Bounded Groups and Norm-Hermitian Matrices

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ABSTRACT

An elementary proof is given that a bounded multiplicative group of complex (real) $n \times n$ nonsingular matrices is similar to a unitary (orthogonal) group. Given a norm on a complex n-space, it follows that there exists a nonsingular $n \times n$ matrix L (the Loewner-John matrix for the norm) such that LHL^{-1} is Hermitian for every norm-Hermitian H. Numerous applications of this result are given.

1. INTRODUCTION

A bounded multiplicative group of complex (or real) $n \times n$ nonsingular matrices is similar to a group of unitary (or orthogonal) matrices. For the case of infinite groups, this theorem is due to Auerbach [1]. His proof depends on the existence of the centroid of a compact convex set in a real space (see also [29, p. 57]). A related proof [10, p. 70, 23, p. 51, 34, p. 220, 44, p. 70] makes use of the existence and invariance of Haar measure of a compact group. The purpose of §3 of this paper is to give an elementary and self-contained proof of Auerbach's theorem, which does not involve integration. For the special case of the group of isometries for a given norm on n-space, a geometric proof, essentially the same as our algebraic one, has been given by Rolewicz [36, p. 251] and Gromov [17]. Their main tool is the Loewner ellipsoid corresponding to the unit ball of the norm [4, 5, 11, p. 90, 22, 37]. This is the unique ellipsoid of minimal volume containing a given balanced convex body.[1] However, in order to make our entire presentation

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[1]This ellipsoid is usually called after Loewner, who did not publish this result (cf. [11, pp. 90 and 414]). The existence of this ellipsoid for general real n-space was first published by John [22] (see also [21]). Thus, we shall refer to a corresponding matrix as the Loewner-John matrix.

as elementary as possible, we have given a volume-free version of the ellipsoid theorem.

In §4 we use Auerbach's theorem to show the following result on norm-Hermitian matrices (Proposition 4.1): If ν is a norm and L is the Loewner-John matrix for ν, defined in §3, then LHL^{-1} is Hermitian for every ν-Hermitian matrix H. If the set of all ν-Hermitian matrices is denoted by $\mathcal{H}_\nu$, and $\mathcal{J}_\nu = \{H + iK : H,\ K \in \mathcal{H}_\nu\}$, a more precise statement is (Theorem 4.4)

$$L\mathcal{H}_\nu L^{-1} = \mathcal{H} \cap \mathcal{J}_\nu,$$

where $\mathcal{H}$ is the set of Hermitian matrices. Proposition 4.1 leads to the following principle: In any theorem on Hermitians with a conclusion that is invariant under similarity, we may replace the word "Hermitian" by "ν-Hermitian" in the hypothesis. In §5 we give numerous applications of this principle (e.g. Proposition 5.1) and of the stronger result of Theorem 4.4 (e.g. Proposition 5.2). We show that many results on Hermitian and normal matrices have analogues for norm-Hermitian and norm-normal matrices. Our final applications are to inertia theorems.

2. DEFINITIONS AND NOTATIONS

Let $\mathbf{K}$ denote either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$, let $M_n(\mathbf{K})$ denote the algebra of $n \times n$ matrices over $\mathbf{K}$, and let I denote the identity matrix in $M_n(\mathbf{K})$. If $A \in M_n(\mathbf{K})$, then we denote by A^* the adjoint (conjugate transpose) of A, by $\det A$ the determinant of A, by $\Lambda(A)$ the spectrum of A, and by $r(A)$ the spectral radius of A. The range and the null-space of $A \in M_n(\mathbf{K})$ will be denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively.

If $A \in M_n(\mathbf{C})$, then we define the *inertia* of A, denoted In A, to be the ordered triple $(\pi^+(A), \pi^-(A), \delta(A))$, where $\pi^+(A)$, $\pi^-(A)$, and $\delta(A)$ are respectively the numbers of eigenvalues (counting multiplicities) of A with positive, negative, and zero real parts. A matrix $A \in M_n(\mathbf{C})$ is said to be *stable* if $\pi^+(A) = n$, and *semi-stable* if $\pi^-(A) = 0$.

A norm ν on $\mathbf{K}^n$ is said to be *standardized* [40] if $\nu(e_j) = 1$ $(j = 1, \ldots, n)$, where e_j is the unit vector in $\mathbf{K}^n$ whose components are equal to zero, except for the jth one, which is equal to 1. A norm ν on $\mathbf{K}^n$ is said to be *absolute* [20, p. 47], [40] if $\nu(|x|) = \nu(x)$ for all $x \in \mathbf{K}^n$, where $|x|$ denotes the vector whose components are the absolute values of the components of x. A norm ν on $\mathbf{K}^n$ is said to be *symmetric* if it is absolute and if $\nu(Sx) = \nu(x)$ for all $x \in \mathbf{K}^n$

and for all permutation matrices $S \in M_n(\mathbf{K})$.

If ν is a norm on $\mathbf{K}^n$, then a matrix $A \in M_n(\mathbf{K})$ will be said to be a *ν-isometry* if $\nu(Ax) = \nu(x)$ for all $x \in \mathbf{K}^n$.

