The Fuglede-Putnam Theorem and Normal Products of Matrices

Dedicated to Olga Taussky Todd

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ABSTRACT

The rectangular matrix version of the Fuglede-Putnam theorem is used to prove that, for rectangular complex matrices A and B, both AB and BA are normal if and only if $A^*AB=BAA^*$ and $B^*BA=ABB^*$. We deduce some results relating the rank of A and the factors in a polar decomposition of A to the normality of AB and BA.

1

Under the assumption that A and B are normal $n\times n$ complex matrices, N. A. Wiegmann [12] proved that AB and BA are normal if and only if $A^*AB=BAA^*$ and $B^*BA=ABB^*$. In [13], Wiegmann improved this by omitting the requirement that B be normal. In this note, we show that the

* The research of this author was supported in part by NSF Grant GP-32834.

†This paper was written while the author was on sabbatical leave at the University of Wisconsin, Madison.

‡The research of this author was supported in part by NSF Grant GP-37978X.

assumption on the normality of A can also be removed. Moreover, we shall assume that A and B are rectangular matrices of appropriate dimensions.

Let A be a nonsingular normal matrix, and let $A = UH$, where U is unitary and H is positive definite Hermitian. Clarifying a result of Wiegmann's [13], Gibson [4] remarks that AB and BA are normal if and only if BU is normal and $HBU = BUH$. In Theorems 3, 4, and 5, we remove the restriction that A be normal, and we use Theorem 2 to investigate to what extent this result can be generalized to singular A.

For the sake of completeness, we give an elementary proof of the rectangular matrix version of the Fuglede-Putnam theorem [2; 8; 9; 10, p.300;11], which is essentially to be found in [6, p. 65]. Our principal result (Theorem 2) will follow immediately from this theorem. For a related application of the Fuglede-Putnam theorem see [5; 6, p. 68].

2

Denote by $\mathbf{C}^{mn}$ the set of all $m \times n$ complex matrices.

THEOREM 1 (Fuglede-Putnam). *Let $P \in \mathbf{C}^{mm}$, $Q \in \mathbf{C}^{nn}$, $T \in \mathbf{C}^{mn}$. If P and Q are normal and $PT = TQ$, then $P^* T = TQ^*$.*

Proof. Since the matrix $P \oplus Q$ is normal, there exists a scalar polynomial g such that $(P \oplus Q)^* = g(P \oplus Q)$. This implies that $P^* = g(P)$ and $Q^* = g(Q)$. Hence, $P^* T = g(P)T = Tg(Q) = TQ^*$. ∎

REMARK 1. Let f be a function defined on the spectra of $P \in \mathbf{C}^{mm}$ and $Q \in \mathbf{C}^{nn}$, in the sense of Gantmacher [3, p. 96]. Then there exists a polynomial g such that $f(P) = g(P)$ and $f(Q) = g(Q)$. Hence, if $T \in \mathbf{C}^{mn}$, it follows that $PT = TQ$ implies that

$$f(P)T = g(P)T = Tg(Q) = Tf(Q).$$

Letting $f(\lambda) = \bar{\lambda}$ for the normal matrices P and Q of Theorem 1, we obtain our proof of that theorem. In the proof of Theorem 3 we use another application of this result. Let $f(\lambda) = \lambda^{1/2} \geqslant 0$ for $\lambda \geqslant 0$. If $H \in \mathbf{C}^{mm}$ and $K \in \mathbf{C}^{nn}$ are positive semidefinite Hermitian, then $H = f(H^2)$ and $K = f(K^2)$. Hence, if $T \in \mathbf{C}^{mn}$ with $H^2 T = TK^2$, then $HT = TK$.

THEOREM 2. *Let $A \in \mathbf{C}^{mn}$ and $B \in \mathbf{C}^{nm}$. Then AB and BA are normal if and only if $A^* AB = BAA^*$ and $ABB^* = B^* BA$.*

Proof. Assume that AB and BA are normal. Then $(AB)^*$ and $(BA)^*$ are normal. Hence, since

$$A^*(AB)^* = A^*B^*A^* = (BA)^*A^*,$$

by Theorem 1, $A^*AB = BAA^*$. Similarly, from $(AB)^*B^* = B^*(BA)^*$, we obtain $ABB^* = B^*BA$. Conversely, if $A^*AB = BAA^*$ and $ABB^* = B^*BA$, then multiplying the first equation by B^* and the second one by A^* we see that AB and BA are normal. ∎

REMARK 2. A result of J. Williamson [14] can be used instead of Theorem 1 to obtain a proof of Theorem 2. Assume that AB and BA are normal. It follows from Williamson's Theorem 2 that there exist unitary matrices $U \in \mathbf{C}^{mm}$, $V \in \mathbf{C}^{nn}$ and rectangular diagonal matrices $F, G \in \mathbf{C}^{mn}$ such that $A = UFV$ and $B^* = UGV$. Then

$$A^*AB = V^*F^*FG^*U^* = V^*G^*FF^*U^* = BAA^*,$$

$$ABB^* = UFG^*GV = UGG^*FV = B^*BA.$$

REMARK 3. The result that Theorem 1 implies Theorem 2 may be put into a more general context. Let $\mathfrak{A}$ be an algebra over the complex numbers with an involution * (see [1]). An element $P \in \mathfrak{A}$ is called normal if $PP^* = P^*P$. We define $\mathfrak{A}$ to be a Fuglede-Putnam algebra if, for all normal $P, Q \in \mathfrak{A}$ and $T \in \mathfrak{A}$, the relation $PT = TQ$ implies $P^*T = TQ^*$. Let $\mathfrak{A}$ be a Fuglede-Putnam algebra and let $A, B \in \mathfrak{A}$. We have shown that AB and BA are normal if and only if $A^*AB = BAA^*$ and $ABB^* = B^*BA$. An example of a Fuglede-Putnam algebra is the algebra of all bounded operators on a Hilbert space, with involution the usual adjoint. Other examples may be found in [7].

