51

On the Geometry of Dual Pairs

By B. E. Cain, B. D. Saunders,* and Hans Schneider**

The set of dual pairs of any norm ν equivalent to a Hilbert norm is shown to be naturally homeomorphic to the sphere of the Hilbert space. The proof begins with a known result showing the representability of every vector as a sum of two orthogonal vectors, one coming from a cone and the other from its dual (a generalization of representation by orthogonal subspaces). The key theorem, showing that every non-zero vector has a positive multiple which is the sum of two ν-dual vectors, follows from this and in turn provides the required homeomorphism. One consequence of this topological equivalence is the arc-connectedness of the numerical range determined by ν.

I. Introduction

If ν is a norm equivalent to the Hilbert norm on a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$, then the numerical range with respect to ν of an operator A is the image under the continuous mapping $(x,y)\to\langle Ax,y\rangle$ of the set of dual pairs,

$$\Pi(\nu)=\{(x,y)\in\mathcal{H}\times\mathcal{H}:\langle x,y\rangle=\nu(x)\nu^*(y)=1\}.$$

Here ν^* is the norm dual to ν (definitions follow). The same numerical range results when $\Pi(\nu)$ is replaced by its subset,

$$\Pi_0(\nu)=\{(x,y)\in\mathcal{H}\times\mathcal{H}:\langle x,y\rangle=\nu(x)=\nu^*(y)=1\}.$$

We develop natural homeomorphisms between $\Pi_0(\nu)$ and the unit sphere S of $\mathcal{H}$, and between $\Pi(\nu)$ and the cylinder $S\times\mathbf{R}$; cf. Theorem 7. These sets are

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arc-connected when $\dim_{\mathbf{R}}\mathcal{H}>1$; consequently, the numerical range is arcwise connected. This is a partial solution to the following question: Is the numerical range of a bounded linear operator on a normed linear space arcwise connected (Bonsall and Duncan [1], p. 129)?

Our main tool is Theorem 6, which is a direct application of the orthogonal decomposition of a Hilbert space with respect to dual (polar) cones. This decomposition is a generalization of the classical decomposition with respect to closed orthogonal subspaces.[1]

Several duality notions play a role in this paper. First we have the dual norms (more generally, dual Minkowski functionals) and their associated dual pairs used in defining the numerical range. This duality is in correspondence with the notion of dual or polar convex bodies. Secondly, as mentioned above, dual cones are used. We employ a notion of dual set which encompasses both the cone duality and the norm ball duality.

We are indebted to Chr. Zenger for pointing out a connection between our work and the theory of monotone sets (e.g., Brezis [2]). This theory employs another duality concept, that of conjugate convex functions (cf. Fenchel [3], Moreau [8], Rockafellar [9]), and, as indicated in the sequel, provides alternate proofs of some of our results. However, we are able to give particularly direct and geometric arguments because we restrict our attention to positive homogeneous convex functions (Fenchel et. al. treat a larger class of convex functions).

The numerical range is most useful when $\mathcal{H}$ is a complex space. Then, for example, its closure contains the spectrum. However, most of the arguments used here are real in spirit. For clarity, then, we use real Hilbert spaces in Sec. II, and only bring in the complex numbers in Sec. III, where the numerical range is discussed.

II. Decompositions, homeomorphisms

Let $(\mathcal{H},\langle\cdot,\cdot\rangle)$ be a real Hilbert space. We denote by $\mathcal{P}$ the set of functions $\rho:\mathcal{H}\to[0,\infty]$ which satisfy

(1) $\rho(\lambda x+(1-\lambda)y)\leqslant\lambda\rho(x)+(1-\lambda)\rho(y)$ for $x,y\in\mathcal{H}$, $\lambda\in(0,1)$ (convexity),
(2) $\rho(\lambda x)=\lambda\rho(x)$ for $\lambda>0$ (positive homogeneity),
(3) $\{x:\rho(x)\leqslant\lambda\}$ is closed for all $\lambda\in[0,\infty]$ (lower semicontinuity).
(4) $\rho(0)=0$ (hence $\rho\not\equiv\infty$).