For a given norm ν on $\mathbf{C}^n$ and a given $A \in M_n(\mathbf{C})$, we denote by $V_\nu(A)$ the *numerical range* of A (field of values in [2], Bauer field of values in [30], spatial numerical range in [9]), i.e.

$$V_\nu(A) = \{ y^*Ax : \nu(x) = \nu^D(y) = y^*x = 1 \},$$

where ν^D denotes the *dual norm* of ν [20, p. 43]. A matrix $A \in M_n(\mathbf{C})$ is said to be *ν-Hermitian* if $V_\nu(A) \subset \mathbf{R}$, *$\nu$-positive definite* if $V_\nu(A) \subset (0, +\infty)$, and *$\nu$-positive semidefinite* if $V_\nu(A) \subset [0, +\infty)$. The set of all ν-Hermitian matrices will be denoted by $\mathcal{H}_\nu$. We also denote

$$\mathcal{J}_\nu = \{ H + iK : H, K \in \mathcal{H}_\nu \},$$

i.e. $\mathcal{J}_\nu = \mathcal{H}_\nu + i\mathcal{H}_\nu$. Taking into account some well-known properties of the numerical range, it can be easily seen that every member of $\mathcal{J}_\nu$ has a unique representation of the form $H + iK$ with $H, K \in \mathcal{H}_\nu$. Consequently, if $A = H + iK (H, K \in \mathcal{H}_\nu)$, then we can define unambiguously $A^c = H - iK$. It is obvious that the set $\mathcal{J}_\nu$ is a subspace of $M_n(\mathbf{C})$ and that we have $(A+B)^c = A^c + B^c$ and $(\alpha A)^c = \bar{\alpha} A^c$ for all $A, B \in \mathcal{J}_\nu$ and all $\alpha \in \mathbf{C}$. It is also clear that a ν-Hermitian matrix A is characterized by the relations $A \in \mathcal{J}_\nu$ and $A^c = A$. A matrix $A \in \mathcal{J}_\nu$ is said to be *ν-normal* if $HK = KH$, where $A = H + iK$ with $H, K \in \mathcal{H}_\nu$. This is equivalent to the condition $A^cA = AA^c$. If χ denotes the standard Euclidean norm on $\mathbf{C}^n$ (i.e., $\chi(x) = (x^*x)^{1/2}$ for $x \in \mathbf{C}^n$), then it is easy to show that the classes of χ-isometries, χ-Hermitian matrices, χ-positive definite matrices, χ-positive semidefinite matrices, and χ-normal matrices coincide with the classes of unitary, Hermitian (self-adjoint), positive definite, positive semidefinite, and normal matrices, respectively. The set of all Hermitian matrices will be denoted by $\mathcal{H}$.

If Ω is a set of complex numbers, then $\bar{\Omega}$ will denote the set of the complex conjugates of the members of Ω and co Ω will denote the convex hull of Ω.

3. ELEMENTARY PROOF OF AUERBACH'S THEOREM

First we quote a lemma, due essentially to Fan [15] (see also [3, p. 63, 6, 7, p. 128, 25, p. 115, 27, 31]. It is an application of the inequality between the algebraic and geometric mean of two positive numbers.

LEMMA 3.1. *Let Q and Q_1 be positive definite matrices in $M_n(\mathbf{K})$. Then*

$$\det \tfrac{1}{2}(Q+Q_1) \geqslant (\det Q \det Q_1)^{1/2}.$$

The equality holds if and only if $Q=Q_1$.

THEOREM 3.2. *Let $\mathcal{C}$ be a compact convex set of positive semidefinite matrices in $M_n(\mathbf{K})$ and suppose that at least one matrix in $\mathcal{C}$ is positive definite. Then, $\mathcal{C}$ contains a unique matrix Q such that*

$$\det Q = \sup\{\det P : P \in \mathcal{C}\},$$

and Q is positive definite.

Proof. The function $\det : \mathcal{C} \to [0, +\infty)$ is continuous, hence it attains its supremum at an element Q of the compact set $\mathcal{C}$. Clearly, Q is positive definite. In order to prove uniqueness, assume that $Q_1 \in \mathcal{C}$ and the $\det Q_1 = \det Q$. Then Q_1 is positive definite and by Lemma 3.1,

$$\det \tfrac{1}{2}(Q+Q_1) \geqslant (\det Q \det Q_1)^{1/2} = \det Q.$$

But $\frac{1}{2}(Q+Q_1) \in \mathcal{C}$, whence

$$\det \tfrac{1}{2}(Q+Q_1) = \det Q,$$

and so, by the condition for equality in Lemma 3.1, we have $Q_1 = Q$. This proves the theorem.

COROLLARY 3.3. *Let ν be a norm on $\mathbf{K}^n$. Then, the set $\mathcal{E} = \{P \in M_n(\mathbf{K})$: P positive semidefinite and $x^*Px \leqslant \nu^2(x)\ \forall x \in \mathbf{K}^n\}$ contains a unique element Q such that*

$$\det Q = \sup\{\det P : P \in \mathcal{E}\}.$$

The matrix Q is positive definite.

Proof. It is easy to see, making use of the equivalence of norms on $\mathbf{K}^n$, that for β sufficiently small and positive, we have $\beta I \in \mathcal{E}$ and that $\mathcal{E}$ is compact. The result follows from Theorem 3.2.

