_{\langle n\rangle}$.

Much work has recently been done on the Hadamard–Fischer inequality, particularly for classes of matrices which contain the totally nonnegative matrices. We mention in this connection the sign-symmetric and weakly sign symmetric matrices introduced by Kotelyanski (1953) [see also Gantmacher and Krein (1960, p. 111)] and studied by Carlson (1967) and, under the name of GKK matrices, by Fan (1967). In Section 7 we give examples to show how τ-matrices are related to these classes.

Our paper may be viewed as an exploration of the relationship of the Hadamard–Fischer inequality to the Perron–Frobenius Theorem, see Theorem (3.12) and the comments following it.

1. NOTATIONS AND DEFINITIONS

1.1) By $\mathbb{R}$ and $\mathbb{C}$ we denote the real and complex field respectively.

1.2) The set of all $m\times n$ matrices with elements in $\mathbb{R}$ (resp. $\mathbb{C}$) is denoted by $\mathbb{R}^{mn}$ (resp. $\mathbb{C}^{mn}$).

1.3) We use $\subseteq$ for set inclusion and $\subset$ for proper set inclusion.

1.4) If n is a positive integer, then $\langle n\rangle = \{1, \ldots, n\}$.

1.5) If $A\in\mathbb{C}^{nn}$, and $\phi \subseteq \mu \subseteq \langle n\rangle$, then

$$A[\mu] = (a_{ij}),\ i,j\in\mu;\ A[\mu]\in\mathbb{C}^{|\mu||\mu|},$$
$$A(\mu) = (a_{ij}),\ i,j\in\mu' = \langle n\rangle\backslash\mu;\ A(\mu)\in\mathbb{C}^{|\mu'||\mu'|},$$
$$A[\mu|\nu] = (a_{ij}),\ i\in\mu, j\in\nu,\ A[\mu|\nu]\in\mathbb{C}^{|\mu||\nu|},\ \nu,\mu\subseteq\langle n\rangle.$$

(We write $A[1]$, $A[12]$ for $A[\{1\}]$, $A[\{12\}]$, etc.)

1.6) A *closed path* on $\langle n\rangle$ is a sequence† $\gamma = (i_1, \ldots, i_k)$ of integers $i_p\in\langle n\rangle$, $p = 1, \ldots, k$, where $k \geqslant 2$, and a *cycle* is a closed path whose elements are distinct. We identify the cycles $(i_1, \ldots, i_k)$ and $(i_p, \ldots, i_k, i_1, \ldots i_{p-1})$, where $1 \leqslant p \leqslant k$.

1.7) A *full cycle* on $\langle n\rangle$ is a cycle $\gamma = (i_1, \ldots, i_k)$ where $k = n$.

1.8) If $\gamma = (i_1, \ldots, i_k)$ is a cycle on $\langle n\rangle$, then the *support* $\bar\gamma$ of γ is defined by $\bar\gamma = \{i_1, \ldots, i_k\}$.

1.9) If $A\in\mathbb{C}^{nn}$ and $\gamma = (i_1, \ldots, i_k)$ is a closed path (on $\langle n\rangle$) then we put

$$\Pi_\gamma(A) = a_{i_1i_2}\ldots a_{i_{k-1}i_k}a_{i_ki_1}.$$

† In Engel and Schneider (1973b), we denoted the same closed path by $(i_1i_2, \ldots, i_k, i_1)$.

[Note the last factor of this product.] If γ is a cycle, the product is called a *cyclic product*.

1.10) A cycle γ is *nonzero for A* if $\Pi_\gamma(A) \neq 0$.

1.11) Let $A, B \in \mathbb{C}^{nn}$. Then $A \sim_c B$ (A is c-equivalent to B) is defined by:

b) $a_{ii} = b_{ii}, i = 1, \ldots, n,$ and b) $\Pi_\gamma(A) = \Pi_\gamma(B)$, for all cycles γ.

[cf. Engel and Schneider (1973b). Observe that in the present notation a_{ii} is not a cyclic product for A].

1.12) If $A \in \mathbb{C}^{nn}$, then Spec A denotes the spectrum of A.

2. PRELIMINARY RESULTS ON CYCLES

In this section we develop some lemmas needed to discuss the cases of equality in the Hadamard and Fischer inequalities.

2.1) LEMMA *Let β, γ be two distinct full cycles on $\{1, \ldots, n\}$. Then there exist nonfull cycles $\sigma_1, \ldots, \sigma_k$, where $k \geqslant 3$, such that*

2.1.1) $$\Pi_\beta \Pi_\gamma = \Pi_{\sigma_1}\Pi_{\sigma_2} \ldots \Pi_{\sigma_k},$$

(*viz. for all $A \in \mathbb{C}^{nn}$, $\Pi_\beta(A)\,\Pi_\gamma(A) = \Pi_{\sigma_1}(A)\Pi_{\sigma_2}(A) \ldots \Pi_{\sigma_k}(A)$.*)

Proof Let $\beta = (i_1, \ldots, i_n)$, $\gamma = (j_1, \ldots, j_n)$, where $i_1 = j_1$. Then there exists a smallest integer p such that $i_p \neq j_p$, and clearly $p \geqslant 2$. There also exist integers q, r, $p < q$, $r \leqslant n$ such that $i_p = j_q$ and $j_p = i_r$. Let $\sigma_1 = (i_1, \ldots, i_{p-1}, j_q, \ldots, j_n)$ and $\sigma_2 = (j_1, \ldots, j_{p-1}, i_r, \ldots, i_n)$. Then σ_1 and σ_2 are cycles and, since $|\sigma_s| < n$, $s = 1, 2$, the cycles σ_1, σ_2 are not full. Let $\sigma = (i_p, \ldots, i_{p-1}, j_p, \ldots, j_{q-1})$. This is a closed path, hence† there exists cycle $\sigma_3, \ldots, \sigma_k$, such that

a) $$\Pi_\sigma = \Pi_{\sigma_3} \ldots \Pi_{\sigma_k},$$

b) $$\bar{\sigma}_s \subseteq \bar{\sigma}, s = 3, \ldots, k.$$

Since $i_1 \notin \bar{\sigma}$, the cycles σ_s, $s = 3, \ldots, k$ are not full. We now have

$$\Pi_\beta \Pi_\gamma = \Pi_{\sigma_1}\Pi_{\sigma_2}\Pi_\sigma = \Pi_{\sigma_1} \ldots \Pi_{\sigma_k}.$$

2.2) COROLLARY *Let $A, B \in \mathbb{C}^{nn}$ and suppose that for all nonfull cycles σ, $\Pi_\sigma(A) = \Pi_\sigma(B)$. Then for any two distinct full cycles β, γ,*

$$\Pi_\beta(A)\,\Pi_\gamma(A) = \Pi_\beta(B)\,\Pi_\gamma(B).$$

Proof Immediate by Lemma (2.1).

2.3) COROLLARY *Let $A, B \in \mathbb{C}^{nn}$ and suppose that for all nonfull cycles σ, $\Pi_\sigma(A) = \Pi_\sigma(B)$. If there exist at least three distinct full nonzero cycles for*

† See Engel and Schneider (1973b, Lemma 2.4). Conclusion (b) is in fact not stated there, but follows by a slightly more precise version of the proof given there.

A, *then* either

2.3.1) $$\Pi_\sigma(A) = \Pi_\sigma(B), \textit{ for all full cycles } \sigma,$$

or

2.3.2) $$\Pi_\sigma(A) = -\Pi_\sigma(B), \textit{ for all full cycles } \sigma.$$

Proof If β is a full nonzero cycle for A, let $r_\beta = \Pi_\beta(B)\Pi_\beta(A)^{-1}$. Suppose that β, γ, σ are distinct full nonzero cycles for A. By Corollary (2.2), $\Pi_\beta(B) \neq 0$ and so $r_\beta \neq 0$. Further, by the same Corollary, $r_\beta = r_\gamma^{-1}$. Thus $r_\beta = r_\gamma^{-1} = r_\sigma = r_\beta^{-1}$, whence $r_\beta = \pm 1$, and $r_\sigma = r_\beta$.