Let $\mathcal{B}$ denote the collection of closed, convex subsets of $\mathcal{H}$ containing the origin. For $\rho\in\mathcal{P}$, the set $B_\rho\equiv\{x:\rho(x)\leqslant 1\}$ is in $\mathcal{B}$, and conversely the Minkowski functional ρ_B of a set $B\in\mathcal{B}$ is in $\mathcal{P}:\rho_B(x)\equiv\inf\{t>0:x\in tB\}$ (by convention $\inf\varnothing=\infty$). This correspondence is one-to-one and onto. It commutes with the taking of duals, i.e., for

$$B^*\equiv\{y\in\mathcal{H}:\langle x,y\rangle\leqslant 1\quad\text{for all}\quad x\in B\},$$

and

$$\rho^*(y)\equiv\inf\{t>0:\langle y,x\rangle\leqslant t\rho(x)\quad\text{for all}\quad x\in\mathcal{H}\},$$

[1]Further applications of this decomposition may be found in [10] and [11]

we have

LEMMA 1. $\rho_{B^*}=(\rho_B)^*$ *and* $B_{\rho^*}=(B_\rho)^*$.

Proof:

$$\begin{aligned}\rho_{B^*}(y)&=\inf\{t>0:y\in tB^*\}\\&=\inf\{t>0:\langle y,x\rangle\leqslant t \quad \text{for all} \quad x\in B\}\\&=\inf\{t>0:\langle y,z\rangle\leqslant t\rho_B(z) \quad \text{for all} \quad z\in\mathcal{H}\}\\&=(\rho_B)^*(y).\end{aligned}$$

(Note: $\rho_B(z)=0$ implies $\{sz:s>0\}\subset B$; so $\langle y,x\rangle\leqslant t\ \forall x\in\mathcal{H}$ implies $\langle y,z\rangle\leqslant 0 = t\rho_B(z)$.)

The commutativity which holds for the correspondence must also hold for its inverse; so $B_{\rho^*}=(B_\rho)^*$. ■

Associated with dual functionals is a Schwarz inequality $\langle y,x\rangle\leqslant\rho(x)\rho^*(y)$, which follows immediately from the definition of ρ^* except when $0\cdot\infty$ occurs on the right-hand side. In the sequel we apply the Schwarz inequality only when we know $\rho(x)$ and $\rho^*(y)$ are both finite.

It is readily verified for A, $B\in\mathcal{B}$ that $A^*\in\mathcal{B}$, that $A=A^{**}$, and that $A\subseteq B$ implies $B^*\subseteq A^*$. Similar facts may be established for the elements of $\mathcal{P}$ by using the correspondence (which is order-reversing).

We single out certain elements of $\mathcal{P}$ and $\mathcal{B}$. *Quasinorms* are those $\rho\in\mathcal{P}$ for which $\rho(x)>0$ for all $x\neq 0$; *seminorms* satisfy $\rho(x)<\infty$ for all $x\in\mathcal{H}$. *Norms* are those $\rho\in\mathcal{P}$ which are both quasinorms and seminorms. *Homogeneous norms* are norms which satisfy $\rho(tx)=|t|\rho(x)$ for all $t\in\mathbf{R}$, $x\in\mathcal{H}$. We call the sets $B\in\mathcal{B}$ for which $tB\subseteq B$ for all $t\geqslant 0$ *cones*; closed subspaces of $\mathcal{H}$, for example, are cones.

We remark that when $K\in\mathcal{B}$ is a cone, $K^*=\{y\in\mathcal{H}:\langle x,y\rangle\leqslant 0$ for all $x\in K\}$, since $\langle x,y\rangle>0$ implies $\langle tx,y\rangle>1$ for large enough t. Also, when $\rho\in\mathcal{P}$ is a homogeneous norm,

$$\rho^*(y)=\sup_{x\neq 0}\left\{\frac{|\langle x,y\rangle|}{\rho(x)}\right\}.$$

Thus the star operations we have defined coincide with the more customary ones in these two situations. In particular, when K is a cone, K^* is also.

The following lemma enables us to obtain information about dual functions in $\mathcal{P}$ from information about dual cones. Here $\mathcal{H}\oplus\mathbf{R}$ is a Hilbert space under the inner product $\langle x\oplus r, y\oplus s\rangle=\langle x,y\rangle+rs$ for x, $y\in\mathcal{H}$; r, $s\in\mathbf{R}$. For $B\in\mathcal{B}$ and $\rho=\rho_B$,

$$K^+(B)=\{x\oplus r:r\geqslant\rho(x)\}$$

and

$$K^-(B)=\{x\oplus r:r\leqslant-\rho(x)\}$$

are cones of $\mathcal{H}\oplus\mathbf{R}$.