If ν is a norm on $\mathbf{K}^n$ and Q is the positive definite matrix of Corollary 3.3, then the set $\{x \in \mathbf{K}^n : x^*Qx \leqslant 1\}$ is the unique ellipsoid of minimal volume containing the unit ball of ν, i.e., the Loewner ellipsoid corresponding to the unit ball of ν. The unique positive definite matrix L such that $L^2 = Q$, will be called *the Loewner-John matrix associated with ν*.

PROPOSITION 3.4. [36, p. 251, 17]. *Let ν be a norm on $\mathbf{K}^n$ and let L be the Loewner-John matrix associated with ν. Then, for every ν-isometry A we have*

(i) $A^*L^2A = L^2$;
(ii) LAL^{-1} *is unitary.*

Proof. Let A be a ν-isometry. Denote $B = A^*L^2A$. Then B is positive definite and

$$x^*Bx = (Ax)^*L^2(Ax) \leqslant \nu^2(Ax) = \nu^2(x)$$

for all $x \in \mathbf{K}^n$, i.e., B belongs to the set $\mathfrak{E}$ of Corollary 3.3. Since

$$\det B = \det(A^*L^2A) = |\det A|^2 \det(L^2) = \det(L^2),$$

from the uniqueness property of L^2 it follows that $B = L^2$, i.e., $A^*L^2A = L^2$. Statement (ii) follows at once from (i).

PROPOSITION 3.5. *Let $\mathcal{G}$ be a bounded multiplicative group of matrices in $M_n(\mathbf{K})$ such that $I \in \mathcal{G}$. Then, there exists a norm ν on $\mathbf{K}^n$ such that every member of $\mathcal{G}$ is a ν-isometry.*

Proof. Let $\|\ \|$ be any norm on $\mathbf{K}^n$. Since $\mathcal{G}$ is bounded and $I \in \mathcal{G}$, the mapping

$$\nu : \mathbf{K}^n \to \mathbf{R}$$

$$\nu(x) = \sup_{G \in \mathcal{G}} \|Gx\| \quad (x \in \mathbf{K}^n)$$

is a norm on $\mathbf{K}^n$. Let $A \in \mathcal{G}$. Then

$$\nu(Ax) = \sup_{G \in \mathcal{G}} \|GAx\| = \sup_{G \in \mathcal{G}} \|Hx\| = \nu(x) \quad \forall x \in \mathbf{K}^n,$$

i.e., A is a ν-isometry.

REMARK. The norm ν is the smallest norm larger than $\|\ \|$ such that every member of $\mathcal{G}$ is a ν-isometry. We could also have used the norm ν', where

$$\nu'(x) = \inf\left\{ \sum_{i=1}^{k} \|G_i x_i\| : k \text{ positive integer}, \sum_{i=1}^{k} x_i = x,\ G_i \in \mathcal{G}\ (i = 1, \ldots, k) \right\}.$$

The norm ν' is the largest norm smaller than $\| \ \|$ such that every member of $\mathcal{G}$ is a ν'-isometry.

THEOREM 3.6. *Let $\mathcal{G}$ be a bounded multiplicative group of matrices in $M_n(\mathbf{K})$ such that $I \in \mathcal{G}$. Then, there exists a positive definite $L \in M_n(\mathbf{K})$ such that LAL^{-1} is unitary for every $A \in \mathcal{G}$.*

Proof. By Proposition 3.5, there exists a norm ν on $\mathbf{K}^n$ such that every member of $\mathcal{G}$ is a ν-isometry. Now, if L is the Loewner-John matrix associated with this ν, then by Proposition 3.4, LAL^{-1} is unitary for every $A \in \mathcal{G}$.

In the following propositions we determine the Loewner-John matrix associated with certain classes of norms on $\mathbf{K}^n$.

PROPOSITION 3.7. *Let ν be a standardized norm on $\mathbf{K}^n$ such that $\nu \geqslant \chi$. Then, the Loewner-John matrix associated with ν is I.*

Proof. We use the notations of Corollary 3.3. The inequality $\chi \leqslant \nu$ implies $I \in \mathfrak{S}$. Denoting $Q = (q_{ij})$, we have $0 < q_{jj} = e_j^* Q e_j \leqslant \nu^2(e_j) = 1$ $(j = 1,\dots,n)$. Making use of Hadamard's inequality for a positive definite matrix [25, p. 114, 26, p. 199], we obtain

$$1 = \det I \leqslant \det Q \leqslant q_{11} \cdots q_{nn} \leqslant 1, \tag{3.1}$$

whence $\det Q = q_{ll} \cdots q_{nn} = 1$. Since equality prevails in (3.1), Q is diagonal. Clearly, $Q = I$ and so the Loewner-John matrix associated with ν is also equal to I.

In order to prove a proposition similar to the previous one, we need part of the following lemma.

LEMMA 3.8. *Let ν be a standardized norm on $\mathbf{K}^n$.*
(i) *If $\nu \leqslant \chi$, then ν^D is standardized.*
(ii) *If $\nu \geqslant \chi$, then ν^D is standardized.*

Proof. (i) For each $j \in \{1,\dots,n\}$, there exists $f_j \in \mathbf{K}^n$ such that $\nu^D(f_j) = f_j^* e_j = 1$. Then

$$1 = f_j^* e_j \leqslant \chi(f_j)\chi(e_j) = \chi(f_j) \leqslant \nu^D(f_j) = 1.$$

Hence $\chi(f_j)=\nu^D(f_j)=1$ and so $f_j=e_j$. Consequently, $\nu^D(e_j)=\nu^D(f_j)=1$, i.e., ν^D is standardized.