3

It is well known that every $A \in \mathbf{C}^{nn}$ has a polar decomposition as $A = UH$ where $H \in \mathbf{C}^{nn}$ is positive semidefinite Hermitian and $U \in \mathbf{C}^{nn}$ is unitary. If A is singular, U is not unique. We have the following theorem.

THEOREM 3. *Let $A = UH$, where $H \in \mathbf{C}^{nn}$ is positive semidefinite Hermitian and $U \in \mathbf{C}^{nn}$ is unitary, and let $B \in \mathbf{C}^{nn}$.*

(a) *If BU is normal and $HBU = BUH$, then AB and BA are normal.*
(b) *If AB and BA are normal, then $HBU = BUH$.*

Proof. Suppose that BU is normal and $HBU = BUH$. Then

$$BAA^* = BUH(UH)^* = BUH^2U^* = H^2BUU^* = H^2B = (UH)^*UHB = A^*AB. \tag{1}$$

Since BU is normal and $HBU = BUH$, from Theorem 1, we also have $H(BU)^* = (BU)^*H$. Hence,

$$\begin{aligned} ABB^* &= UHBU(BU)^* = UBU(BU)^*H = U(BU)^*BUH \\ &= UU^*B^*BUH = B^*BA. \end{aligned} \tag{2}$$

Therefore, by Theorem 2, AB and BA are normal. This proves (a). To prove (b), let AB and BA be normal and note that there exists a positive semidefinite Hermitian $K \in \mathbf{C}^{nn}$ such that $A = KU$. Using Theorem 2, we obtain

$$H^2B = A^*AB = BAA^* = BK^2.$$

Hence, since H and K are positive semidefinite Hermitian, $HB = BK$ (see Remark 1). Then $HBU = BKU = BUH$. ∎

THEOREM 4. *Let $A = UH$, where $H \in \mathbf{C}^{nn}$ is positive semidefinite Hermitian and $U \in \mathbf{C}^{nn}$ is unitary. The following are equivalent:*

(a) $\operatorname{rank}(A) \geqslant n-1$;
(b) *if $B \in \mathbf{C}^{nn}$ such that AB and BA are normal, then BU is normal.*

Proof. Let $\operatorname{rank}(A) \geqslant n-1$, and let $B \in \mathbf{C}^{nn}$ be such that AB and BA are normal. From Theorem 2 and part (b) of Theorem 3, we see that

$$(BU)^*BUH = U^*B^*BA = U^*ABB^* = HBU(BU)^* = BU(BU)^*H. \tag{3}$$

Hence, if $\operatorname{rank}(H) = \operatorname{rank}(A) = n$, then BU is normal. Suppose that $\operatorname{rank}(A) = n-1$. Then there exist a unitary $V \in \mathbf{C}^{nn}$ and a positive definite Hermitian matrix L of order $n-1$ such that

$$VHV^* = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Let

$$VBUV^* = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where $G_{22} \in \mathbf{C}$. Since L is nonsingular, from part (b) of Theorem 3 we see that $G_{12} = 0$ and $G_{21} = 0$. Then Eq. (3) implies that G_{11} is normal. Moreover, since $G_{22} \in \mathbf{C}$, we see that BU is normal. Hence (a)$\Rightarrow$(b).

Let $\operatorname{rank}(A) = k < n-1$. There exist $L \in \mathbf{C}^{kk}$ and unitary $V \in \mathbf{C}^{nn}$ such that VHV^* has the form (4). Since $m = n-k \geqslant 2$, there exists $R \in \mathbf{C}^{mm}$ such that R is not normal. Let

$$B = V^* \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix} VU^*.$$

Then

$$BUH = HBU, \qquad BU(BU)^*H = (BU)^*BUH.$$

These equations imply $A^*AB = BAA^*$ and $B^*BA = ABB^*$ by an argument similar to that at the beginning of the proof of Theorem 3 [see (1) and (2)]. Hence, by Theorem 2, AB and BA are normal. However, BU is not normal. Therefore, (b)$\Rightarrow$(a). ∎

Clearly, Theorems 3 and 4 imply the following theorem.

THEOREM 5. *Let $A = UH$, where $H \in \mathbf{C}^{nn}$ is positive semidefinite Hermitian and $U \in \mathbf{C}^{nn}$ is unitary. The following are equivalent*:

(a) $\operatorname{rank}(A) \geqslant n-1$;

(b) *if $B \in \mathbf{C}^{nn}$, then AB and BA are normal if and only if BU is normal and $HBU = BUH$.*

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Received 20 September 1974