LEMMA 2. *If $B\in\mathfrak{B}$, then $K^+(B)^*=K^-(B^*)$.*

Proof: The lower half space $K^-(\mathcal{H})=\{x\oplus r: x\in\mathcal{H},\ r\leqslant 0\}$ contains $K^-(B^*)$. Since $0\in B$, $K^+(\{0\})\subseteq K^+(B)$, so that $K^+(B)^*\subseteq K^+(\{0\})^*=K^-(\mathcal{H})$. Now since both $K^-(B^*)$ and $K^+(B)^*$ are cones in the lower half space, it suffices to show both have the same intersection with $\mathcal{H}\oplus\{-1\}$. Note that $\{x\oplus 1: x\in B\}$ generates $K^+(B)$, and thus

$$\begin{aligned} y\oplus -1\in K^-(B^*) \quad &\Leftrightarrow \quad y\in B^* \\ &\Leftrightarrow \quad \langle x,y\rangle\leqslant 1 \quad \text{for all} \quad x\in B \\ &\Leftrightarrow \quad \langle x\oplus 1, y\oplus -1\rangle\leqslant 0 \quad \text{for all} \quad x\in B \\ &\Leftrightarrow \quad y\oplus -1\in K^+(B)^*. \quad \blacksquare \end{aligned}$$

The sets of dual pairs $\Pi(\nu)$ and $\Pi_0(\nu)$ have been defined for norms ν. These definitions apply equally well for arbitrary $\rho\in\mathcal{P}$. We need, however, yet another set of pairs, namely $M(\rho)=\{(x,y)\in\mathcal{H}\times\mathcal{H}: \langle x,y\rangle=\rho(x)\rho^*(y)$ and $\rho(x)=\rho^*(y)\}$. Note that $\Pi_0(\rho)=\Pi(\rho)\cap M(\rho)$.

LEMMA 3. *If $\rho\in\mathcal{P}$, then $M(\rho)$ is homeomorphic (with respect to the topology inherited from the product topology on $\mathcal{H}\times\mathcal{H}$) to its image under the addition map, $(x,y)\overset{\alpha}{\rightarrow}x+y$.*

Proof: Clearly α is continuous. If (x,y) and (x',y') are points of $M(\rho)$, then by the Schwarz inequality

$$\begin{aligned} \langle x-x',y-y'\rangle &= \langle x,y\rangle-\langle x,y'\rangle-\langle x',y\rangle+\langle x',y'\rangle \\ &\geqslant \rho(x)\rho^*(y)-\rho(x)\rho^*(y')-\rho(x')\rho^*(y)+\rho(x')\rho^*(y') \\ &= \left[\rho(x)-\rho(x')\right]\left[\rho^*(y)-\rho^*(y')\right] \\ &= \left[\rho(x)-\rho(x')\right]^2 \\ &\geqslant 0. \end{aligned}$$

Thus for $z=x+y$ and $z'=x'+y'$,

$$\begin{aligned} \|z-z'\|^2 &= \|x-x'\|^2+\|y-y'\|^2+2\langle x-x',y-y'\rangle \\ &\geqslant \|x-x'\|^2+\|y-y'\|^2. \end{aligned}$$

Then α is 1-1, since $z=z'$ forces $x=x'$ and $y=y'$. Further, α^{-1} is continuous, because z near z' forces x near x' and y near y'. ■

The first part of the above proof shows that $M(\rho)$ is a monotone set in the sense of Minty [6]. The second part gives a direct proof that the lemma holds for monotone sets; cf. Minty [5], Theorem 3.

When K is a cone, $(x,y)\in M(\rho_K)$ if and only if $x\in K$, $y\in K^*$ and $\langle x,y\rangle=0$. This observation together with Lemma 3 proves all but the existence assertion of the following orthogonal decomposition lemma. This lemma was proven by Moreau [7] and independently rediscovered by Schneider and Vidyasagar [12].

LEMMA 4. *If K is a cone in $\mathcal{H}$, then for each $z \in \mathcal{H}$ there is a unique pair x, $y \in \mathcal{H}$ such that*

(i) $z = x + y$,
(ii) $x \in K$, $y \in K^*$, *and*
(iii) $\langle x,y \rangle = 0$.