(ii) We have for each $j\in\{1,\ldots,n\}$,

$$1=e_j^*e_j\leqslant \nu(e_j)\nu^D(e_j)=\nu^D(e_j)\leqslant\chi(e_j)=1,$$

whence $\nu^D(e_j)=1$, i.e., ν^D is standardized.

Proposition 3.9. *Let ν be a norm on $\mathbf{K}^n$. If $\nu\geqslant\chi$ and ν^D is standardized, then the Loewner-John matrix associated with ν is I.*

Proof. Since ν^D is standardized and $\nu^D\leqslant\chi$, by Lemma 3.8 (i), ν is standardized. Now, Proposition 3.7 implies that the Loewner-John matrix associated with ν is I.

Proposition 3.10. *Let ν be an absolute norm on $\mathbf{K}^n$. Then, the Loewner-John matrix associated with ν is diagonal.*

Proof. Let L be the Loewner-John matrix associated with ν and let $L^2=(q_{ij})$. Fixing $i\in\{1,\ldots,n\}$, consider the matrix $A=\operatorname{diag}(a_1,\ldots,a_n)\cdot\in M_n(\mathbf{K})$, where $a_i=-1$ and $a_k=1$ for all $k\neq i$. Then A is a ν-isometry and so, by Proposition 3.4 (i), $L^2A=AL^2$. But this implies $q_{ij}=0$ if $i\neq j$. Thus, L is diagonal.

Proposition 3.11. *Let ν be a symmetric norm on $\mathbf{K}^n$. Then, the Loewner-John matrix associated with ν is αI, where*

$$\alpha=\inf\left\{\frac{\nu(x)}{\chi(x)}:0\neq x\in\mathbf{K}^n\right\}.$$

Proof. Let L be the Loewner-John matrix associated with ν. By Proposition 3.10, L is diagonal. Let $L^2=\operatorname{diag}(q_1,\ldots,q_n)$ and fix $i,j\in\{1,\ldots,n\}$, $i\neq j$. If S is the permutation matrix obtained by interchanging the ith and the jth rows of I, then S is a ν-isometry and so, by Proposition 3.4 (i), $L^2S=SL^2$. But this implies that $q_i=q_j$ and, consequently, $L=\alpha I$ for some $\alpha\in\mathbf{R}$. The value of α follows at once from the definition of L.

Corollary 3.12. *The Loewner-John matrix associated with the Hölder* *norm* $l_p (1 \leqslant p \leqslant \infty)$ *is* αI, *where*

$$\alpha = \begin{cases} l & \text{if } l \leqslant p \leqslant 2, \\ n^{(2-p)/2p} & \text{if } p > 2. \end{cases}$$

Proof. Clearly, the Hölder norms are symmetric. To obtain the value of α, note that from Proposition 3.11 we have $\alpha^{-1} = \sup\{l_2(x)/l_p(x): 0 \neq x \in \mathbf{K}^n\}$ (see relation (3.22) of [32]).

4. MAPPING NORM-HERMITIANS INTO HERMITIANS

PROPOSITION 4.1. *Let* ν *be a norm on* $\mathbf{C}^n$ *and let* L *be the Loewner-John matrix associated with* ν. *Then*

(i) LAL^{-1} *is Hermitian for every* ν*-Hermitian* $A \in M_n(\mathbf{C})$;

(ii) LAL^{-1} *is normal for every* ν*-normal* $A \in M_n(\mathbf{C})$;

(iii) LAL^{-1} *is positive (semi) definite for every* ν*-positive (semi) definite* $A \in M_n(\mathbf{C})$.

Proof. (i) Let A be ν-Hermitian and let α be any real number. Then, $\exp(i\alpha A)$ is a ν-isometry [9] and now, making use of Theorem 3.6, it follows that $\exp(i\alpha LAL^{-1}) = L\exp(i\alpha A)L^{-1}$ is unitary. Consequently, LAL^{-1} is Hermitian.

(ii) Let A be ν-normal and let $A = H + iK$, where $H, K \in \mathcal{H}_\nu$ and $HK = KH$. Since LHL^{-1} and LKL^{-1} commute and are Hermitian (by part (i)), it follows that LAL^{-1} is normal.

(iii) Let A be ν-positive (semi) definite. By part (i), LAL^{-1} is Hermitian and since it has positive (nonnegative) eigenvalues, it follows that it is positive (semi) definite.

PROPOSITION 4.2. *Let* ν *be a norm on* $\mathbf{C}^n$ *and let* L *be the Loewner-John matrix associated with* ν. *If* $A \in \mathcal{J}_\nu$, *then* $LA^cL^{-1} = (LAL^{-1})^*$.

Proof. Let $A = H + iK$ where $H, K \in \mathcal{H}_\nu$. Then

$$LA^cL^{-1} = LHL^{-1} - iLKL^{-1} = (LHL^{-1} + iLKL^{-1})^* = (LAL^{-1})^*.$$

COROLLARY 4.3. *Let* ν *be a norm on* $\mathbf{C}^n$, *let* L *be the Loewner-John* *matrix associated with* ν, *let* $Q = L^2$, *and let* $A \in \mathcal{J}_\nu$. *Then*,

(i) $A^c = Q^{-1}A^*Q$;

(ii) $\Lambda(A^c) = \overline{\Lambda(A)}$;

(iii) $r(A^c) = r(A)$.