2.4) *Example* If we drop the assumption that there exist three full nonzero cycles for A, then Corollary (2.3) no longer holds. As an example let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & .25 \\ .5 & 1 & 2 \\ 4 & .5 & 1 \end{bmatrix}.$$

2.5) LEMMA *Let $A, B \in \mathbb{C}^{nn}$, and suppose that for all nonfull cycles σ, $\Pi_\sigma(A) = \Pi_\sigma(B)$ and $a_{ii} = b_{ii}$, $i = 1, \ldots, n$. If for all full cycles σ*

2.5.1) $$\begin{cases} \text{either} & |\Pi_\sigma(A)| > |\Pi_\sigma(B)|, \\ \text{or} & \Pi_\sigma(A) = \Pi_\sigma(B), \end{cases}$$

then

2.5.2) $$\begin{cases} \text{either} & A \sim_c B \\ \text{or} & \textit{there exists at most one full nonzero cycle for } A. \end{cases}$$

Proof Suppose β is a nonzero full cycle for A. If there is another full nonzero cycle γ for A, then, by Corollary (2.2), $\Pi_\beta(A)\,\Pi_\gamma(A) = \Pi_\beta(B)\,\Pi_\gamma(B)$. Since $|\Pi_\sigma(B)| \leqslant |\Pi_\sigma(A)|$ for all cycles σ, it follows by (2.5.1) that $\Pi_\beta(A) = \Pi_\beta(B)$ and $\Pi_\sigma(A) = \Pi_\sigma(B)$. Hence, if there are two nonzero full cycles for A, then $A \sim_c B$.

3. INEQUALITIES FOR τ-MATRICES

To prove many of our results we shall use a well known expansion of the determinant of a matrix. If $A \in \mathbb{C}^{nn}$ and $t \in \mathbb{C}$ then

3.1) $$\det(A+tI) = \sum_{\phi \subseteq \mu \subseteq \langle n \rangle} t^{|\mu|} \det A(\mu),$$

where $|\mu|$ is the cardinality of μ and $\det A(\langle n \rangle) = 1$ by convention. We shall call (3.1) the *fundamental expansion* of $\det(A+tI)$. We first consider matrices $A \in \mathbb{C}^{nn}$ which satisfy

3.2) $$\operatorname{spec} A[\mu] \cap \mathbb{R} \neq \phi, \quad \text{for all } \mu, \quad \phi \subset \mu \subseteq \langle n \rangle.$$

For all such matrices, we define, as in the introduction,

3.3) $$l(A[\mu]) = \min\{\operatorname{Spec} A[\mu] \cap \mathbb{R}\}, \ \phi \subset \mu \subseteq \langle n \rangle.$$

3.3) LEMMA *Let* $A \in \mathbb{C}^{nn}$ *satisfy* (3.2). *Then* $\det A \in \mathbb{R}$.

Proof By induction on n. The result is obvious if $n = 1$. So let $n > 1$, and suppose that $\det A[\mu] \in \mathbb{R}$, if $|\mu| < n$. Let $l = l(A)$ and $B = A - lI$. Then, since $\det B = 0$ and B satisfies (3.2), it follows from the fundamental expansion that

$$\det A = \sum_{\phi \neq \mu \subseteq \langle n \rangle} l^{|\mu|} \det B(\mu),$$

which is real by inductive assumption.

3.4) LEMMA *Let* $A \in \mathbb{C}^{nn}$ *satisfy* (3.2) *and suppose that* $l(A) \geqslant 0$ ($l(A) > 0$). *Then* $\det A \geqslant 0$ ($\det A > 0$).

Proof By (3.3) and (3.1) the characteristic polynomial $\det(A - tI)$ has real coefficients. Hence the nonreal eigenvalues occur in conjugate pairs. The result follows.

3.5) DEFINITIONS i) *Let* $A \in \mathbb{C}^{nn}$. *If* (3.2) *holds and*

3.5.1) *If* $\phi \subset \nu \subseteq \mu \subseteq \langle n \rangle$, *then* $l(A[\mu]) \leqslant l(A[\nu])$,

then we call A *an* ω*-matrix. The set of all* ω*-matrices in* $\mathbb{C}^{nn}$ *will be denoted by* $\omega_{\langle n \rangle}$.

We shall refer to the property, (3.5.1) briefly as *eigenvalue monotonicity*.

ii) *We define a reflexive, transitive relation on* $\omega_{\langle n \rangle}$ *thus*: *If* $A, B \in \omega_{\langle n \rangle}$, *then* $A \leqslant_\tau B$ *if*

3.5.2) $l(A[\mu]) \leqslant l(B)[\mu])$, *for all* μ, $\phi \subseteq \mu \subseteq \langle n \rangle$.

iii) *If* $A, B \in \omega_{\langle n \rangle}$, $A \leqslant_\tau B$ *and* $B \leqslant_\tau A$ *then we write* $A \sim_\tau B$. *Clearly* $\sim_\tau$ *is an equivalence relation on* $\omega_{\langle n \rangle}$.

iv) *If* $A \in \omega_{\langle n \rangle}$, *and* $l(A) \geqslant 0$ (*viz.* $0 \leqslant_\tau A$) *then we call* A *a* τ*-matrix. We denote the set of all* τ*-matrices in* $\mathbb{C}^{nn}$ *by* $\tau_{\langle n \rangle}$.

Remark Let $\phi \subset \mu \subset \langle n \rangle$. If $A \in \tau_{\langle n \rangle}$, then $A[\mu] \in \tau_\mu$.

3.6) THEOREM *Let* $A \in \omega_{\langle n \rangle}$. *Then the following are equivalent*

3.6.1) $A \in \tau_{\langle n \rangle}$,

3.6.2) $\det A[\mu] \geqslant 0$, *for all* μ, $\phi \subset \mu \subseteq \langle n \rangle$.

Proof (3.6.1) $\Rightarrow$ (3.6.2). This follows immediately from Lemma (3.4). (3.6.2) $\Rightarrow$ (3.6.1). By induction on n. The result is obvious if $n = 1$. So suppose it is true for $A[\mu]$, where $|\mu| < n$. By the fundamental expansions, for $t > 0$, $\det(A + tI) = \sum_{\phi \subseteq \mu \subseteq \langle n \rangle} t^{|\mu|} \det A(\mu) \geqslant t^n > 0$. Hence A has no negative eigenvalues, whence $l(A) \geqslant 0$.

3.7 *Remarks* i) A similar argument shows: Let $A \in \omega_{\langle n \rangle}$. Then the following are equivalent:

3.7.1) $l(A) > 0$,

3.7.2) $\det A[\mu] > 0$, for all μ, $\phi \subseteq \mu \subseteq \langle n \rangle$.

ii) Further, let $A \in \tau_{\langle n\rangle}$. Then clearly $l(A) > 0$ if and only if $\det A > 0$.

Our main result is the next theorem.

3.8) THEOREM *Let $A, B \in \tau_{\langle n\rangle}$ and $A \leqslant_\tau B$.*

i) *Then*

3.8.1) $$0 \leqslant \det A \leqslant \det B - \det(B - l(A)\, I) \leqslant \det B.$$

ii) *Let* $\det A > 0$. *Then the following are equivalent*:

3.8.2) $$\det A = \det B,$$

3.8.3) $$\det A[\mu] = \det B[\mu], \quad \textit{for} \quad \phi \subseteq \mu \subseteq \langle n\rangle,$$

3.8.4) $$A \sim_\tau B.$$

iii) *Let* $\det A > 0$. *Then the following are equivalent*:

3.8.5) $$\det A = \det B - \det(B - l(A)\, I),$$

3.8.6) $$\det A[\mu] = \det B[\mu], \quad \textit{for} \quad \phi \subseteq \mu \underset{\neq}{\subset} \langle n\rangle,$$

3.8.7) $$A[\mu] \sim_\tau B[\mu], \quad \textit{for} \quad \phi \subset \mu \underset{\neq}{\subset} \langle n\rangle.$$

Proof i) In view of Lemma (3.4) we need to prove only the middle inequality of (3.8.1). We proceed by induction on n. The result holds if $n = 1$. So let $n > 1$, and $A, B \in \tau_{\langle n\rangle}$ where $A \leqslant_\tau B$. Put $l = l(A)$. Since $B - lI \in \tau_{\langle n\rangle}$, $\det(A - lI)(\mu) \leqslant \det(B - lI)(\mu)$ for $\phi \subset \mu \subseteq \langle n\rangle$, by our inductive assumption. But $\det(A - lI) = 0$ and hence it follows from the fundamental expansion (3.1) that

$$\begin{aligned}
\det B &= \sum_{\phi \subseteq \mu \subseteq \langle n\rangle} l^{|\mu|} \det(B - lI)(\mu) \\
&= \sum_{\phi \subset \mu \subseteq \langle n\rangle} l^{|\mu|} \det(B - lI)(\mu) + \det(B - lI) \\
&\geqslant \sum_{\phi \subset \mu \subseteq \langle n\rangle} l^{|\mu|} \det(A - lI)(\mu) + \det(B - lI) \\
&= \sum_{\phi \subseteq \mu \subseteq \langle n\rangle} l^{|\mu|} \det(A - lI)(\mu) + \det(B - lI) \\
&= \det A + \det(B - lI),
\end{aligned}$$

and (i) is proved.

ii) (3.8.2) $\Rightarrow$ (3.8.3). By inspection of the proof of (i), since $l > 0$, we see that $\det A = \det B$ implies that

$$\det(A - lI)(\mu) = \det(B - lI)(\mu), \ \phi \subseteq \mu \subset \langle n\rangle.$$

