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Positive Operators on the n-Dimensional Ice Cream Cone*

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Let $K_n = \{x \in \mathbb{R}^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leqslant x_n\}$ be the n-dimensional ice cream cone, and let $\Gamma(K_n)$ be the cone of all matrices in $\mathbb{R}^{nn}$ mapping K_n into itself. We determine the structure of $\Gamma(K_n)$, and in particular characterize the extreme matrices in $\Gamma(K_n)$.

INTRODUCTION

In the vector space $\mathbb{R}^n$ of real n-tuples we consider the *cone*

$$K_n = \{x \in \mathbb{R}^n: (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leqslant x_n\}.$$

A matrix $A \in \mathbb{R}^{nn}$ is a *positive operator* on K_n if $AK_n \subseteq K_n$. In this paper, we determine the structure of the cone $\Gamma(K_n)$ of positive operators on K_n. In Section 2, we show that $AK_n = K_n$ or $AK_n = -K_n$ if and only if $A^tJ_nA = \mu J_n$, for some $0 < \mu \in \mathbb{R}$, where $J_n = \operatorname{diag}(-1,..., -1, 1)$. Further, if rank $A \neq 1$, then $AK_n \subseteq K_n$ or $AK_n \subseteq -K_n$, if and only if, for some $\mu \geqslant 0$, $A^tJ_nA - \mu J_n$ is a positive semidefinite symmetric matrix. These are related to the notion of copositivity for K_n. Section 3 contains some basic lemmas. Lemmas 3.5 and 3.7 exhibit interesting properties of matrices mapping K_n onto itself. In Section 4 we prove our main result: The extreme matrices in the cone $\Gamma(K_n)$ are precisely the matrices A which map

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K_n onto K_n together with the matrices of the form uv^t, where u and v belong to the boundary of K_n. As a corollary, we show that $\Gamma(K_n)$ is the closure of the convex hull of all matrices mapping K_n onto K_n.

We have the following *conjecture*.[1] Let C be a cone in $\mathbb{R}^n$ (see Definition 1.2), and let $\Gamma(C)$ be the cone of positive operators on C. If A is an extreme matrix of $\Gamma(C)$, then A maps the set of extreme vectors of C into itself. The converse of this conjecture is proved in Loewy and Schneider [5] for a nonsingular A and an indecomposable cone C. The converse is in general false for a singular matrix A or a decomposable cone C.

1. Preliminaries

Let $\mathbb{R}$ be the real field, and let $\mathbb{R}^n$ be the vector space of all real column vectors $x = (x_1, ..., x_n)^t$. (We use x^t for the transpose of x and we shall always assume that $n \geqslant 2$.) By e^i we denote the ith standard unit vector in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, let the vectors $\tilde{x}$ and $\hat{x}$ in $\mathbb{R}^{n-1}$ be given by

$$\tilde{x}^t = (x_1, x_2, ..., x_{n-1}), \qquad \hat{x}^t = (x_2, x_3, ..., x_n). \tag{1.1}$$

Definition 1.2. A *cone* C in $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ such that

(i) $C + C \subseteq C$,
(ii) $\lambda C \subseteq C$, for $0 \leqslant \lambda \in \mathbb{R}$,
(iii) $C \cap (-C) = \{0\}$,
(iv) $C - C = \mathbb{R}^n$,
(v) C is closed.

We write $x \geqslant y$ if $x - y \in C$.

If C is a cone, then so is

$$C' = \{y \in \mathbb{R}^n : y^t x \geqslant 0, \forall x \in C\},$$

and C' is called the *cone dual to* C. For the sake of precision, we define:

Definition 1.3. Let C be a cone in $\mathbb{R}^n$. If $x \in C$, then x is an *extreme vector* in C if $0 \leqslant y \leqslant x$ implies that $y = \alpha x$, for some $\alpha \in \mathbb{R}$.

Definition 1.4. Let S be a nonempty subset of $\mathbb{R}^n$. Then $x \in \operatorname{hull} S$ if and only if there exist $x^i \in S$, $0 \leqslant \lambda_i \in \mathbb{R}$, $i = 1, ..., r$, such that

$$x = \sum_{i=1}^{r} \lambda_i x^i.$$

[1] After the completion of this paper R. C. O'Brien has shown in an interesting counterexample (private communication) that the conjecture is false in general.