Given a norm ν on $\mathbf{C}^n$ and denoting by L the Loewner-John matrix associated with ν, it seems convenient to introduce the similarity mapping

$$(\ )_\varphi : M_n(\mathbf{C}) \to M_n(\mathbf{C})$$

defined by

$$A_\varphi = LAL^{-1}, \qquad (A \in M_n(\mathbf{C})).$$

Obviously, $(\ )_\varphi$ is an algebra automorphism, i.e. $(\ )_\varphi$ is a bijection and

$$(A+B)_\varphi = A_\varphi + B_\varphi, \quad (\alpha A)_\varphi = \alpha A_\varphi, \quad (AB)_\varphi = A_\varphi B_\varphi$$

for all $A, B \in M_n(\mathbf{C})$ and all $\alpha \in \mathbf{C}$. Moreover, by Proposition 4.2,

$$(A^c)_\varphi = (A_\varphi)^*, \quad \forall A \in \mathcal{J}_\nu.$$

Making use of Proposiition 4.2, we can strengthen Proposition 4.1.

THEOREM 4.4. *Let* ν *be a norm on* $\mathbf{C}^n$ *and let* L *be the Loewner-John matrix associated with* ν. *Then*,

(i) *A is ν-Hermitian if and only if $A \in \mathcal{J}_\nu$ and LAL^{-1} is Hermitian*;

(ii) *A is ν-normal if and only if $A \in \mathcal{J}_\nu$ and LAL^{-1} is normal*;

(iii) *A is ν-positive (semi) definite if and only if $A \in \mathcal{J}_\nu$ and LAL^{-1} is positive (semi) definite.*

Proof. The necessity part of each statement follows from Proposition 4.1. Sufficiency: (i) we have $(A^c)_\varphi = (A_\varphi)^* = A_\varphi$, whence $A^c = A$, which, in turn, implies that A is ν-Hermitian; (ii) we have $(A^cA)_\varphi = (A^c)_\varphi A_\varphi = (A_\varphi)^* A_\varphi = A_\varphi (A_\varphi)^* = A_\varphi (A^c)_\varphi = (AA^c)_\varphi$, whence $A^cA = AA^c$, which implies that A is ν-normal; (iii) by part (i), A is ν-Hermitian and, clearly, it has positive (nonnegative) eigenvalues.

PROPOSITION 4.5. *Let* ν *be a standardized norm on* $\mathbf{K}^n$ *such that either* $\nu \geqslant \chi$ *or* $\nu \leqslant \chi$. *Then*

(i) *every ν-isometry matrix is unitary*;

(ii) *every ν-Hermitian matrix is Hermitian*;

(iii) *every ν-normal matrix is normal*;

(iv) *every ν-positive (semi) definite matrix is positive (semi) definite.*

Proof. If $\nu \geqslant \chi$, then the proposition follows from Propositions 3.7, 3.4, and 4.1. Suppose $\nu \leqslant \chi$ and let A be a ν-isometry. Then, it can be easily seen that A^* is a ν^D-isometry. By Lemma 3.8, ν^D is standardized and, since $\nu^D \geqslant \chi$, it follows, from what has been already proved in this proposition, that A^* is unitary. Hence A is unitary. Parts (ii), (iii), and (iv) can be proved in the same manner or, alternatively, they follow from part (i) by arguments similar to those given in the proof of Proposition 4.1.

5. APPLICATIONS

In the sequel, ν is a given norm on $\mathbf{C}^n$.

PROPOSITION 5.1. *If A, B, and AB are ν-Hermitian matrices, then $AB = BA$.*

Proof. The given conditions imply that A_φ, B_φ, and $A_\varphi B_\varphi$ are Hermitian. Therefore, $A_\varphi B_\varphi = B_\varphi A_\varphi$, whence $AB = BA$.

Remark. Proposition 5.1 is due to M. J. Crabb (private communication), who gives a different proof.

PROPOSITION 5.2. *If A, B are ν-Hermitian matrices such that $AB = BA$ and $AB \in \mathfrak{J}_\nu$, then AB is ν-Hermitian.*

Proof. The given conditions imply that A_φ and B_φ are Hermitian and that they commute. Therefore, $A_\varphi B_\varphi$ is Hermitian and then, by Theorem 4.4, AB is ν-Hermitian.

PROPOSITION 5.3 [8, (2.11)]. *If $A_1, \ldots, A_k$ are commuting ν-Hermitian matrices and $A_1 \cdots A_k \in \mathfrak{J}_\nu$, then $A_1 \cdots A_k$ is ν-Hermitian.*

Proof. The proof is similar to that of Proposition 5.2.

PROPOSITION 5.4. *If A is a ν-Hermitian matrix, p is a polynomial over* $\mathbf{R}$ *and $p(A) \in \mathfrak{J}_\nu$, then $p(A)$ is ν-Hermitian.*

Proof. The proof is similar to that of Proposition 5.2.