Hence

$$\begin{aligned}
\det B[\mu] &= \sum_{\phi \subseteq \nu \subseteq \mu} l^{|\nu|} \det(B - lI)(\nu) \\
&= \sum_{\phi \subseteq \nu \subseteq \mu} l^{|\nu|} \det(A - lI)(\nu) = \det A[\mu].
\end{aligned}$$

(3.8.3) $\Rightarrow$ (3.8.4). By induction on $\langle n\rangle$. The result is clearly true for $n = 1$. So suppose that $n > 1$, and suppose that

$$\det A[\nu] = \det B[\nu], \quad \text{for} \quad \phi \subset \nu \subseteq \mu \subset \langle n\rangle,$$

implies that $A[\mu] \sim_\tau B[\mu]$. By (3.8.3), $\det(B-lI) = 0$. Since $0 \leqslant_\tau (B-lI)$, it follows that $l(B) = l$, whence $A \sim_\tau B$. (3.8.4.) $\Rightarrow$ (3.8.2). This is obvious by (i).

iii) (3.8.5) $\Rightarrow$ (3.8.6). By inspection of the proof of (i), (3.8.5) implies that $\det(A-lI)(\mu) = \det(B-lI)(\mu)$ for $\phi \subset \mu \subset \langle n\rangle$. Hence, as in the proof of (3.8.2) $\Rightarrow$ (3.8.3), we obtain (3.8.6).

(3.8.6) $\Rightarrow$ (3.8.7). By (ii).

(3.8.7) $\Rightarrow$ (3.8.5). By (3.8.7),

$$(A-lI)[\mu] \sim_\tau (B-lI)[\mu], \quad \text{for} \quad \phi \subset \mu \subset \langle n\rangle.$$

Hence by (ii), $\det(A-lI)(\mu) = \det(B-lI)(\mu)$ for $\phi \subset \mu \subsetneqq \langle n\rangle$, and the result follows from the fundamental expansion, see the proof of (i).

We now show that the Hadamard–Fischer inequality characterizes τ-matrices in a certain sense. Let $A \in \mathbb{C}^{nn}$ be a matrix with real principal minors, and for $t \in \mathbb{R}$ put $A_t = A - tI$. Observe that for t negative and $|t|$ sufficiently large, the principal minors of A_t are nonnegative, while for $t > \min\{a_{ii}, i \in \langle n\rangle\}$, some principal minor of A_t is negative. Hence we may make the following definitions.

3.9) Definitions

i) $\mathbf{P}_{\langle n\rangle} = \{A \in \mathbb{C}^{nn}$: *all principals minors of A are positive*$\}$.

ii) $\mathbf{P}^0_{\langle n\rangle} = \{A \in \mathbb{C}^{nn}$: *all principal minors of A are nonnegative*$\}$.

iii) *Let* $A \in \mathbb{C}^{nn}$ *have real principal minors. Then, for* μ, $\phi \subset \mu \subset \langle n\rangle$, $m(A[\mu]) = \sup\{t \in \mathbb{R}: A_t[\mu] \in \mathbf{P}^0_{\langle n\rangle}\}$ *where* $A_t = A - tI$.

We note that if $A \in \mathbf{P}^0_\mu$ and $s < 0$

3.10) $$\det A_s[\mu] = \sum_{\phi \subseteq \nu \subseteq \mu} (-s)^{|\nu|} \det A[\mu\backslash\nu] \geqslant (-s)^{|\mu|} > 0$$

so that $A_s \in \mathbf{P}_{\langle n\rangle}$.

Also it is clear that if A has all principal minors real then

3.11) $$\phi \subset \nu \subseteq \mu \subseteq \langle n\rangle \text{ implies that } m(A[\mu]) \leqslant m(A[\nu]).$$

3.12) Theorem *Let $A \in \mathbb{C}^{nn}$ be a matrix with all principal minors real. Then the following are equivalent:*

3.12.1) $$A \in \omega_{\langle n\rangle}$$

3.12.2) *For all* μ, $\phi \subset \mu \subseteq \langle n\rangle$ *and all* $t \in \mathbb{R}$ *such that* $A_t[\mu] = A[\mu] - tI[\mu] \in \mathbf{P}^0_\mu$ *and all* ν, $\phi \subset \nu \subseteq \mu$,

$$\det A_t[\mu] \leqslant \det A_t[\nu] \det A_t[\mu\backslash\nu],$$

3.12.3) *For all* μ, $\phi \subset \mu \subseteq \langle n\rangle$ *and all* $t \in \mathbb{R}$ *such that* $A_t[\mu] = A[\mu] - tI[\mu] \in \mathbf{P}^0_\mu$ *and all* i, $i \in \mu$,

$$\det A_t[\mu] \leqslant \det A_t[i] \det A_t[\mu\backslash\{i\}]$$

Proof (3.12.1) $\Rightarrow$ (3.12.2).

Suppose $A \in \omega_{\langle n\rangle}$ and let $t \in \mathbb{R}$ be such that $A_t \in \mathbf{P}^0_{\langle n\rangle}$. It follows from (3.10) that $l(A_t) \geqslant 0$, whence $A_t \in \tau_{\langle n\rangle}$. Let $\phi \subset \mu \subseteq \langle n\rangle$ and put $B_t = A_t[\mu] \oplus A_t(\mu)$. Then clearly also $B_t \in \tau_{\langle n\rangle}$, and since for $\phi \subset \kappa \subseteq \langle n\rangle$ and $\kappa_1 = \kappa \cap \mu$, $\kappa_2 = \kappa \cap \mu'$,

$$l(B_t[\kappa]) = \min(l(B_t[\kappa_1])), l(B_t[\kappa_2])) \geqslant l(A_t(\kappa)),$$

it follows that $A_t \leqslant {}_\tau B_t$. Thus (3.12.2) follows from Theorem (3.8.1).

(3.12.2) $\Rightarrow$ (3.12.3). Trivial.

(3.12.3) $\Rightarrow$ (3.12.1).

Suppose (3.12.3) holds. It is enough to prove that $m(A[\mu])$ is an eigenvalue of $A[\mu]$, $\phi \subset \mu \subseteq \langle n\rangle$, for then by (3.10), $m(A[\mu])$ is the least real eigenvalue of $A[\mu]$, and (3.11) shows that the eigenvalue monotonicity property holds. Our proof that $m(A[\mu])$ is an eigenvalue of $A[\mu]$ is by induction on n. Clearly the result is true if $n = 1$. So suppose the result is true for $A[\mu]$, if $|\mu| = n-1$. Assume that $m(A)$ is not an eigenvalue of A. Then, by (3.10), $\det A_t > 0$ for $t \leqslant m(A)$. Let $\phi \subset \mu \subset \langle n\rangle$, where $|\mu| = n-1$. By (3.12.2), $\det A_t[\mu] > 0$, for $t \leqslant m(A)$. Since $A[\mu] \in \omega_\mu$, we deduce that $l(A[\mu]) > m(A)$. But then $\det A_t[v] > 0$, for all v, $\phi \subset v \subset \langle n\rangle$ and $t \leqslant m(A)$. Thus we can find $s > m(A)$ such that $A_s \in \mathbf{P}^0_{\langle n\rangle}$. This contradicts the definition of $m(A)$.

In (3.12.2) and (3.12.3) we can evidently replace $\mathbf{P}^0_{\langle n\rangle}$ by $\mathbf{P}_{\langle n\rangle}$.

The last part of our proof of Theorem (3.12) may be compared with Frobenius' (1908) proof of the Perron–Frobenius theorem for a matrix Q with positive entries [see also Gantmacher and Krein (1960, pp. 97–99) for essentially the same proof as Frobenius']. Though we know of no evidence that Frobenius was familiar with Hadamard (1893), with unhistorical hindsight we may regard Frobenius' proof of the existence of a positive eigenvalue of Q thus: first establish Hadamard's inequality for M-matrices in the form (3.12.2). Then use this inequality to prove that an M-matrix is also an ω-matrix. We have found no counterpart for ω-matrices to the result that the Perron–Frobenius root of Q is also the spectral radius of Q, and this has led to the open problem (7.5.ii).

4. CONDITIONS FOR EQUALITY IN $\det A = \det B$

4.1) THEOREM *Let $A, B \in \tau_{\langle n\rangle}$ and suppose that, for all cycles γ,*

4.1.1)
$$\begin{cases} \text{either } |\Pi_\gamma(B)| < |\Pi_\gamma(A)|, \\ \text{or} \quad \Pi_\gamma(B) = \Pi_\gamma(A). \end{cases}$$

Let $A \leqslant_\tau B$ and let $\det A > 0$. *Then the following are equivalent.*

4.1.2) $$\det A = \det B,$$

4.1.3) $$A \sim_\tau B,$$

4.1.4) $$A \sim_c B.$$

Proof (4.1.2) $\Leftrightarrow$ (4.1.3). By Theorem (3.8) (ii).

