Linear and Multilinear Algebra, 1974, Vol. 2, pp. 135–140

Printed in Great Britain

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Primes in the Semigroup of Non-Negative Matrices

DANIEL J. RICHMAN and HANS SCHNEIDER†

Department of Mathematics University of Wisconsin

(*Received September 12, 1973*)

A matrix A in the semigroup $\mathbb{N}_n$ of non-negative $n \times n$ matrices is prime if A is not monomial and $A = BC$, B, $C \in \mathbb{N}_n$ implies that either B or C is monomial. One necessary and another sufficient condition are given for a matrix in $\mathbb{N}_n$ to be prime. It is proved that every prime in $\mathbb{N}_n$ is completely decomposable.

1. INTRODUCTION

Let $\mathbb{N}_n$ be the semigroup of $n \times n$ non-negative matrices. Although non-negative matrices have been studied from many points of view, relatively little work has been done on the semigroup properties of $\mathbb{N}_n$. In an interesting recent paper Plemmons [2] classifies the regular elements, and hence the regular $\mathscr{D}$-classes of $\mathbb{N}_n$. In this note, we attempt the first steps towards a theory of factorization in $\mathbb{N}_n$. In Section 2, we define the concept of a prime matrix in $\mathbb{N}_n$. In Theorem (2.4) we find a necessary condition, and in Theorem (2.6) a sufficient condition of a combinatorial nature, for a matrix to be prime in $\mathbb{N}_n$. By Theorem (2.9), all primes can be derived from the fully indecomposable primes, and all primes are completely decomposable (Corollary (2.10)). In Section 3 we list all primes of orders 2 and 3 and some of order 4 and in Section 4 we state two open questions and point out a generalization to totally ordered division rings.

If $A \in \mathbb{N}_n$, then a_j denotes the jth column of A. By A^* we denote the (0, 1) matrix defined by $a^*_{ij} = 1$ if $a_{ij} > 0$ and $a^*_{ij} = 0$ if $a_{ij} = 0$. Further we define a^*_j to be the jth column of A^*. We use the component-wise partial order on $\mathbb{N}_n$ and on the set of column n-tuples. The transpose of a_j will be denoted by $(a_j)^T$.

† The research of these authors was sponsored in part by the National Science Foundation under Grant GP–379X, the Science Research Council, United Kingdom and the United States Army under Contract No. DA–31–124–ARO–D–462.

2. FACTORIZATION THEOREMS

DEFINITION 2.1 *Let $M \in \mathbb{N}_n$. Then M is called* monomial *if M has exactly one positive element in each row and column.*

We call $M \in \mathbb{N}_n$ *invertible* if M is non-singular and $M^{-1} \in \mathbb{N}_n$. The following lemma is very easy, and we omit the proof.

LEMMA 2.2 *Let $M \in \mathbb{N}_n$. Then M is invertible if and only if M is monomial.*

DEFINITION 2.3 *Let $P \in \mathbb{N}_n$. Then P is called a* prime *if*

i) *P is not monomial, and*

ii) *$P = BC$, where $B, C \in \mathbb{N}_n$, implies that either B is monomial or C is monomial.*

If P is neither a prime nor monomial, then P is called factorizable.

THEOREM 2.4 *Let $A \in \mathbb{N}_n$. Let $1 \leqslant i, k \leqslant n$,* and *$i \neq k$. If $a_i^* \geqslant a_k^*$, then A is factorizable.*

Proof By reordering the columns of A, we may assume without loss of generality that $a_1^* \geqslant a_2^*$. Hence there exists a positive $\in$ such that $b_1 = a_1 - a_2 \in \geqslant 0$ and $b_1^* = a_1^*$. Let $b_i = a_i$, $i = 2, \ldots, n$. Then $B = [b_1, \ldots, b_n] \in \mathbb{N}_n$. We shall prove that B is not monomial.

Either $b_2 = 0$ or $b_2 \neq 0$. If $b_2 = 0$, then B is not monomial. If $b_2 \neq 0$, then there is an r, $1 \leqslant r \leqslant n$, such that $b_{r2} > 0$. Since $a_1^* \geqslant a_2^* = b_2^*$, we have $a_{r1} > 0$ and since $b_1^* = a_1^*$, it follows that $b_{r1} > 0$. Thus in both cases, B is not monomial.