COROLLARY 5.5. *If $\mathfrak{J}_\nu$ is an algebra, A is a ν-Hermitian matrix and p is a polynomial over* $\mathbf{R}$*, then $p(A)$ is ν-Hermitian.*

Remark. Corollary 5.5 follows also from Theorem 6.3 of [9].

Proposition 5.6 [8, (2.16), (2.17)]. *Let E and F be ν-Hermitian projections. The following statements are equivalent:*

(i) $EF=E$;
(ii) $FE=E$;
(iii) $\mathfrak{R}(E)\subseteq\mathfrak{R}(F)$;
(iv) $\mathfrak{N}(F)\subseteq\mathfrak{N}(E)$;
(v) $\nu(Ex)\leqslant\nu(Fx) \quad \forall x\in\mathbf{C}^n$;
(vi) $F-E$ is ν-positive semidefinite;
(vii) $F-E$ is a ν-Hermitian projection.

Proof. The implications (i)$\Leftrightarrow$(iv), (ii)$\Leftrightarrow$(iii), (v)$\Rightarrow$(iv) hold for any projections E and F. The implications (i)$\Leftrightarrow$(ii) follow at once from Proposition 5.1. The implication (i)$\Rightarrow$(v) follows from the fact that the operator norm of E is equal to 1 if $E\neq 0$. (i)$\Rightarrow$(vii): Clearly, $F-E$ is ν-Hermitian and, taking into account the equivalence between (i) and (ii), we obtain $(F-E)^2=F-E$. (vii)$\Rightarrow$(vi): By Sinclair's theorem [41], $V_\nu(F-E)=\operatorname{co}\Lambda(F-E)\subseteq[0,1]$. (vi)$\Rightarrow$(i): E_φ and F_φ are Hermitian projections and $F_\varphi-E_\varphi$ is Hermitian. Since $\Lambda(F_\varphi-E_\varphi)=\Lambda(F-E)\subset[0,+\infty)$, $F_\varphi-E_\varphi$ is positive semidefinite. Then [18, p. 148] $E_\varphi F_\varphi=E_\varphi$, whence $EF=E$.

Remark. The implication (vi)$\Rightarrow$(i) has been known only in the case when the norm ν is strictly convex [8, (2.17)].

Proposition 5.7. *Let $E_1,\dots,E_k$ be ν-Hermitian projections. Then $E_1+\cdots+E_k$ is a ν-Hermitian projection if and only if $E_iE_j=0$ for $i\neq j, i,j,=1,\dots,k$.*

Proof. Sufficiency: trivial. Necessity: $(E_1)_\varphi,\dots,(E_k)_\varphi$, and $(E_1)_\varphi+\cdots+(E_k)_\varphi$ are Hermitian projections and therefore [18, p. 148] $(E_i)_\varphi(E_j)_\varphi=0$ for $i\neq j$. Then $E_iE_j=0$ for $i\neq j$.

Proposition 5.8. *If $A,B,AB\in\mathcal{J}_\nu$, then $(AB)^c=B^cA^c$.*

Proof. We have $[(AB)^c]_\varphi=[(AB)_\varphi]^*=(B_\varphi)^*(A_\varphi)^*=(B^c)_\varphi(A^c)_\varphi=(B^cA^c)_\varphi$, whence $(AB)^c=B^cA^c$.

Proposition 5.9. *If $A\in\mathcal{J}_\nu$ and A is a ν-isometry, then $A^cA=I$.*

Proof. We have $(A^cA)_\varphi=(A^c)_\varphi A_\varphi=(A_\varphi)^*A_\varphi=I$, since A_φ is unitary.

REMARK. The question whether the converse of the last proposition is true, remains open. In other words, if $A \in \mathcal{J}_\nu$ and $A^cA = I$, does it follow that A is a ν-isometry?

COROLLARY 5.10. *If A is a ν-isometry in $\mathcal{J}_\nu$, then A is ν-normal.*

It is well known [19, p. 215] that a complex $n \times n$ matrix A is diagonable (i.e., similar to a diagonal matrix) if and only if there exist distinct scalars $\lambda_1,\dots,\lambda_k$ and projections $E_1,\dots,E_k$ such that

$$\begin{aligned} A &= \lambda_1 E_1 + \cdots + \lambda_k E_k, \\ I &= E_1 + \cdots + E_k, \\ E_i E_j &= 0 \qquad (i \neq j;\ i,j = 1,\dots,k). \end{aligned} \tag{5.1}$$

Since the decomposition (5.1) is uniquely determined by the matrix A (the scalars, for example, are the distinct eigenvalues of A), we will call (5.1) the *spectral resolution* of A.

PROPOSITION 5.11. *Let A be a ν-normal matrix. Then,*
 (i) *A is diagonable;*
 (ii) *if the spectral resolution of A is given by*

$$A = \lambda_1 E_1 + \cdots + \lambda_k E_k,$$

then

$$A^c = \bar{\lambda}_1 E_1 + \cdots + \bar{\lambda}_k E_k. \tag{5.2}$$

Proof. (i) By Proposition 4.1 (ii), A_φ is normal. If the spectral resolution of A_φ is given by

$$A_\varphi = \lambda_1 F_1 + \cdots + \lambda_k F_k, \tag{5.3}$$

then

$$A = \lambda_1 E_1 + \cdots + \lambda_k E_k, \tag{5.4}$$

where E_j is the inverse image of F_j under the mapping $(\ )_\varphi$, i.e. $(E_j)_\varphi = F_j$ $(j = 1,\dots,k)$. It is easy to verify that (5.4) gives the spectral resolution of A, consequently, A is diagonable. (ii) From (5.3), taking into account that the F_j's are Hermitian, we obtain $(A^c)_\varphi = (A_\varphi)^* = \bar{\lambda}_1 F_1 + \cdots + \bar{\lambda}_k F_k$, from where (5.2) follows at once.