(4.1.4) $\Rightarrow$ (4.1.2). Obvious, since permutation products may be factorized into cyclic product and products of diagonal elements.

(4.1.3) $\Rightarrow$ (4.1.4). The proof is by induction on n. The result is clearly true if $n = 1$. So suppose that $n > 1$, and that for $\phi \subset \mu \subset \langle n \rangle$, $A[\mu] \sim_\tau B[\mu]$ implies $A[\mu] \sim_c B[\mu]$. Let $A \sim_\tau B$. By inductive assumption $A[\mu] \sim_c B[\mu]$, for $\phi \subset \mu \subset \langle n \rangle$. We also know that $\det A = \det B$. It follows that

$$\sum_{\gamma \in \Gamma} \Pi_\gamma(A) = \sum_{\gamma \in \Gamma} \Pi_\gamma(B), \tag{4.1.5}$$

where Γ is the set of full cycles on $\langle n \rangle$, since, for any matrix $C \in \mathbb{C}^{nn}$

$$\det C = (-1)^{n-1} \sum_{\gamma \in \Gamma} \Pi_\gamma(C) + \sum_{\kappa \in S_n \setminus \Gamma} (-1)^{\operatorname{sgn} \kappa} \Pi_\kappa(C)$$

where S_n is the symmetric group on $\langle n \rangle$, and Γ is also interpreted as a set of permutations. First, suppose there exist at most one nonzero full cycle β for A. Since $|\Pi_\gamma(B)| \leqslant |\Pi_\gamma(A)|$, it follows that $\Pi_\gamma(B) = 0$, for γ a full-cycle, $\gamma \neq \beta$, and by (4.1.5), $\Pi_\beta(B) = \Pi_\beta(A)$. Hence $A \sim_c B$. Next, suppose there is more than one nonzero full cycle for A. Then $A \sim_c B$, by Lemma (2.5) since $A \sim_\tau B$ implies that $a_{ii} = b_{ii}$, $i = 1, \ldots, n$.

4.2) COROLLARY *Let* $\phi \subset \mu \subset \langle n \rangle$. *Let* $A \in \tau_{\langle n \rangle}$ *and* $\det A > 0$.
Then the following are equivalent:

4.2.1) $\quad \det A = \det A[\mu] \det A(\mu)$,

4.2.2) $\quad A \sim_\tau A[\mu] \oplus A(\mu)$,

4.2.3) $\quad A \sim_c A[\mu] \oplus A(\mu)$,

4.2.4) *If* γ *is a nonzero cycle for* A, *then either* $\bar{\gamma} \subseteq \mu$ *or* $\bar{\gamma} \subseteq \mu' = \langle n \rangle \setminus \mu$.

Proof Let $B = A[\mu] \oplus A(\mu)$. We observe that $B \in \tau_{\langle n \rangle}$ and $A \leqslant_\tau B$, compare the proof of Theorem (3.12). The equivalence of the first three conditions now follows from Theorem (4.1), since condition (4.1.1) obviously holds.

(4.2.3) $\Rightarrow$ (4.2.4). Let γ be a cycle and let $\Pi_\gamma(A) \neq 0$. If $A \sim_c B$, then $\Pi_\gamma(B) \neq 0$, whence either $\bar{\gamma} \subseteq \mu$ or $\bar{\gamma} \subseteq \mu'$.

(4.2.4) $\Rightarrow$ (4.2.3). Let γ be any cycle on $\langle n \rangle$. If $\bar{\gamma} \subseteq \mu$ or $\bar{\gamma} \subseteq \mu'$ then $\Pi_\gamma(A) = \Pi_\gamma(B)$. If $\bar{\gamma} \cap \mu \neq \phi$ and $\bar{\gamma} \cap \mu' \neq \phi$, then $\Pi_\gamma(B) = 0$. But by (4.2.4), $\Pi_\gamma(A) = 0$, whence $A \sim_c B$.

By applying (4.2) to the submatrices of A, we obtain a result on partitions on $\langle n \rangle$.

4.3) THEOREM *Let* $(\mu_1, \ldots, \mu_k)$ *be a partition of* $\langle n \rangle$ (*into* non-empty *subsets*). *Let* $A \in \tau_{\langle n \rangle}$. *Then*

i) $\det A \leqslant \det A[\mu_1] \ldots \det A[\mu_k]$.

ii) *Let* $\det A > 0$.

Then the following are equivalent:

4.3.1) $$\det A = \det A[\mu_1] \ldots \det A[\mu_k]$$

4.3.2) $$A \sim_\tau A[\mu_1] \oplus \ldots \oplus A[\mu_k]$$

4.3.3) $$A \sim_c A[\mu_1] \oplus \ldots \oplus A[\mu_k].$$

4.3.4) *If γ is a nonzero cycle for A, then for some i, $1 \leqslant i \leqslant k$, $\bar{\gamma} \subseteq \mu_i$.*

4.4) COROLLARY *Let $A \in \tau_{\langle n\rangle}$ and let* $\det A > 0$. *Then the following are equivalent*:

4.4.1) *For all partitions $(\mu_1, \ldots, \mu_k)$ of $\langle n\rangle$, $(k \geqslant 2)$,*

$$\det A < \det A[\mu_1] \ldots \det A[\mu_k],$$

4.4.2) A *is irreducible.*

Proof For all μ, $\phi \subset \mu \subset \langle n\rangle$, $A \nsim_c A[\mu] \oplus A(\mu)$ if and only if A is irreducible.

To obtain an explicit form for A in the equality case we need to assume that the $A[\mu_i]$ are irreducible. We first state formally,

4.5) DEFINITION *Let $(\mu_1, \ldots, \mu_k)$ be a partition of $\langle n\rangle$. Then $A \in \mathbb{C}^{nn}$ is said to be* triangulable *for $(\mu_1, \ldots, \mu_k)$ if there exists a permutation κ of $\langle k\rangle$ such that $A[\mu_i \mid \mu_j] = 0$ if $\kappa(i) > \kappa(j)$.*

Let $A' \in \mathbb{C}^{kk}$ be the matrix derived from A thus: $a'_{ij} = 1$ if $A[\mu_i \mid \mu_j] \neq 0$ and $a'_{ij} = 0$ otherwise. Then A is triangulable for $(\mu_1, \ldots, \mu_k)$ if and only if $P^{-1}A'P$ is triangular, for some permutation matrix P.

4.6) THEOREM *Let $(\mu_1, \ldots, \mu_k)$ be a partition of $\langle n\rangle$. Let $A \in \mathbb{C}^{nn}$ and suppose that $A[\mu_i]$, $i = 1, \ldots, k$ is irreducible. Let $A \in \tau_{\langle n\rangle}$ and suppose that* $\det A > 0$. *Then the following are equivalent*:

4.6.1) $$\det A = \det A[\mu_1] \ldots \det A[\mu_k],$$

4.6.2) *A is triangulable for $(\mu_1, \ldots, \mu_k)$.*

Proof (4.6.2) $\Rightarrow$ (4.6.1). Is obvious.

(4.6.1) $\Rightarrow$ (4.6.2). Suppose A is not triangulable with respect to $(\mu_1, \ldots, \mu_k)$. Then the derived matrix A' is not triangulable by a permutation matrix P. Hence there exists a cycle γ nonzero for A' (Harary, 1969, p. 200 or Engel and Schneider, 1973a). Without loss of generality, we suppose $\gamma = (1, 2, \ldots, m)$, where $m \geqslant 2$. Thus there exist p_i, q_i with $p_i, q_i \in \mu_i$, $i = 1, \ldots, m$ such that $a_{p_i q_{i+1}} \neq 0$. (We use the convention $q_{m+1} = q_1$). Since $A[\mu_i]$ is irreducible, $i = 1, \ldots, m$, there exist distinct $r_i, t_i, \ldots, s_i \in \mu_i$ such that $a_{q_i r_i} a_{r_i t_i} \ldots a_{s_i p_i} \neq 0$. Then

$$\beta = (q_1, r_1, \ldots, s_1, p_1, q_2, \ldots, s_m, p_m)$$

is a nonzero cycle for A, and $\bar{\beta} \cap \mu_i \neq \phi$, $i = 1, \ldots, m$. Hence by Theorem (4.3)(ii), $\det A < \det A[\mu_1] \ldots A[\mu_k]$.