Let

$$C = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} \oplus I_{n-2},$$

where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix. Then C is not monomial. Since $A = BC$, it follows that A is factorizable. Q.E.D.

COROLLARY 2.5 *If A is prime, then A^* has a 0 and a 1 in every row and column.*

The converse of (2.4) is false.

A counter-example is:

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{bmatrix}.$$

A matrix $A \in \mathbb{N}_n$ is called *fully indecomposable* if there do *not* exist permutation matrices M, N such that

$$\mathrm{MAN} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is square.

A matrix A is *completely decomposable* if there exist permutation matrices M, N such that $\mathrm{MAN} = A_1 \oplus \ldots \oplus A_s$, where A_i is fully indecomposable, $i = 1, \ldots, s$ and $s \geqslant 1$. (Note that a fully indecomposable matrix is completely decomposable).

We now state a sufficient condition for A to be prime in $\mathbb{N}_n$.

THEOREM 2.6 *Let $n > 1$ and let $A \in \mathbb{N}_n$. If*

i) *A is fully indecomposable, and*

ii) *$(a_i^*)^T a_k^* \leqslant 1$ for all i, k such that $1 \leqslant i, k \leqslant n$, and $i \neq k$,*

then A is prime.

Proof By (i), A is not monomial.

Let $A = BC$, where $B, C \in \mathbb{N}_n$. Let $\mathbb{Z}_n = \{1, \ldots, n\}$ and let $J = \{j \in \mathbb{Z}_n: b_j$ has at most one positive entry$\}$.

We now assert that:

($\ddagger$) *If $j \in \mathbb{Z}_n \backslash J$, then there is at most one $i \in \mathbb{Z}_n$ such that $c_{ji} > 0$.* For suppose that $j \in \mathbb{Z}_n \backslash J$ and that $c_{ji} > 0$, $c_{jk} > 0$, where $i, k \in \mathbb{Z}_n$, $i \neq k$. Then $a_i = \sum_{l=1}^{n} b_l c_{li} \geqslant b_j c_{ji}$, whence $a_i^* \geqslant b_j^*$. Similarly, $a_k^* \geqslant b_j^*$. Hence $(a_i^*)^T a_k^* \geqslant 2$, which contradicts (ii). Thus ($\ddagger$) is proved.

If E is a set, let $|E|$ denote the number of elements in E.

We shall next show that

$$0 < |J| < n \text{ is impossible.}$$

Let $|J| = q$, and put $I = \{i \in \mathbb{Z}_n: c_{ji} = 0 \text{ for all } j \in \mathbb{Z}_n \backslash J\}$. Suppose that $|I| = r$. Let $d = \sum_{i \in I} a_i$. By (i), d has at least $r + 1$ positive entries. Since for every $i \in I$, we have $a_i = \sum_{i \in J} b_j c_{ji}$ it follows that d has at most q positive entries. Hence $r < q$. Let $I' = \mathbb{Z}_n \backslash I$ and $J' = \mathbb{Z}_n \backslash J$. By definition of I, for each $i \in I'$ there exists a $j \in J'$ such that $c_{ji} > 0$. Since $|I'| = n - r > n - q = |J'|$, there exists a $j \in J'$ such that $c_{ji} > 0$ and $c_{jk} > 0$ for distinct i, k in $\mathbb{Z}_n$. But this contradicts ($\ddagger$). Hence $0 < |J| < n$ is impossible.

There are two remaining possibilities:

a) $$|J| = n.$$

Then each column of B has at most one positive entry. But by (i), every row of B is non-zero. Hence B is monomial.

b) $$|J| = 0.$$

By ($\ddagger$), each row of C has at most one positive entry. But by (i), every column of C is non-zero, whence C is monomial. Q.E.D.

Remark It is clear that Theorem (2.4), Corollary (2.5) and Theorem (2.6) have analogues for rows instead of columns.