Moreover, x is the nearest point of K to z, and y is the nearest point of K^* to z. Also x and y are continuous functions of z.

Now that we have some information about dual cones, we can use Lemma 1 to obtain information about dual pairs.

THEOREM 5. *If $\rho \in \mathcal{P}$, then $M(\rho)$ is homeomorphic to $\mathcal{H}$ via the addition map, $(x,y) \overset{\alpha}{\to} x + y$.*

Proof: In view of Lemma 3, it suffices to show that every point z of $\mathcal{H}$ may be written as $z = x + y$, where $(x,y) \in M(\rho)$. This is achieved by applying Lemma 4 to $z \oplus 0$ in $\mathcal{H} \oplus \mathbf{R}$ with respect to the cone $K^+(B)$, where $B = B_\rho$. Since by Lemma 2, $K^+(B)^* = K^-(B^*)$, the decomposition is $z \oplus 0 = x \oplus r + y \oplus -r$, where $\langle x,y \rangle = r^2$ and $\rho(x) \leqslant r$, $\rho^*(y) \leqslant r$. But, by the Schwarz inequality, $r^2 = \langle x,y \rangle \leqslant \rho(x)\rho^*(y) \leqslant r^2$, which forces $\rho(x) = r = \rho^*(y)$. Thus $(x,y) \in M(\rho)$ as required. ■

Theorem 5 may be obtained by applying Proposition 4.a of Moreau [8] to the function $\frac{1}{2}\rho^2$. Another approach is via a theorem of Rockafellar [9]. He proves that a set M in $\mathcal{H} \times \mathcal{H}$ is the graph of the subgradient of a l.s.c. proper convex function if and only if M is a maximal cyclically monotone set. Then Minty [5] has shown that maximal monotone sets (which include all maximal cyclically monotone sets in the Hilbert-space setting) are homeomorphic to $\mathcal{H}$ under the addition map α. $M(\rho)$ is the graph of the subgradient of $\frac{1}{2}\rho^2$, so that Theorem 5 follows.

We have included Theorem 5 mainly to point up the relationship of our work to the monotone-function theory. Theorem 5 is not used in the sequel, but could be used to prove Theorem 7(a) and hence Theorem 9. On the other hand, Theorems 6 and 7(b) do not appear to follow from monotone-function theory. Indeed, $\Pi(\rho)$ is not a monotone set, though it is a monotone set in the sense of Zenger [13].

When ρ is a norm, $x \oplus r$ is on the algebraic boundary of $K^+(B_\rho)$ just in case $r = \rho(x)$. This observation facilitates a more complete exploitation of Lemma 4.

THEOREM 6. *If $\rho \in \mathcal{P}$ and ρ or ρ^* is a norm, then for $z \in \mathcal{H}$ and $\delta \in \mathbf{R}$ the following statements are equivalent*:

(1) $-\rho^*(z) \leqslant \delta \leqslant \rho(z)$.
(2) *There exists a pair $(x,y) \in \mathcal{H} \times \mathcal{H}$ such that*

   (i) $z = x + y$,
   (ii) $\langle x,y \rangle = \rho(x)\rho^*(y)$,
   (iii) $\delta = \rho(x) - \rho^*(y)$.

When (1) *holds, the pair (x,y) of* (2) *is unique. Furthermore, x, y, $\rho(x)$, and $\rho^*(y)$ are continuous functions of z and δ.*

Proof: Without loss of generality, suppose ρ is a norm. To show (1) implies

(2), we apply Lemma 4 to the cone $K = K^+(B_\rho)$, obtaining $z\oplus\delta = x\oplus r + y\oplus -s$ with $r \geqslant \rho(x)$, $s \geqslant \rho^*(y)$, and $\langle x,y\rangle = rs$. Since $\delta \leqslant \rho(z)$, $z\oplus\delta$ is not in the interior of K, whence $x\oplus r$, the nearest point of K, is on the boundary, i.e., $r=\rho(x)$. If $-\rho^*(z)=\delta$, then $z\oplus\delta \in K^*$, $x=0$, and (2) follows. If $-\rho^*(z)<\delta$, then $z\oplus\delta \not\in K^*$, $x\neq 0$, hence $\rho(x)\neq 0$. Then $rs=\langle x,y\rangle \leqslant \rho(x)\rho^*(y)\leqslant rs$ implies $\rho^*(y)=s$, so that (2) follows.