4.7) COROLLARY *Let* $A \in \tau_{\langle n \rangle}$.

i) *Then* $\det A \leqslant a_{11} \ldots a_{nn}$.

ii) *If* $\det A > 0$, *the following are equivalent*:

4.7.1) $$\det A = a_{11} \ldots a_{nn},$$

4.7.2) *For some permutation matrix* P, $P^{-1}AP$ *is triangular.*

A matrix $A \in \mathbb{C}^{nn}$ is said to be *completely reducible* if it is the direct sum of irreducible matrices. The class of all completely reducible matrices is denoted by $\mathscr{C}$ in Engel and Schneider (1973b, Definition 2.14). A matrix $A \in \mathbb{C}^{nn}$ said to be *combinatorially symmetric* if $a_{ij} \neq 0$ implies that $a_{ji} \neq 0$, $1 \leqslant i, j \leqslant n$. It is easy to see that a combinatorially symmetric matrix is completely reducible, and that a Hermitian matrix is combinatorially symmetric. Hence known results for Hermitian matrices are immediate consequences of the corollaries below.

4.8) COROLLARY *Let* A *and* B *be completely reducible. Let* $A \leqslant_\tau B$, *and let* $\det A > 0$. *If* (4.1.1) *holds, then the following are equivalent.*

4.8.1) $$\det A = \det B.$$

4.8.2) *For some nonsingular diagonal matrix* X, $B = X^{-1}AX$.

Proof By the equivalence of (4.1.2) and (4.1.4) and Engel and Schneider (1973b, Theorem 4.1).

4.9) COROLLARY *Let* $(\mu_1, \ldots, \mu_k)$ *be a partition of* $\langle n \rangle$. *Let* $A \in \mathbb{C}^{nn}$ *be completely reducible. If* $A \in \tau_{\langle n \rangle}$, *and* $\det A > 0$ *then the following are equivalent:*

4.9.1) $$\det A = \det A[\mu_1] \ldots \det A[\mu_k],$$

4.9.2) $$A = A[\mu_1] \oplus \ldots \oplus A[\mu_k].$$

Proof Since A is completely reducible, every nonzero nondiagonal element lies on a nonzero cycle. The result follows by the equivalence of (4.3.1) and (4.3.4).

A lower bound for $\det A$ is given by the next theorem.

4.10) THEOREM *Let* $A \in \tau_{\langle n \rangle}$.

i) *Then*

4.10.1) $$\det A \geqslant l(A) \det A(n).$$

ii) *If* $\det A > 0$, *then the following are equivalent*:

4.10.2) $$\det A = l(A) \det A(n),$$

4.10.3) $$A \sim_c A(n) \oplus l(A).$$

Proof i) Let $B = A(n) \oplus l(A)$. It is easy to show that $B \in \tau_{\langle n \rangle}$ and $B \leqslant_\tau A$. Hence by Theorem (3.8) (i), $\det A \geqslant l(A) \det A(n)$.

ii) (4.10.2) $\Rightarrow$ (4.10.3). Since $\det A > 0$, and $l(B) = l(A)$ it follows that $l(B) > 0$. Assume that $\det A = \det B$. Then by Theorem (3.8) (ii), $A \sim_\tau B$.

Further A and B satisfy (4.1.1). Therefore by Theorem (4.1), $A \sim_c B$. (4.10.3) $\Rightarrow$ (4.10.2). Obvious.

4.11) COROLLARY *Let $A \in \tau_{\langle n \rangle}$, and let $l = l(A)$.*

i) *Then* $\det A \geqslant l^n$.

ii) *If* $\det(A) > 0$, *then* $\det A = l^n$ *if and only if there is a permutation matrix P such that $P^T AP = lI + U$, where U is strictly upper triangular.*

Proof i) By induction on n, using (4.10.i).

ii) Suppose $0 < \det A = l^n$. Then by induction on n, using (4.10.ii), we obtain that $A \sim_c lI$. But this implies that $P^T AP = lI + U$, where U is strictly upper triangular, see Harary (1969, Theorem 16.3).

The converse direction is trivial.

5. CONDITIONS FOR EQUALITY IN $\det A = \det B - \det(B - l(A)I)$

5.1) THEOREM *Let $A, B \in \tau_{\langle n \rangle}$ and suppose that* (4.1.1) *holds. If $A \leqslant_\tau B$ and* $\det A > 0$ *then the following are equivalent*:

5.1.1) $$\det A = \det B - \det(B - l(A)\, I) < \det B,$$

5.1.2) $\begin{cases} \text{a)} & A[\mu] \sim_\tau B[\mu], \quad \phi \subset \mu \subset \langle n \rangle, \\ \text{b)} & A \nsim_\tau B, \end{cases}$

5.1.3) $\begin{cases} \text{a)} & A[\mu] \sim_c B[\mu], \quad \phi \subset \mu \subset \langle n \rangle, \\ \text{b)} & \textit{There exists a unique nonzero full cycle } \delta \textit{ for } A \textit{ and, further,} \\ & \Pi_\delta(A) \neq \Pi_\delta(B). \end{cases}$

Proof (5.1.1) $\Leftrightarrow$ (5.1.2). By Theorem (3.8) (iii), $A[\mu] \sim_\tau B[\mu]$, for $\phi \subset \mu \subset \langle n \rangle$, and by Theorem (3.8) (ii), $A \nsim_\tau B$.

(5.1.2) $\Rightarrow$ (5.1.3). By Theorem (4.1), (4.1.3) $\Rightarrow$ (4.1.4), we have $A[\mu] \sim_c B[\mu]$, for $\phi \subset \mu \subset \langle n \rangle$. Further $A \nsim_c B$, since otherwise again by Theorem (4.1), $\det A = \det B$. Hence there exist at least one full cycle γ, such that $\Pi_\gamma(A) \neq \Pi_\gamma(B)$, and by (4.1.1) it follows that $\Pi_\gamma(A) \neq 0$. But by Lemma (2.5), since $A \nsim_c B$, γ is the unique full nonzero cycle for A. (5.1.3) $\Rightarrow$ (5.1.2). By (4.1.3) $\Leftrightarrow$ (4.1.4).

5.2) COROLLARY *Let $(\mu_1, \ldots, \mu_k)$ be a partition of $\langle n \rangle$. Let $A \in \tau_{\langle n \rangle}$.*

i) *Then*

5.2.1) $$\det A \leqslant \det A[\mu_1] \ldots \det A[\mu_k] - \det((A - l(A)I)[\mu_1] \ldots \det((A - l(A)I)[\mu_k])$$

ii) *If* $\det A > 0$, *the following are equivalent*:

5.2.2) $$\det A = \det A[\mu_1] \ldots \det A[\mu_k] - \det((A - l(A)I)[\mu_1]) \ldots \det((A - l(A)I)[\mu_k]) < \det A[\mu_1] \ldots \det A[\mu_k]$$

5.2.3) $\begin{cases} \text{a)} & A[\mu] \sim_\tau (A[\mu_1] \oplus \ldots \oplus A[\mu_k])[\mu], \quad \phi \subset \mu \subset \langle n \rangle \\ \text{b)} & A \nsim_\tau (A[\mu_1] \oplus \ldots \oplus A[\mu_k]) \end{cases}$

5.2.4) $\begin{cases} \text{a)} & A[\mu] \sim_c (A[\mu_1] \oplus \ldots \oplus A[\mu_k])[\mu], \quad \phi \subset \mu \subset \langle n \rangle \\ \text{b)} & \textit{There exists a unique nonzero full cycle } \gamma \textit{ for } A. \end{cases}$

5.2.5) $\begin{cases} \text{a)} & \textit{There exists a unique nonzero full cycle } \gamma \textit{ for } A, \\ \text{b)} & \textit{If } \beta \textit{ is any nonzero cycle for } A, \beta \neq \gamma, \textit{ then, for some } i, 1 \leqslant i \leqslant k, \\ & \beta \leqslant \mu_i. \end{cases}$

Let $(\nu_1, \ldots, \nu_k)$ be a partition of $\langle n \rangle$, and let $G \in \mathbb{C}^{nn}$. We define a condition on $(\nu_1, \ldots, \nu_k)$ and G, in order to investigate the form of a matrix A satisfying (5.2.2).

5.3) *Condition* on $(\nu_1, \ldots, \nu_k)$ and G.

5.3.1) The partition $(\nu_1, \ldots, \nu_k)$ of $\langle n \rangle$ satisfies:

There exist $p_0, \ldots, p_k$, $1 < p_0 < p_1 < \ldots < p_k = n$ such that

$$\nu_t = \{p_{t-1}+1, \ldots, p_t\}, \; t = 1, \ldots, k.$$

5.3.2) The matrix $G \in \mathbb{C}^{nn}$ satisfies:

a) If $j = i+1$ then $g_{ij} \neq 0$. Also $g_{n1} \neq 0$.

b) If $j \neq i+1$, then $g_{ij} \neq 0$ implies $j \leqslant i$ and, $i, j \in \nu_t$ for some t, $1 \leqslant t \leqslant k$.