THEOREM 2.7 *Let $A \in \mathbb{N}_n$ and let A be prime. Then there exists an r, $1 \leqslant r \leqslant n$, and a fully indecomposable prime $P \in \mathbb{N}_r$ such that*

$$\text{MAN} = P \oplus D,$$

where M, N are permutation matrices in $\mathbb{N}_n$ and D is a non-singular diagonal matrix in $\mathbb{N}_{n-r}$.

Proof The proof is by induction on n. If $n = 1$, the result is trivial, since there are no primes in $\mathbb{N}_1$. So suppose that $n > 1$, and that the theorem holds for $\mathbb{N}_{n-1}$. Let A be a prime in $\mathbb{N}_n$. If A is fully indecomposable, there is no more to prove. So suppose that, for suitable permutation matrices R, S,

$$\text{RAS} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} is $s \times s$, $0 < s < n$.

We shall show that $A_{12} = 0$.

Suppose $A_{12} \neq 0$, say $a_{ij} > 0$, $1 \leqslant i \leqslant s$ and $s + 1 \leqslant j \leqslant n$. It follows that A_{11} is not monomial, for otherwise we would have $a_j^* \geqslant a_k^*$, where $1 \leqslant k \leqslant s$, and by Theorem (2.4) A would not be prime.

Let I_s denote the $s \times s$ identity matrix.

Then

$$\text{RAS} = \begin{bmatrix} I_s & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n-s} \end{bmatrix},$$

with neither factor monomial, which is again a contradiction.

Hence $A_{12} = 0$, and

$$\text{RAS} = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

If either A_{11} or A_{22} is factorizable, then it is easily seen that RAS is factorizable. Hence since

$$\text{RAS} = \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n-s} \end{bmatrix} \begin{bmatrix} I_s & 0 \\ 0 & A_{22} \end{bmatrix},$$

either

a) A_{22} is monomial and A_{11} is prime, or

b) A_{11} is monomial and A_{22} is prime.

Suppose (a) holds. By inductive hypothesis we permute the rows and columns of A_{11} to obtain $P \oplus D_1$, where P is a fully indecomposable prime in $\mathbb{N}_r$ where $1 \leqslant r \leqslant s$, and D_1 is a non-singular diagonal matrix in $\mathbb{N}_{s-r}$. We also permute the rows and columns of A_{22} to obtain a non-singular diagonal matrix D_2 in $\mathbb{N}_{n-s}$. Thus, for suitable permutation matrices M and N,

$$\text{MAN} = P \oplus D,$$

where $D = D_1 \oplus D_2$ is a non-singular diagonal matrix in $\mathbb{N}_{n-r}$. The proof in case (b) is similar. Q.E.D.

THEOREM 2.8 *If P is a prime in $\mathbb{N}_r$ and Q is monomial in $\mathbb{N}_{n-r}$, where $1 \leqslant r \leqslant n$, then $P \oplus Q$ is a prime in $\mathbb{N}_n$.*

Proof Let $A = P \oplus Q$ and let $A = BC$. Partition

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad C = [C_1 \ C_2]$$

where B_1 is $r \times n$ and C_1 is $n \times r$. Replacing B by BN and C by $N^{-1}C$, where N is a permutation matrix, we may suppose that any zero columns of B_1 are at the right. Thus

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where C_{11} is $s \times r$, B_{11} is $r \times s$, $B_{12} = 0$ and no column of B_{11} is 0. Clearly $s > 0$, since A has no zero row. We have

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} = A = BC = \begin{bmatrix} B_{11}C_{11} & B_{11}C_{12} \\ B_{21}C_{11} + B_{22}C_{21} & B_{21}C_{12} + B_{22}C_{22} \end{bmatrix}$$

whence $0 = B_{11}C_{12}$.

Since no column of B_{11} is 0, it follows that $C_{12} = 0$. Hence $s < n$, since A has no zero column. Thus $0 < s < n$.

We now have $P = B_{11}C_{11}$ and $Q = B_{22}C_{22}$.

We next show that $r = s$.