We denote the set of extreme vectors in C by $E(C)$. It is known that $E(C) \subseteq \partial C$, the boundary of C, and that $C = \mathrm{hull}(E(C))$ (cf. [6, p. 167]). Note that hull S satisfies (i) and (ii) of 1.2, but not necessarily the others. The interior of C is denoted by int C.

DEFINITION 1.5. A cone C in $\mathbb{R}^n$ is called *decomposable* if there exist nonzero subcones C_1 and C_2 such that

(i) $C_1 + C_2 = C$, and

(ii) span $C_1 \cap$ span $C_2 = \{0\}$.

(Here span X is the linear span of a subset X of $\mathbb{R}^n$.) A cone that is not decomposable is called *indecomposable.*

Notation 1.6. Let C be a cone in $\mathbb{R}^n$. Then

(i) $\Gamma(C) = \{A \in \mathbb{R}^{nn}: AC \subseteq C\}$,

(ii) $\Theta(C) = \{A \in \mathbb{R}^{nn}: AC = C\}$,

(iii) $\Delta(C) = \{A \in E(\Gamma(C))$: rank $A \leqslant 1\}$.

It is known that $\Gamma(C)$ is a cone in $\mathbb{R}^{nn}$ (Schneider and Vidyasagar [8]). For $A, B \in \mathbb{R}^{nn}$ we write $A \geqslant B$ if $A - B \in \Gamma(C)$.

2. COPOSITIVITY AND THE ICE CREAM CONE

For $x \in \mathbb{R}^n$, we shall always use the Euclidean norm:

$$\| x \| = (x_1{}^2 + \cdots + x_n{}^2)^{1/2}.$$

The n-dimensional *ice cream cone* K_n is defined by

$$K_n = \{x \in \mathbb{R}^n: (x_1{}^2 + \cdots + x_{n-1}^2)^{1/2} \leqslant x_n\}.$$

Equivalently, $x \in K_n$ if and only if $\| x \| \leqslant \sqrt{2}\, x_n$. It is well known that K_n is a cone. Further, K_n is self-dual, viz. $K_n' = K_n$. Since the norm $\| \cdot \|$ is strictly convex (i.e., if $\| x \| = \| y \| = 1$, then $\| x + y \| = 2$ if and only if $x = y$), it follows that $\partial K_n = E(K_n)$ and that the cone K_n is indecomposable for $n \geqslant 3$ (Barker and Schneider [1], Loewy and Schneider [5]).

Let Σ_n be the set of symmetric matrices in $\mathbb{R}^{nn}$. If $H_1, H_2 \in \Sigma_n$, we shall write

$$H_1 \geqq H_2,$$

to mean that $H_1 - H_2$ is positive semidefinite (p.s.d.) and

$$H_1 \gg H_2,$$

to mean that $H_1 - H_2$ is positive definite (p.d.).

Let $J_n = \operatorname{diag}(-1, -1, \ldots, -1, 1) \in \Sigma_n$. We have

$$K_n = \{x \in \mathbb{R}^n : x^t J_n x \geqslant 0 \text{ and } x_n \geqslant 0\}.$$

DEFINITION 2.1. Let C be a cone in $\mathbb{R}^n$, and let $M \in \Sigma_n$. Then M is called *copositive for* C if $x^t M x \geqslant 0$ for all $x \in C$.

LEMMA 2.2. *Let K_n be the ice cream cone in $\mathbb{R}^n$, and let $M \in \Sigma_n$. Then M is copositive for K_n if and only if there is a $\mu \in \mathbb{R}$, $\mu \geqslant 0$, such that $M \geqq \mu J_n$.*

Proof. Let $\mu \geqslant 0$ and let $M \geqq \mu J_n$. Then $x^t J_n x \geqslant 0$ implies that

$$x^t M x \geqslant \mu x^t J_n x \geqslant 0.$$

Conversely, suppose that M is copositive for K_n. Let $x^t J_n x \geqslant 0$. Then $x \in K_n \cup (-K_n)$, whence $x^t M x \geqslant 0$.