If $(\nu_1, \ldots, \nu_k)$ and G satisfy (5.3), then clearly G satisfies condition (5.2.5) for $(\nu_1, \ldots, \nu_k)$, since $(1, 2, \ldots, n)$ is the unique nonzero full cycle for G. Note G is of the form

$$G = \begin{bmatrix} G_{11} & G_{12} & 0 & & & 0 \\ 0 & G_{22} & G_{23} & & & 0 \\ & & x & x & & \\ & & & x & x & \\ 0 & & & & x & G_{k-1k} \\ G_{k1} & 0 & 0 & & & G_{kk} \end{bmatrix}$$

where $G[\mu_t] = G_{tt}$ has zeros above the first super diagonal, and G_{tt+1} and G_{k1} have all elements equal to zero, except for the bottom left hand element. We shall show that any matrix A that satisfies the equality (5.2.2) may be put into the above form by simultaneous permutation of rows and columns and changing the indexing of $\mu_1, \ldots, \mu_k$, provided that the $A[\mu_i]$ are irreducible. We denote by P_π the permutation matrix associated with the permutation π on $\langle n \rangle$.

5.4) THEOREM *Let $(\mu_1 \ldots, \mu_k)$ be a partition of $\langle n \rangle$. Let $A \in \mathbb{C}^{nn}$, and suppose that $A[\mu_t]$, $t = 1, \ldots, k$ is irreducible. If $A \in \tau_{\langle n \rangle}$ and* $\det A > 0$, *then the following are equivalent*:

5.4.1) $\det A =$

$$\det A[\mu_1] \ldots \det A[\mu_k] - \det(A - l(A)\,I)[\mu_1] \ldots \det(A - l(A)\,I)[\mu_k]$$
$$< \det A[\mu_1] \ldots \det A[\mu_k],$$

5.4.2) *There exists a permutation* π *of* $\langle n\rangle$ *and a permutation* σ *on* $\langle k\rangle$ *such that if* $\nu_t = \pi(\mu_{\sigma(t)})$, $t = 1, \ldots, k$, *then the partition* $(\nu_1, \ldots, \nu_t)$ *and the matrix* $G = P_\pi^{-1}AP_\pi$ *satisfy* (5.3).

Proof We need only prove (5.4.1) $\Rightarrow$ (5.4.2). By (5.2.5) there exists a unique nonzero full cycle β for A. Suppose $\beta = (i_1, \ldots, i_n)$, where $i_1 \in \mu_p$, $i_n \in \mu_q$ $p \neq q$. Let π be the permutation on $\langle n\rangle$ given by $\pi(i_s) = s$, $s = 1, \ldots, n$ Let $G = P_\pi^{-1}AP_\pi$. Then the unique nonzero full cycle for G is $\gamma = (1, \ldots, n)$.

Let $\nu_1 = \pi(\mu_1)$. Let $p_1 = \{\max i{:}i \in \nu_1\}$. Then $p_1 < n$. We shall prove that $\nu_1 = \{1, \ldots, p_1\}$. For suppose this is false. Then there exists i, $1 < i < p_i$, such that $i \notin \nu_1$. Since $G[\nu_1]$ is irreducible, there is a nonzero cycle $\beta = (1, m_1, \ldots, m_1, p_1, p_1+1, \ldots, n\}$ for G, where $m_1, \ldots, m_r$ belong to ν_1. But β is not-full since $i \notin \bar\beta$, and hence β violates (5.2.5) (b). Thus $\nu_1 = \{1, \ldots, p_1\}$. Suppose $p_1+1 \in \pi(\mu_t)$. Define $\sigma(1) = 1$, $\sigma(2) = t$, and put $\nu_2 = \pi(\mu_t)$. As above, it can be shown that $\nu_2 = \{p_1+1, \ldots, p_2\}$ for some p_2; $p_1 < p_2 \leqslant n$. Continuing thus we obtain a permutation σ of $\{1, \ldots, k\}$ and integers $p_0, p_1, \ldots, p_k$, $1 = p_0 < p_1 < p_2 < p_k = n$ such that $\nu_t = \pi(\mu_{\sigma(t)}) = \{p_{t-1}+1, \ldots, p_t\}$, $t = 1, \ldots, k$. Hence (5.3.1) is satisfied, and $g_{ij} \neq 0$ for $j = i+1$ or $j = n$ and $i = 1$.

We shall now show that (5.3.2)(b) is also satisfied. Suppose $j > i+1$. If $g_{ij} \neq 0$, then $\beta = (1, \ldots, i, j, \ldots, n)$ is a nonzero cycle for G, and $\bar\gamma \cap \nu_1 \neq \phi$, $\bar\gamma \cap \nu_k \neq \phi$. This contradicts (5.2.5). It follows that $g_{ij} = 0$ whenever $j > i+1$. Now suppose $j \leqslant i$, and $j \in \nu_j$, $i \in \nu_t$, $s \neq t$. If $g_{ij} \neq 0$, then $\beta = (j, j+1, \ldots, i)$ is a nonzero cycle for G, with either $1 \in \bar\beta$ or $n \notin \bar\beta$ and $\bar\beta \cap \nu_s \neq \phi$, $\beta \cap \nu_t \neq \phi$. This again contradicts (5.2.5) and so $g_{ij} = 0$.

A matrix C is a *full cycle matrix* if there is a full cycle $(h_1, \ldots, h_n)$ such that $c_{ij} \neq 0$ if and only if $i = h_p$, $j = h_{p+1}$, $1 \leqslant p \leqslant n$, or $i = h_n$, $j = h_1$. The following result is related to a result for M-matrices stated in Engel–Schneider (1973a, Theorem 1 (IV)).

5.5) COROLLARY *Let* $A \in \tau_{\langle n\rangle}$ *and suppose* $\det A > 0$. *If* $l = l(A)$ *then the following are equivalent.*

5.5.1) $$\det A = \prod_{i=1}^{n} a_{ii} - \prod_{i=1}^{n} (a_{ii}-l) \equiv \prod_{i=1}^{n} a_{ii},$$

5.5.2) $$A = D+C$$

where $D = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$ *and* C *is a full cycle matrix.*

6. CONDITIONS ON ω-MATRICES

6.1) DEFINITION *A cycle* γ *on* $\langle n\rangle$ *is called a minimal cycle for* A *if* $\Pi_\gamma(A) \neq 0$ *and, for every cycle* β, *with* $\bar\beta \subset \bar\gamma$, $\Pi_\beta(A) = 0$.

6.2) LEMMA *Let $A \in \mathbb{C}^{nn}$, and suppose that for all μ, $\phi \subset \mu \subset \langle n\rangle$ and ν, $\phi \subset \nu \subset \mu$, $B = A[\mu]$ satisfies*

6.2.1) $$0 \leqslant \det B \leqslant \det B[\nu] \det B(\nu).$$

6.2.2) *Then, for every cycle γ minimal for A, $(-1)^{|\gamma|-1}\Pi_\gamma(A) < 0$.*

Proof Let γ by a minimal cycle for A. Since (6.2.1) holds for $A[\bar{\gamma}]$, $\prod_{i\in\bar{\gamma}} a_{ii} \geqslant \det A[\bar{\gamma}]$. But $\det A[\bar{\gamma}] = \prod_{i\in\bar{\gamma}} a_{ii} + (-1)^{|\gamma|-1}\Pi_\gamma(A)$.

We define Γ_μ to be the set of cycles γ with $\bar{\gamma} = \bar{\mu}$.

6.3) LEMMA *Let $A \in \mathbb{C}^{nn}$, then the following are equivalent*

6.3.1) $$\det A[\mu] \in \mathbb{R}, \quad \textit{for} \quad \phi \subset \mu \subseteq \langle n\rangle,$$

6.3.2) $$\begin{cases} \text{a)} & a_{ii} \in \mathbb{R},\ i = 1, \ldots, n, \\ \text{b)} & \textit{For all } \mu,\ \phi \subset \mu \subseteq \langle n\rangle,\ \sum_{\gamma\in\Gamma_\mu} \Pi_\gamma(A) \in \mathbb{R}. \end{cases}$$

Proof (6.3.1) $\Rightarrow$ (6.3.2). By induction on n. The result is clearly true if $n = 1$. So let $n > 1$, and suppose $\sum_{\gamma\in\Gamma_\mu} \Pi_\gamma(A) \in \mathbb{R}$, for $\phi \subset \mu \subset \langle n\rangle$. Since [see Engel (1973) or Maybee–Quirk (1969)]

6.3.3) $$\det A = a_{11}\det A(1) + \sum_{\{1\}\subset\mu\subseteq\langle n\rangle} (-1)^{|\mu|-1}\det A(\mu)\Big(\sum_{\gamma\in\Gamma_\mu}\Pi_\gamma(A)\Big),$$

it follows that $\sum_{\gamma\in\Gamma\langle n\rangle} \Pi_\gamma(A) \in \mathbb{R}$.

(6.3.2) $\Rightarrow$ (6.3.1). By induction on n. The result is true for $n = 1$, and the inductive step follows immediately from (6.3.3).