If $s < r$, we have $P = B'_{11}C'_{11}$, where $B'_{11} = [B_{11}\, 0] \in \mathbb{N}_r$ and $C'_{11} = \begin{bmatrix} C_{11} \\ 0 \end{bmatrix} \in \mathbb{N}_r$. But this factorization contradicts that P is prime. Similarly, if $s > r$, we obtain $n - r < n - s$, a contradiction to $Q = B_{22}C_{22}$ and that Q is monomial. Hence $r = s$. But

$$A = BC = \begin{bmatrix} B_{11}C_{11} & 0 \\ B_{21}C_{11} + B_{22}C_{21} & B_{22}C_{22} \end{bmatrix},$$

and so $B_{21}C_{11} = 0$ and $B_{22}C_{21} = 0$. Since $P = B_{11}C_{11}$ is a factorization in $\mathbb{N}_r$ it follows that C_{11} is either prime or monomial. Thus C_{11} has no zero row. Hence it follows from $B_{21}C_{11} = 0$ that $B_{21} = 0$. Similarly, we deduce from $B_{22}C_{21} = 0$ and the fact that B_{22} is monomial that $C_{21} = 0$. Hence $B = B_{11} \oplus B_{22}$ and $C = C_{11} \oplus C_{22}$. Since B_{22}, C_{22} are monomial and one of B_{11}, C_{11} is monomial, it follows that either B or C is monomial. Q.E.D.

THEOREM 2.9 *Let $A \in \mathbb{N}_n$. Then A is prime if and only if there exists an r, $1 \leqslant r \leqslant n$, and a fully indecomposable prime $P \in \mathbb{N}_r$, such that*

$$\mathrm{MAN} = P \oplus D,$$

where M, N are permutation matrices in $\mathbb{N}_n$ and D is a non-singular diagonal matrix in $\mathbb{N}_{n-r}$.

Proof Immediate by Theorems (2.7) and (2.8). Q.E.D.

Remark Since there are no primes in $\mathbb{N}_1$ and $\mathbb{N}_2$ (see Section 3), we can improve the inequality in Theorem (2.7) and Theorem (2.9) to $3 \leqslant r \leqslant n$.

COROLLARY 2.10 *Every prime in $\mathbb{N}_n$ is completely decomposable.*

3. PRIMES OF ORDER 2, 3, 4

We use Theorems (2.4), (2.6), (2.9) to classify all primes of orders 2 and 3, and to list some primes of order 4.

$n = 2$. There are no primes in $\mathbb{N}_2$. This follows from Theorem (2.4).

$n = 3$. The matrix $A \in \mathbb{N}_3$ is a prime if and only if

$$\mathrm{MA^*N} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

for suitable permutation matrices M and N.

This follows from Theorems (2.4) and (2.6).

$n = 4$. Let $A \in \mathbb{N}_4$. If for suitable permutation matrices M, N, MA*N is one of the following three matrices:

$$P_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then A is prime. This follows from Theorems (2.6) and (2.8).

Note that P_1 is singular, P_1 and P_2 are fully indecomposable, and P_3 is completely decomposable. A matrix $A \in \mathbb{N}_n$ has a *non-negative rank factorization* if there exist $B, C \in \mathbb{N}_n$ such that $A = BC$ and rank A = rank B = rank C (see Berman-Plemmons [1], Plemmons [2]). It is clear that a singular prime has no non-negative rank factorization. Indeed, our matrix P_1 is used as an example in [1], due to J. S. Montague, of a matrix with no non-negative rank factorization.

We do not know whether there is a prime of order 4 with a different (0, 1) pattern.

4. OPEN QUESTIONS AND GENERALIZATION

(5.1) Does every prime in $\mathbb{N}_n$ satisfy Condition (ii) of Theorem (2.6)?

(5.2) If $A \in \mathbb{N}_n$, is A prime if and only if A^* is prime?

If $n = 2$ or $n = 3$, the answer is affirmative for both questions.

Remark Let $\mathbb{F}$ be a totally ordered division ring. Our results and their proofs remain valid if $\mathbb{N}_n$ is the semigroup of all $n \times n$ matrices with non-negative entries from $\mathbb{F}$.

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