Case 1. Let $n \geqslant 3$. We first consider the special case when

$$x^t J_n x \geqslant 0 \quad \text{and} \quad x \neq 0 \quad \text{imply that} \quad x^t M x > 0. \tag{2.2.1}$$

Then, by a result due to Greub–Milnor [4, p. 256] it follows that there exists a nonsingular T in $\mathbb{R}^{nn}$ such that $T^t J_n T$ and $T^t M T$ are diagonal matrices. By Sylvester's law of inertia (cf. [3, Vol. I, p. 297]) we may suppose that $T^t J_n T = J_n$. Let

$$T^t M T = D = \operatorname{diag}(d_1, d_2, \ldots, d_n).$$

It follows from (2.2.1) that

$$x^t J_n x \geqslant 0 \quad \text{and} \quad x \neq 0 \quad \text{imply that} \quad x^t D x > 0. \tag{2.2.2}$$

Let $x = x_i e^i + e^n$, where $1 \leqslant i < n$ and $|x_i| \leqslant 1$. Then

$$x^t J_n x = 1 - x_i^2 \geqslant 0.$$

Hence

$$x^t D x = d_n + d_i x_i^2 > 0.$$

Thus $d_n > 0$ and $d_n + d_i > 0$. Hence

$$D = d_n J_n + \operatorname{diag}(d_n + d_1, \ldots, d_n + d_{n-1}, 0) \geqq d_n J_n.$$

Thus,

$$M = (T^{-1})^t D T^{-1} \geqq d_n (T^{-1})^t J_n T^{-1} = d_n J_n.$$

We now turn to the general case. Let $P^{(r)} = M + (1/r)I$, for $r = 1, 2, \ldots$. Then $x^t J_n x \geq 0$, $x \neq 0$ imply that $x^t P^{(r)} x > 0$. Hence there exists a $\mu_r \geq 0$ and a p.s.d. $H^{(r)} \in \Sigma_n$ such that $P^{(r)} = \mu_r J_n + H^{(r)}$. Then

$$m_{nn} + 1 \geq p_{nn}^{(r)} = \mu_r + h_{nn}^{(r)} \geq \mu_r .$$

Hence $\mu_1, \mu_2, \ldots$, are bounded and we may select a subsequence that converges to a nonnegative μ. Since the limit of the same subsequence of $P^{(1)}, P^{(2)}, \ldots$, is M, it follows that $M \geqq \mu J_n$.

Case 2. Let $n = 2$. In this case the Greub–Milnor theorem cannot be applied. Let

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix},$$

be copositive and let $x^t = (x_1, 1)$, where $|x_1| \leq 1$. We have

$$f(x_1) = x^t M x = m_{11} x_1^2 + 2m_{12} x_1 + m_{22} \geq 0.$$

Let

$$f(c) = \min\{f(x_1): -1 \leq x_1 \leq 1\}.$$

Case 2a. $|c| < 1$. In this case we must have $m_{11} > 0$ and $|m_{12}| < m_{11}$. Thus $c = -m_{12}/m_{11}$ and $f(c) = m_{11}^{-1} \det M \geq 0$. Hence $M \geqq 0 = \mu J_2$, where $\mu = 0$.

Case 2b. $|c| = 1$. In this case either $m_{11} \leq 0$ or $0 \leq m_{11} \leq |m_{12}|$. Then $f(c) = m_{11} - 2|m_{12}| + m_{22} \geq 0$. Choose $\mu = \frac{1}{2}(m_{22} - m_{11})$. Then

$$\mu \geq |m_{12}| - m_{11} \geq 0$$

and

$$M - \mu J_2 = \begin{bmatrix} m_{11} + \mu & m_{12} \\ m_{12} & m_{22} - \mu \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(m_{11} + m_{22}) & m_{12} \\ m_{12} & \frac{1}{2}(m_{11} + m_{22}) \end{bmatrix} \geqq 0.$$

The next theorem essentially characterizes them atrices which map K_n into K_n or K_n into $-K_n$.

THEOREM 2.3. *Let $A \in \mathbb{R}^{nn}$. If $A \in \Gamma(K_n) \cup (-\Gamma(K_n))$, then*

$$A^t J_n A \geqq \mu J_n \qquad \text{for some } \mu \geq 0. \tag{2.3.1}$$

Conversely, if rank $A \neq 1$ *and there is a $\mu \geq 0$ such that* (2.3.1) *holds, then*

$$A \in \Gamma(K_n) \cup (-\Gamma(K_n)).$$

Proof. Let $A \in \Gamma(K_n) \cup (-\Gamma(K_n))$. Then $x^t J_n x \geqslant 0$ implies that

$$x^t A^t J_n A x = (Ax)^t J_n (Ax) \geqslant 0,$$

whence $A^t J_n A$ is copositive for K_n. Thus (2.3.1) holds, by Lemma 2.2.