To prove the converse, consider a pair (x,y) satisfying (2). Observe that $x\oplus\rho(x)+y\oplus-\rho^*(y)=z\oplus\delta$ is the decomposition of $z\oplus\delta$ with respect to K. If $x\neq 0$ and $y\neq 0$, then $z\oplus\delta$ is in neither K nor K^*, so that $-\rho^*(z)<\delta<\rho(z)$. But if $x=0$, $\delta=-\rho^*(z)$ and if $y=0$, $\delta=\rho(z)$. This proves that (2) implies (1).

The uniqueness of x and y and continuity of $x, y, \rho(x)$, and $\rho^*(y)$ follow from the corresponding properties of the Lemma 4 decomposition. ■

We turn our attention now to the sets of dual pairs, $\Pi(\rho)$ and $\Pi_0(\rho)$.

THEOREM 7. *Let* $\rho\in\mathcal{P}$.

(a) *If* ρ *or* ρ^* *is a norm, then* $\Pi_0(\rho)$ *is homeomorphic to the unit sphere* S *of* $\mathcal{H}$.

(b) *If* ρ *and* ρ^* *are both norms, then* $\Pi(\rho)$ *is homeomorphic to the cylinder* $S\times\mathbf{R}$.

Proof: First note that the norms in $\mathcal{P}$ are always continuous functions. For if $\rho\in\mathcal{P}$ is a norm, then $\mathcal{H}=\cup_n\{x:\rho(x)\leqslant n\}$ expresses $\mathcal{H}$ as a union of closed sets, since ρ is l.s.c. By the Baire category theorem, these sets have interior; in fact, $\lambda S\subseteq B_\rho$ for some $\lambda>0$. Then $\rho(x)\leqslant\lambda^{-1}\|x\|$ for $x\in\mathcal{H}$, which implies ρ is continuous. We now assume without loss of generality that ρ is a norm.

Let C be the subset of the cylinder $S\oplus\mathbf{R}$ in $\mathcal{H}\oplus\mathbf{R}$ given by $C=\{z\oplus\delta\in\mathcal{H}\oplus\mathbf{R}:\|z\|=1,\ -\rho^*(z)<\delta<\rho(z)\}$. We shall show that C is homeomorphic to $\Pi(\rho)$. Let $z\oplus\delta\in C$. Then we may apply Theorem 6 to get a unique pair $x,y\in\mathcal{H}$ such that $z=x+y$, $\rho(x)-\rho^*(y)=\delta$, and $\langle x,y\rangle=\rho(x)\rho^*(y)$. Similarly each point of the ray $t(z\oplus\delta)$, $t\geqslant 0$, is decomposed by the pair (tx,ty). Since $\langle tx,ty\rangle = t^2\rho(x)\rho^*(y)$, it is clear that $t_0=[\rho(x)\rho^*(y)]^{-1/2}$ is the only value of t for which $(tx,ty)\in\Pi(\rho)$. Let ϕ be the map which takes $z\oplus\delta\in C$ to $(t_0x,t_0y)\in\Pi(\rho)$. The continuity of ϕ follows from that of $x, y, \rho(x)$, and $\rho^*(y)$ in z and δ. [To see that $t_0<\infty$ note that ρ^* is a quasinorm. Thus $\rho(x)\rho^*(y)=0$ would imply that x or y is 0, and hence that $\delta=-\rho^*(z)$ or $\rho(z)$.]

Let $\psi:\Pi(\rho)\to S\oplus\mathbf{R}$ be given by

$$\psi(x,y)=\frac{1}{\|x+y\|}\{[x+y]\oplus[\rho(x)-\rho^*(y)]\}.$$

Then ψ is continuous. [Note that ρ is continuous and $\rho^*(y)=1/\rho(x)$ when $(x,y)\in\Pi(\rho)$.].

For $(x,y)\in\Pi(\rho)$ let $z=x+y$ and $\delta=\rho(x)-\rho^*(y)$. Then Theorem 6 shows that $-\rho^*(z)\leqslant\delta\leqslant\rho(z)$, and its proof explains that since $x\neq 0\neq y$, we have $-\rho^*(z)<\delta<\rho(z)$. Thus the range of ψ lies in C.