Let $A \in \omega_{\langle n\rangle}$. Since $A + tI \in \tau_{\langle n\rangle}$ for sufficiently large t, by (6.2),

$$(-1)^{|\gamma|-1}\Pi_\gamma(A) < 0$$

for each minimal cycle γ, and by (6.3), $\sum_{\gamma\in\Gamma_\mu} \Pi_\gamma(A) \in \mathbb{R}$, for each

$$\mu,\ \phi \subset \mu \subseteq \langle n\rangle.$$

If $A \in \mathbb{C}^{nn}$, the above conditions do not imply that $A \in \omega_{\langle n\rangle}$. Examples are given in (7.2) and (7.3). However, if we replace (6.3.2) by a stronger condition we obtain:

6.4) THEOREM *Let $A \in \mathbb{C}^{nn}$. If*

6.4.1) $$\begin{cases} \text{a)} & a_{ii} \in \mathbb{R}, \\ \text{b)} & (-1)^{|\mu|+1}\sum_{\gamma\in\Gamma_\mu}\Pi_\gamma(A) \leqslant 0, \quad \textit{for all} \quad \mu,\ \mu \subseteq \langle n\rangle,\ |\mu| > 1, \end{cases}$$

then $A \in \omega_{\langle n\rangle}$.

Proof The proof is by induction. The result is true if $n = 1$. So let $n > 1$ and suppose that $A[\mu] \in \omega_\mu$, if $\phi \subset \mu \subset \langle n\rangle$. We note

$$\det(A+tI) = (a_{11}+t)\det(A+tI)(1)$$
$$+ \sum_{\{1\}\subset\mu\subseteq\langle n\rangle} (-1)^{|\mu|-1} \det(A+tI))(\mu).(\sum_{\gamma\in\Gamma_\mu} \Pi_\gamma(A))$$

By Lemma (6.3), $\det A \in \mathbb{R}$, and so, for large $t > 0$, $\det(A+tI) > 0$. Let $s = -l(A(1))$. Then by inductive assumption, $\det(A+sI)(\mu) \geqslant 0$, for $\{1\} \subset \mu \subseteq \langle n\rangle$. Hence $\det(A+sI) \leqslant 0$. It follows that there is an t, $t \geqslant -l(A(1))$ such that $\det(A+tI) = 0$.

The condition (6.4.1) is not necessary for $A \in \omega_{\langle n\rangle}$, see (7.4).

Remark Suppose that $B \in \mathbb{R}^{nn}$ is nonnegative (viz. $b_{ij} \geqslant 0$, $i, j = 1, \ldots, n$). Then $A = -B$ obviously satisfies (6.4.1). Thus Theorem (6.4) implies a weak form of the Perron–Frobenius theorem: The matrix B has a real eigenvalue $m = -l(A)$, and $m \geqslant 0$ since $l(A) \leqslant \min\{a_{ii}: i \in \langle n\rangle\}$. Further, Theorems (3.6) and (6.4) combined furnish a proof that $A \in \tau_{\langle n\rangle}$ if A is an M-matrix as originally defined by Ostrowski (1937), viz. $a_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, \ldots, n$, and $\det A[\mu] \geqslant 0$, for all μ, $\phi \subset \mu \subseteq \langle n\rangle$.

Our last two theorems will characterize matrices which are c-equivalent to matrices with nonpositive off-diagonal elements.

6.5) LEMMA *Let $B \in \mathbb{C}^{nn}$ and suppose that $\Pi_\gamma(B) \in \mathbb{R}$, for all cycles γ. Then the following are equivalent*:

6.5.1) $\quad \Pi_\gamma(B) \geqslant 0$, *for all cycles* γ,

6.5.2) $\quad \sum_{\gamma\in\Gamma_\mu} \Pi_\gamma(B) \geqslant 0$, *for all* μ, $\phi \subset \mu \subseteq \langle n\rangle$, $|\mu| > 1$.

Proof (6.5.1) $\Rightarrow$ (6.5.2). Trivial.
(6.5.2) $\Rightarrow$ (6.5.1). The result is vacuously satisfied if $n = 1$.

Suppose that $n \geqslant 2$ and that the implication holds for $B[\mu]$, $\phi \subset \mu \subset \langle n\rangle$. If $\Pi_\gamma(B) = 0$, for all full cycles γ, then there is nothing to prove. So suppose there exists a nonzero full cycle for B. Since $\Pi_\gamma(B) \in \mathbb{R}$, for $\gamma \in \Gamma_{\langle n\rangle}$, it follows from (6.5.2) that there is a full cycle β such that $\Pi_\beta(B) > 0$. Let γ be any full cycle, $\gamma \neq \beta$. They by Lemma (2.1) and our inductive hypothesis, $\Pi_\beta(B)\,\Pi_\gamma(B) \geqslant 0$. Hence $\Pi_\gamma(B) \geqslant 0$.

In Lemma (6.5) we cannot drop the hypothesis that $\Pi_\gamma(B) \in \mathbb{R}$. For example let $A \in \mathbb{C}^{33}$ be a Hermitian matrix such that $a_{12} = a_{23} = a_{31} = e^{\pi i/6}$.

6.6) THEOREM *Let $A \in \mathbb{C}^{nn}$ and suppose that $\Pi_\gamma(A) \in \mathbb{R}$ for all cycles γ. Then condition* (6.4.1) *is equivalent to*:

6.6.1) *There exists an $M \in \mathbb{R}^{nn}$ such that $m_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, \ldots, n$, and $M \sim_c A$.*

Proof (6.6.1) $\Rightarrow$ (6.4.1). Put $B = -A$ in Lemma (6.5).

(6.4.1) $\Rightarrow$ (6.6.1). Define $M \in \mathbb{R}^{nn}$ by

$$m_{ii} = a_{ii}, i = 1, \ldots, n \quad \text{and} \quad m_{ij} = -|a_{ij}|, \qquad i \neq j, i, j = 1, \ldots, n.$$

Then, by Lemma (6.5), $\Pi_\gamma(-A) \geqslant 0$ for all cycles γ. Since also $\Pi_\gamma(-M) \geqslant 0$, we deduce (6.6.1).

6.7) COROLLARY *Let $A \in \mathbb{R}^{nn}$ and suppose there is an M-matrix N such that* $\det A[\mu] = \det N[\mu]$, *for all μ, $\phi \subset \mu \subseteq \langle n \rangle$. Then A is c-equivalent to an M-matrix.*

Proof By Theorem (6.6), there is an $M \in \mathbb{R}^{nn}$ with $m_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, \ldots, n$ such that $M \sim_c A$. But, since $\det M[\mu] = \det N[\mu]$, for $\phi \subset \mu \subseteq \langle n \rangle$, it follows that M is an M-matrix.

In fact, the matrix M can be chosen as $m_{ii} = a_{ii}$, $m_{ij} = -|a_{ij}|$, $i \neq j$. A simple example (due to W. Hurewicz) of a matrix $A \in \mathbb{R}^{44}$ such that $\det A[\mu] \geqslant 0$, $\phi \subset \mu \subseteq \langle n \rangle$, but A is not c-equivalent to an M-matrix is given in Samuelson (1944). Another example is the matrix $A_2 \in \mathbb{R}^{33}$ in (7.2) or

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6.8) THEOREM *Let $A \in \mathbb{C}^{nn}$. The following are equivalent:*

6.6.1) *There exists $M \in \mathbb{R}^{nn}$ such that $m_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, \ldots, n$, and $A \sim_c M$,*

6.8.1) *If $B \in \mathbb{C}^{nn}$ satisfies $b_{ij} = a_{ij}$ or $b_{ij} = 0$, $1 \leqslant i, j \leqslant n$, then $B \in \omega_{\langle n \rangle}$.*

Proof (6.6.1) $\Rightarrow$ (6.8.1). Clearly $b_{ii} \in \mathbb{R}$, $i = 1, \ldots, n$. Clearly

$$(-1)^{|\gamma|+1}\,\Pi_\gamma(A) = (-1)^{|\gamma|+1}\,\Pi_\gamma(M) \leqslant 0.$$

Hence also

$$(-1)^{|\gamma|+1}\,\Pi_\gamma(B) \leqslant 0.$$

Thus $B \in \omega_{\langle n \rangle}$, by Theorem (6.4).

(6.8.1) $\Rightarrow$ (6.6.1). Let $\gamma = (p_1, \ldots, p_k)$ be a nonzero cycle for A. Define B by $b_{ij} = a_{ij}$ if for some q, $1 \leqslant q < k$, $i = p_q$ and $j = p_{q+1}$, or $i = p_k$ and $j = p_1$ and $b_{ij} = 0$, otherwise. Thus γ is a minimal cycle for B, and since $B \in \omega_{\langle n \rangle}$, it follows by Lemma (6.2) that

$$(-1)^{|\gamma|-1}\,\Pi_\gamma(A) = (-1)^{|\gamma|-1}\,\Pi_\gamma(B) < 0.$$

Hence for all cycles γ, $(-1)^{|\gamma|-1}\,\Pi_\gamma(A) \leqslant 0$. Thus $A \sim_c M$, where $m_{ii} = a_{ii} \in \mathbb{R}$ and $m_{ij} = -|a_{ij}| \leqslant 0$, for $i \neq j$, $1 \leqslant i, j \leqslant n$.