Conversely, suppose (2.3.1) holds and that rank $A > 1$. Let $x \in K_n$. Then $(Ax)^t J_n A x \geqslant \mu x^t J_n x \geqslant 0$, whence $Ax \in K_n \cup -K_n$. We must show that either $Ax \in K_n$ for all $x \in K_n$ or $Ax \in -K_n$ for all $x \in K_n$. So suppose these exist $x^1, x^2 \in K_n$ such that $y^1 = Ax^1 \in K_n$, $y^2 = Ax^2 \in -K_n$ and $y^1, y^2 \neq 0$. Let $x^\alpha = \alpha x^1 + (1-\alpha) x^2$, where $0 \leqslant \alpha \leqslant 1$, and let

$$F_1 = \{\alpha \in [0, 1]: Ax^\alpha \in K_n\},$$
$$F_2 = \{\alpha \in [0, 1]; Ax^\alpha \in -K_n\}.$$

Since F_1 and F_2 are closed nonempty intervals in $[0, 1]$ and $F_1 \cup F_2 = [0, 1]$, there exists a β, $0 < \beta < 1$, such that $\beta \in F_1 \cap F_2$. Thus $Ax^\beta \in K_n \cap (-K_n)$, whence $Ax^\beta = 0$. Thus, $\beta y^1 + (1-\beta) y^2 = 0$, and y^1 and y^2 are linearly dependent. It follows that rank $A = 1$, contrary to assumption.

In our next theorem we characterize the matrices that map K_n onto itself.

THEOREM 2.4. *Let $A \in \mathbb{R}^{nn}$. Then $A \in \Theta(K_n) \cup (-\Theta(K_n))$ if and only if*

$$A^t J_n A = \mu J_n \qquad \textit{for some } \mu > 0. \tag{2.4.1}$$

Proof. Suppose (2.4.1) holds. Then A is nonsingular and

$$(A^{-1})^t J_n A^{-1} = \mu^{-1} J_n.$$

Hence, by Theorem 2.3, both A and A^{-1} belong to $\Gamma(K_n) \cup (-\Gamma(K_n))$. It follows easily that either both A and A^{-1} belong to $\Gamma(K_n)$ or both A and A^{-1} belong to $-\Gamma(K_n)$. Thus, either $A \in \Theta(K_n)$ or $A \in -\Theta(K_n)$.

Conversely, suppose that $A \in \Theta(K_n) \cup (-\Theta(K_n))$. Then A and A^{-1} belong to $\Gamma(K_n) \cup (-(\Gamma(K_n))$. By Theorem 2.3, there exist $H \geqq 0$, $P \geqq 0$, $\mu \geqslant 0$ and $\nu \geqslant 0$ such that

$$A^t J_n A = \mu J_n + H \qquad \text{and} \qquad (A^{-1})^t J_n A^{-1} = \nu J_n + P.$$

Hence

$$\begin{aligned} J_n &= \mu (A^{-1})^t J_n A^{-1} + (A^{-1})^t H A^{-1} \\ &= \mu\nu J_n + \mu P + (A^{-1})^t H A^{-1}. \end{aligned}$$

So $(1 - \mu\nu) J_n \geqq 0$. Thus $\mu\nu = 1$, whence $\mu P + (A^{-1})^t H A^{-1} = 0$. Hence $P = 0$ and $H = 0$, and (2.4.1) holds.

3. Basic Lemmas on $\Gamma(K_n)$ and $\Theta(K_n)$

Lemma 3.1. *Let* rank $A = 1$. *Then the following are equivalent:*

(1) *There exist* $u, v \in K_n$ *such that* $A = uv^t$,

(2) $A \in \Gamma(K_n)$.

Proof. (1) $\Rightarrow$ (2). If $A = uv^t$, where $u, v \in K_n$, then for every $x \in K_n$ we have $Ax = (v^t x)\, u \in K_n$, since $K_n' = K_n$.