Now it is simple to verify that ϕ and ψ are inverses of each other and hence homeomorphisms. We have proved that C is homeomorphic to $\Pi(\rho)$.

To prove (a), we observe that the restriction of ϕ to $S\oplus 0$ is inverse to the restriction of ψ to $\Pi_0(\rho)$.

Next we prove (b). If ρ and ρ^* are both norms, and hence both finite, continuous functions of z, then C is homeomorphic to the cylinder $X\times\mathbf{R}$ via the map

$$z\oplus\delta\to\begin{cases}\left(z,\dfrac{\delta}{\rho(z)-\delta}\right) & \text{for}\quad 0\leqslant\delta<\rho(z)\\ \left(z,\dfrac{\delta}{\rho^*(z)+\delta}\right) & \text{for}\quad -\rho^*(z)<\delta\leqslant 0.\end{cases}$$

This completes the proof. ■

Remarks:

(1) When ρ and ρ^* are both norms and hence both equivalent to the Hilbert norm, the homeomorphism $\psi:\Pi_0(\rho)\to S$ and its inverse ϕ are uniformly continuous. The map $(x,y)\to\|x+y\|^{-1}\sqrt{1+[\rho(x)-\rho^*(y)]^2}\,(x+y)\oplus[\rho(x)-\rho^*(y)]$ is a uniformly continuous homeomorphism from $\Pi(\rho)$ onto the hyperboloid $\{z\oplus\delta:\|z\|^2-\delta^2=1\}$ with a uniformly continuous inverse.

(2) The dual of a seminorm is a quasinorm. When $\mathcal{H}$ is finite-dimensional, the dual of a quasinorm is also a seminorm (and hence the hypotheses of the two statements in Theorem 7 are equivalent). When $\mathcal{H}$ is infinite-dimensional, this need not be true. In fact, the dual of a norm may fail to be a seminorm. For example, let l_p be the space of real sequences $x=(x_n)$ for which the function $\nu_p(x)=(\Sigma|x_n|^p)^{1/p}$ is finite. Since $l_q\subseteq l_p$ for $q<p$, the functions ν_p restricted to the Hilbert space l_2 are norms when $2<p$. But for such p, $\nu_p^*=\nu_q$ where $pq=p+q$. Then $q<2$, and ν_q is a quasinorm, but not a seminorm on l_2.

(3) The image of $\Pi_0(\rho)$ under the addition map is the boundary of a convex set in many cases. In $\mathbf{R}^2$ this holds, for example, for the Hölder norms ν_p. However, Fig. 1 indicates a case where it fails.