Corollary 5.12. *If A is ν-normal, then* $\mathfrak{N}(A-\lambda I)=\mathfrak{N}(A^c-\bar{\lambda}I)$ *for all* $\lambda\in\mathbf{C}$.

Corollary 5.13. *Let* $A\in\mathcal{J}_\nu$. *Then A is ν-normal if and only if A^c is a polynomial in A.*

Remark. Proposition 5.11 and Corollaries 5.12 and 5.13 can be derived easily without making use of the Loewner-John matrix associated with the norm ν (used implicitly through the mapping $(\ )_\varphi$). Indeed, if $A=H+iK$ $(H,K\in\mathcal{H}_\nu)$ is ν-normal, then H and K are commuting diagonable matrices. Hence, there exists a nonsingular S such that both $C=S^{-1}HS$ and $D=S^{-1}KS$ are diagonal matrices [19, p. 207, 28, p. 318]. Thus, $A=S(C+iD)S^{-1}$ and $A^c=S(C-iD)S^{-1}$, which is easily seen to be equivalent to Proposition 5.11.

Proposition 5.14. *Let A be a diagonable matrix in* $\mathcal{J}_\nu$. *If the spectral resolutions of A and A^c are given by*

$$A=\lambda_1E_1+\cdots+\lambda_kE_k,$$

and

$$A^c=\bar{\lambda}_1E_1+\cdots+\bar{\lambda}_kE_k,$$

respectively, then A is ν-normal.

Proof. The assumptions imply at once that $A^cA=AA^c$.

Proposition 5.15. *Let* $A\in\mathcal{J}_\nu$. *Then $A^cA=I$ if and only if A is ν-normal and every eigenvalue of A has absolute value equal to one.*

Proof. Sufficiency is an immediate consequence of Proposition 5.11. Necessity: we have $AA^c=A^cA$, which implies that A is ν-normal. The matrix A_φ is unitary since $(A_\varphi)^*A_\varphi=(A^c)_\varphi A_\varphi=A^cA)_\varphi=I$. From the similarity of A and A_φ it follows that every eigenvalue of A has absolute value equal to 1.

Proposition 5.16. *If A is a ν-normal matrix and every eigenvalue of A is real, then A is ν-Hermitian.*

Proof. This is an immediate consequence of Proposition 5.11.

REMARK. Proposition 5.16 follows also from the fact that for a ν-normal matrix the numerical range is equal to the convex hull of the eigenvalues [38].

PROPOSITION 5.17. *If A, B, and AB are ν-normal matrices and $BA \in \mathfrak{J}_\nu$, then BA is ν-normal.*

Proof. The given conditions imply that A_φ, B_φ, and $A_\varphi B_\varphi$ are normal. Therefore, by a theorem of Wiegmann [45, Theorem 1], $(BA)_\varphi$ is normal and then, by Theorem 4.4, BA is ν-normal.

PROPOSITION 5.18. *If A, B are ν-normal matrices, $D \in M_n(\mathbf{C})$, and $AD = DB$, then $A^cD = DB^c$.*

Proof. The given conditions imply that A_φ, B_φ are normal and that $A_\varphi D_\varphi = D_\varphi B_\varphi$. Therefore, by the Fuglede-Putnam theorem [16], [35] (for an elementary proof of the finite-dimensional case, see [43]), $(A_\varphi)^* D_\varphi = D_\varphi (B_\varphi)^*$, or $(A^c)_\varphi D_\varphi = D_\varphi (B^c)_\varphi$, whence $A^cD = DB^c$.

PROPOSITION 5.19. *If A, B are ν-normal matrices such that $AB = BA$ and $AB \in \mathfrak{J}_\nu$, then AB is ν-normal.*

Proof. Making use of Proposition 5.18 we obtain $A^cB = BA^c$ and $B^cA = AB^c$. Now, applying Proposition 5.8, we have $(AB)^cAB = B^cA^cAB = B^cAA^cB = AB^cBA^c = ABB^cA^c = AB(AB)^c$, which implies at once that AB is ν-normal.

PROPOSITION 5.20. *If A, B are ν-normal matrices such that $AB = BA$ and $A + B \in \mathfrak{J}_\nu$, then $A + B$ is ν-normal.*

Proof. The proof of this proposition is similar to that of the previous one.

COROLLARY 5.21. *If A is a ν-normal matrix, p is a polynomial over $\mathbf{C}$ and $p(A) \in \mathfrak{J}_\nu$, then $p(A)$ is ν-normal.*

PROPOSITION 5.22. *Let A, B, AB, $BA \in \mathfrak{J}_\nu$. Then, both AB and BA are ν-normal if and only if $A^cAB = BAA^c$ and $B^cBA = ABB^c$.*

Proof. Sufficiency follows easily by verifying that $(AB)^cAB = AB(AB)^c$

and $(BA)^c BA = BA(BA)^c$. The proof of necessity is similar to that of Proposition 5.18, except that one makes use of a proposition in [14].