7. EXAMPLES AND OPEN QUESTIONS

7.1) There is an $A_1 \in \mathbb{C}^{nn}$ such that $A_1 \in \tau_{\langle n \rangle}$, $-A_1 \in \omega_{\langle n \rangle}$ but A_1 is not weakly sign-symmetric in the sense of Kotelyanski (1953) and Carlson (1967). Hence, by Carlson (1967, Theorem 1), the matrix A_1 does not satisfy the generalized

Hadamard inequalities discussed by Carlson (1967) [see formula (2)]. These generalized inequalities were proved for positive definite matrices by Krull (1958). For totally nonnegative matrices see Gantmacher–Krein (1960, p. 111), and for M-matrices by Fan (1960), formula (12), see also Fan (1968). Let

$$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then

$$\operatorname{Spec} A_1[23] = \{2\},$$
$$\operatorname{Spec} A_1[12] = \operatorname{Spec} A_1[13] = \{1, 3\},$$

and

$$\operatorname{Spec} A_1 = \{1, \tfrac{1}{2}(5 \pm \sqrt{5})\}.$$

Observe that

$$9 = \det A_1[12] \det A_1[13] < \det A_1 \det A_1[1] = 10,$$

and that $\det A_1[12 \mid 13] \det A_1[13 \mid 12] < 0$. The matrix A_1 is also an example of a matrix in $\tau_{\langle n\rangle}$ which is not c-equivalent to a Hermitian, totally nonnegative or M-matrix.

7.2) There is an $A_2 \in \mathbb{C}^{33}$ such that A_2 is positive sign-symmetric, [Carlson (1974)], but $A \notin \omega_{\langle n\rangle}$. For, let

$$A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 1 & 5 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Observe that $\operatorname{Spec} A_2[\mu] \cap \mathbb{R} \neq \phi$, for $\phi \subset \mu \subseteq \langle 3\rangle$, but

$$\operatorname{Spec} A_2[12] = \{5 \pm \sqrt{2}\},$$

and

$$\operatorname{Spec} A_2 = \{8, \tfrac{1}{2}(7 \pm \sqrt{3}i)\}.$$

Hence $l(A_2) > l(A_2[12])$.

Another example is the matrix A in Carlson (1974).

7.3) The matrix A_2 in (7.2) is also an example of a matrix which satisfies the conditions (6.2.2), (6.3.1) and (6.3.2), but which is not in $\omega_{\langle n\rangle}$. Another example (which is not weakly sign symmetric is)

$$A_3 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix} = A_2 - 3I.$$

7.4) There is an $A_4 \in \omega_{\langle n\rangle}$ which does not satisfy condition (6.4.1):

$$A_4 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Then $\sum_{|\gamma| = \langle 3 \rangle} \Pi_\gamma(A_4) > 0$. Clearly, A_4 is Hermitian.

7.5) *Open Questions*

i) We do not know how to characterize $A \in \omega_{\langle n \rangle}$ in terms of the cyclic products $\Pi_\gamma(A)$.

ii) We do not know if every $A \in \tau_{\langle n \rangle}$ is nonnegative semistable, viz. has Re $\lambda \geqslant 0$ for $\lambda \in$ Spec A, cf. Johnson (1974) for a related question. For $n \leqslant 3$ the result is a consequence of Carlson (1974, Theorem 3).

iii) If $A \in \tau_{\langle n \rangle}$ and $D = \mathrm{diag}(d_1, \ldots, d_n)$, where $d_i \geqslant 0$, $i = 1, \ldots, n$, is $(A+D) \in \tau_{\langle n \rangle}$?

iv) Let $A \in \omega_{\langle n \rangle}$ and suppose that $\det(A[\mu_i]) > 0$ where $\mu_i = \{1, \ldots, i\}$. Is $A \in \tau_{\langle n \rangle}$ [cf. Theorem (3.6)]?

References

Beckenbach, E. F. and Bellman, R. [1961], Inequalities, Ergeb. Math. and Grenzgeb., Springer-Verlag, Berlin (1961).

Carlson, D. [1967], Weakly sign-symmetric matrices and some determinental inequalities, *Colloquium math.* **17** (1967), 123–129.

Carlson, D. [1974], A class of positive stable matrices, *J. Res. natn. Bur. Stand.* **78B** (1974), 1–2.

Cauchy, A. [1829], Sur l'equation a l'aide de laquelle on détermine les inegalités seculaires de mouvement des planètes, *Oeuv. comp.* **9** (2) (1829), 174–195.

Engel, G. M. [1973], Regular Equimodular sets of matrices for generalized matrix functions, *Linear Alg. Apps.* **7** (1973), 243–274.

Engel, G. M. and Schneider, H. [1973a], Inequalities for determinants and permanents, *Linear and Multilinear Algebra*, **1** (1973), 187–201.

Engel, G. M. and Schneider, H. [1973b], Cyclic and diagonal products on a matrix, *Linear Algebra Apps.* **7** (1973), 301–335.

Engel, G. M. and Schneider, H. [1975], Diagonal similarity and equivalence for matrices over groups with 0, *Czech. Math. J.* 25 (100) (1975), 389–403.

Fan, K. [1960], Note on M-matrices, *Quart. J. Math.* **11** (2) (1960), 43–49.

Fan, K. [1966], Some matrix inequalities, *Abh. Math. Sem. Univ. Hamburg*, **29** (1966), 185–196.

Fan, K. [1967], Subadditive functions on a distributive lattice and an extension of the Szasz inequality, *J. Math. Anal. Appl.* **18** (1967), 262–268.

Fan, K. [1968], An inequality for subadditive functions on a distributive lattice, with applications to determinantal inequalities, *Lin. Alg. Appl.* **1** (1968), 33–38.

Fiedler, M. and Ptak, V. [1962], On matrices with non-positive off-diagonal elements and positive principal minors, *Czech. Math. J.* **12** (87) (1962), 382–400.

Fischer, E. [1908], Uber den Hadamardschen Determinantensatz, *Arch. Math. Phys.* **13** (3) (1908), 32–40.

Frobenius, G. [1908], Über Matrizen aus positiven Elementen, *Sber. preuss. Akad. Wiss.* (1908), 471–476, and *Gesch. Abh.*, Springer-Verlag, Berlin (1968), Vol. 3, 404–409.

Gantmacher, F. R. [1959], *Matrix Theory*, Chelsea Pub. Co., New York (1959).

Gantmacher, F. R. and Krein, M. G. [1935], Uber eine spezielle Klasse von Determinanten, die mit Kelloggschen Integralkernen zusamnenhangen, *Mat. Sb.* **42** (1935), 501–508.

Gantmacher, F. R. and Krein, M. G. [1937], Sur les matrices oscillatoires et completement non-negative, *Comp. Math.* **4** (1937), 445–476.

Gantmacher, F. R. and Krein, M. G. [1960], *Oszillations Matrizen, Oszillationskeme, und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin (1960).

Hadamard, H. [1893], Resolution d'une question relative aux determinants, *Bull. Sci. Math.* **17** (2) (1893), 240–248.
Harary, F. [1969], *Graph Theory*, Addison-Wesley Pub. Co., Reading, Mass., 1969.
Johnson, C. R. [1974], A sufficient condition for matrix stability, *J. Res. natn. Bur. Stand.* **78B** (1974), 103–104.
Karlin, S. [1968], *Total Positivity*, Vol. I, Stanford U.P., Stanford, Cal. (1968).
Kotelyanskii, D. M. [1953], A property of sign symmetric matrices, *Amer. Math. Soc. Transl.*, Ser. 2, **27** (1963), 19–24, orig. in *Usp. mat. nauk* (N.S.) **8** (1953), 163–167.
Krull, W. [1958], Uber eine Verallgemeinerung der Hadamardschen Ungleichung, *Arch. Math.* **91** (1958), 42–45.
Marcus, M. and Minc, H. [1964], *A Survey of Matrix Theory and Matrix Inequalities*, Allyn and Bacon, Boston (1964).
Maybee, J. and Quirk, J. [1969], Qualitative problems in matrix theory, *SIAM Rev.* **11** (1969), 30–51.
Ostrowski, A. M. [1937], Uber die Determinanten mit uberwiegender Hauptdiagonale, *Comm. Math. Helvetici*, **10** (1937), 69–96.
Samuelson, P. A. [1944], The relation between Hicksian stability and true dynamic stability, *Econometrica*, **13** (1944), 256–257.
Taussky, O. [1958], Research problem, *Bull. American Math. Soc.* **64** (1958), 124.