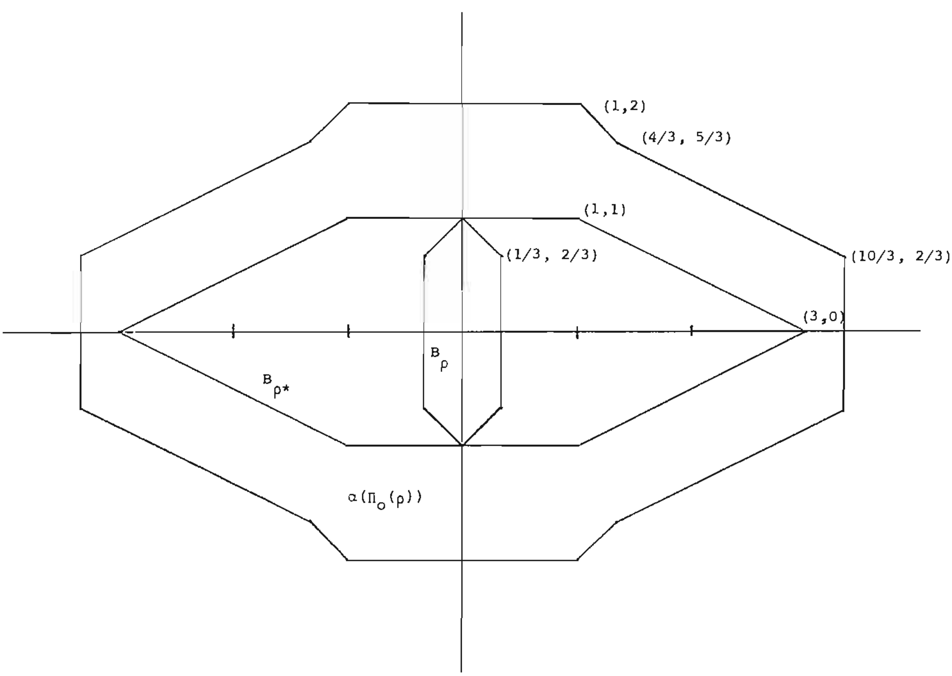

Figure 1

III. Complex spaces, numerical range

If $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a complex Hilbert space, then $(\mathcal{H},\mathrm{Re}\langle\cdot,\cdot\rangle)$ is a real Hilbert space to which the preceeding discussion applies. In particular, for a norm ν on $\mathcal{H}$, $\Pi_0(\nu)$ now has the form

$$\Pi_0(\nu,\mathbf{R})=\{(x,y)\in\mathcal{H}\times\mathcal{H}:\mathrm{Re}\langle x,y\rangle=\nu(x)=\nu^*(y)=1\}.$$

However, we wish to consider the numerical range of the continuous function $A:\mathcal{H}\to\mathcal{H}$, which is the set $\{\langle Ax,y\rangle:(x,y)\in\Pi_0(\nu,\mathbf{C})\}$, where

$$\Pi_0(\nu,\mathbf{C})=\{(x,y)\in\mathcal{H}\times\mathcal{H}:\langle x,y\rangle=\nu(x)=\nu^*(y)=1\}.$$

LEMMA 8. *If ν is a norm on a complex Hilbert space $\mathcal{H}$, and ν is homogeneous (i.e., $\nu(\alpha x)=|\alpha|\nu(x)$ for all $\alpha\in\mathbf{C}$, $x\in\mathcal{H}$), then $\Pi_0(\nu,\mathbf{R})=\Pi_0(\nu,\mathbf{C})$.*

Proof: Since $\mathrm{Re}\,1=1$, $\Pi_0(\nu,\mathbf{C})\subseteq\Pi_0(\nu,\mathbf{R})$. If $(x,y)\in\Pi_0(\nu,\mathbf{R})$, then $\mathrm{Re}\langle x,y\rangle=\nu(x)\nu^*(y)=1$. If $\mathrm{Im}\langle x,y\rangle\neq 0$, then there is a λ with $|\lambda|=1$ such that $1<\mathrm{Re}\lambda\cdot\langle x,y\rangle\leqslant|\langle\lambda x,y\rangle|\leqslant\nu(\lambda x)\nu^*(y)=|\lambda|\nu(x)\nu^*(y)=1$. Thus we must have $\mathrm{Im}\langle x,y\rangle=0$ and $(x,y)\in\Pi_0(\nu,\mathbf{C})$. ■

THEOREM 9. *If ν is a homogeneous norm equivalent to the given norm of a complex Hilbert space $\mathcal{H}$, then for each continuous function $A:\mathcal{H}\to\mathcal{H}$ the numerical range $V_\nu(A)$ is arcwise connected.*

Proof: $\theta(x,y)=\langle Ax,y\rangle$ is continuous on $\mathcal{H}\times\mathcal{H}$. If $\phi:S\to\Pi_0(\nu,\mathbf{R})$ is the homeomorphism of Theorem 7, then

$$S\xrightarrow{\phi}\Pi_0(\nu,\mathbf{R})=\Pi_0(\nu,\mathbf{C})\xrightarrow{\theta}V_\nu(A)$$

is continuous. Since S is arcwise connected, $V_\nu(A)$ is path-connected and hence arcwise connected (cf. Hocking and Young [4]). ■

COROLLARY 10. *For any norm ν on $\mathbf{C}^n$ and any continuous function $A:\mathbf{C}^n\to\mathbf{C}^n$, $V_\nu(A)$ is arcwise connected.*

Proof: Theorem 9 applies since all norms on $\mathbf{C}^n$ are equivalent to the Euclidean norm. ■

Note: After this work was completed the authors learned that C. M. McGregor ("Some results in the theory of numerical ranges," Thesis, University of Aberdeen, 1971) has proved for finite dimensional spaces that $\pi_0(\mu)$ and S are homeomorphic and that V_ν is arcwise connected. He remarks that his method works in some infinite dimensional spaces.

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