PROPOSITION 5.23. *If A and B are ν-normal matrices such that $AB=0$, then $BA=0$.*

Proof. The matrices A_φ and B_φ are normal and $A_\varphi B_\varphi = 0$. Then $B_\varphi A_\varphi = 0$ (see, for example, [14]), whence $BA=0$.

PROPOSITION 5.24. *Let A and B be ν-normal matrices such that $AB \in \mathfrak{J}_\nu$. Then, AB is ν-normal if and only if $A^cAB = BA^cA$ and $AB^cB = B^cBA$.*

Proof. Sufficiency follows easily by verifying that $(AB)^c AB = AB(AB)^c$. The proof of necessity is similar to that of Proposition 5.18, except that one makes use of a result of Wiegmann [45, Theorem 2] (see also [14]).

PROPOSITION 5.25. *If $A \in \mathfrak{J}_\nu$, then*
(i) $(A^cA)_\varphi = (A_\varphi)^* A_\varphi$;
(ii) *A^cA is similar to a positive semidefinite matrix;*
(iii) $\Lambda(A^cA) \subset [0, +\infty)$.

Proof. We have $(A^cA)_\varphi = (A^c)_\varphi A_\varphi = (A_\varphi)^* A_\varphi$, from where we obtain at once (ii) and (iii).

PROPOSITION 5.26. *If both A and A^cA are in $\mathfrak{J}_\nu$, then A^cA is ν-positive semidefinite.*

Proof. This is an immediate consequence of Proposition 5.25 (i) and Theorem 4.4 (iii).

COROLLARY 5.27 ([9, LEMMA 6.7]). *If $\mathfrak{J}_\nu$ is an algebra, then A^cA is ν-positive semidefinite for every $A \in \mathfrak{J}_\nu$.*

PROPOSITION 5.28. *If $A \in \mathfrak{J}_\nu$ and $A^cA = 0$, then $A = 0$.*

Proof. The proof is similar to that of Proposition 5.1.

We next turn to some inertia theorems. (For definitions, see §2).

PROPOSITION 5.29. *Let H be ν-Hermitian and let $A \in \mathcal{J}_\nu$ be nonsingular. Then*

(i) $\Lambda(A^c HA) \subset \mathbf{R}$;

(ii) $\operatorname{In}(A^c HA) = \operatorname{In} H$.

Proof. The matrices H_φ and $(A^c HA)_\varphi$ are Hermitian. Consequently, $\Lambda(A^c HA) \subset \mathbf{R}$ and by Sylvester's theorem [39, p. 338], $\operatorname{In}(A^c HA) = \operatorname{In}[(A_\varphi)^* H_\varphi A_\varphi] = \operatorname{In}(H_\varphi) = \operatorname{In} H$.

PROPOSITION 5.30. *Let H be ν-Hermitian and let $A \in \mathcal{J}_\nu$. If $AH + HA^c$ is stable, then* $\operatorname{In} A = \operatorname{In} H$.

Proof. Let $K = (AH + HA^c)_\varphi$. Then $K = A_\varphi H_\varphi + H_\varphi (A_\varphi)^*$, whence K is positive definite. The result follows from the Ostrowski-Schneider Main Inertia Theorem [33, Theorem 1].

The proofs of the following propositions are similar.

PROPOSITION 5.31. *Let $A \in \mathcal{J}_\nu$. If $A + A^c$ is stable and H is ν-Hermitian, then* $\operatorname{In}(AH) = \operatorname{In} A$.

Proof. By Ostrowski-Schneider [33, Corollary 3].

PROPOSITION 5.32. *Let $A \in \mathcal{J}_\nu$. If H is ν-positive definite and $AH + HA^c$ is semi-stable, then:*

(i) *A is semi-stable;*

(ii) *if λ is a pure imaginary eigenvalue of A, then all elementary divisors belonging to λ are linear.*

Proof. By Carlson-Schneider [13, Corollary III.1].

Finally, we give an application of an interesting generalization of the Stein-Pfeffer theorem [42] due to Carlson-Loewy [12] and Loewy [24]. For $\lambda \in \Lambda(A)$, let $m_\lambda(A)$ be the number of elementary divisors belonging to the eigenvalue λ, i.e., $m_\lambda(A) = \dim \mathfrak{N}(A - \lambda I)$. Put

$$m^+(A) = \max\{m_\lambda(A) : \lambda \in \Lambda(A) \quad \text{and} \quad \operatorname{Re}\lambda > 0\},$$

$$m^-(A) = \max\{m_\lambda(A) : \lambda \in \Lambda(A) \quad \text{and} \quad \operatorname{Re}\lambda < 0\}.$$

PROPOSITION 5.33. *Let $A \in \mathcal{J}_\nu$, and suppose that $\lambda + \bar{\lambda} \neq 0$ for all*

$\lambda \in \Lambda(A)$. *If H is ν-positive definite and $K = AH + HA^c$, then*

$$\pi^+(K) \geqslant m^+(A),$$

and

$$\pi^-(K) \geqslant m^-(A).$$

Proof. By Loewy [24, Theorem 1].

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