(2) $\Rightarrow$ (1). Let $A \in \Gamma(K_n)$. Then $A = uv^t$ for some $u, v \in \mathbb{R}^n$. There exists $x \in K_n$ sueh that $Ax \neq 0$. Then $Ax = (v^t x)\, u \in K_n$, whence $u \in K_n \cup (-K_n)$. If $u \in -K_n$, we replace u by $-u$, v by $-v$, so we can assume that $u \in K_n$. Then for all $x \in K_n$ $Ax = (v^t x)\, u \in K_n$, whence $v \in K_n' = K_n$.

Lemma 3.2. *Let* $A \in \mathbb{R}^{n,n}$. *Then the following are equivalent:*

(1) *There exist* $u, v \in \partial K_n$ *such that* $A = uv^t$,

(2) $A \in \Delta(K_n)$.

Proof. (1) $\Rightarrow$ (2). Let $0 \leqslant B \leqslant uv^t$ in the ordering of $\Gamma(K_n)$. Let $x \in K_n$. Then $0 \leqslant Bx \leqslant (v^t x)\, u$. Hence, as $u \in E(K_n)$, we have $Bx = \alpha_x u$ for some $\alpha_x \geqslant 0$. Thus $B = uw^t$ for some $w \in \mathbb{R}^n$. But by Lemma 3.1 $w \in K_n$. Thus

$$0 \leqslant (w^t x)\, u \leqslant (v^t x)\, u \qquad \text{for all } x \in K_n,$$

so $w^t x \leqslant v^t x$ for all $x \in K_n$. Hence $v - w \in K_n' = K_n$, or, equivalently, $0 \leqslant w \leqslant v$. Since $v \in E(K_n)$, we have $w = \beta v$ for some $\beta \geqslant 0$. Thus $B = \beta A$, and so $A \in \Delta(K_n)$.

(2) $\Rightarrow$ (1). Let $A \in \Delta(K_n)$. If $A = 0$, choose $u = 0$ and any $v \in \partial K_n$. So we may assume that rank $A = 1$. By Lemma 3.1, there exist $u, v \in K_n$ such that $A = uv^t$. We prove that $u, v \in \partial K_n$. Suppose $u \notin \partial K_n$. Then there exists a w, $0 \leqslant w \leqslant u$, such that for any $\beta \geqslant 0$, $w \neq \beta u$. Hence $0 \leqslant wv^t \leqslant uv^t$ and $wv^t \neq \beta A$ for any $\beta \geqslant 0$. It follows that A is not an extreme matrix of $\Gamma(K_n)$, contrary to assumption. Thus $u \in \partial K_n$. Similarly one proves that $v \in \partial K_n$.

Lemma 3.3. *Let* $A \in \Gamma(K_n)$. *If there exists* $x \in \operatorname{int} K_n$ *such that* $Ax \in \partial K_n$, *then* rank $A \leqslant 1$.

Proof. Let $Ax = y$. Let $z \in \mathbb{R}^n$. Then there exists $\epsilon > 0$ such that $x \pm \epsilon z \in K_n$. Hence $A(x \pm \epsilon z) = y \pm \epsilon Az \in K_n$. But,

$$y = \tfrac{1}{2}(y + \epsilon Az) + \tfrac{1}{2}(y - \epsilon Az),$$

whence $\frac{1}{2}(y + \epsilon Az) \leqslant y$. Thus $\epsilon Az \leqslant y$, and since $y \in E(K_n)$, it follows that $Az = \alpha_z y$ for some $\alpha_z \geqslant 0$. Hence rank $A = 1$.

LEMMA 3.4. *Let* Q_1, $Q_2 \in \Theta(K_n)$. *Then* $A \in E(\Gamma(K_n))$ *if and only if* $Q_1 A Q_2 \in E(\Gamma(K_n))$.

Proof. Suppose $A \in E(\Gamma(K_n))$ and let $0 \leqslant B \leqslant Q_1 A Q_2$. Since $Q_1^{-1}, Q_2^{-1} \in \Gamma(K_n)$, it follows that $0 \leqslant Q_1^{-1} B Q_2^{-1} \leqslant A$, whence $Q_1^{-1} B Q_2^{-1} = \beta A$, where $0 \leqslant \beta \leqslant 1$. Hence $B = \beta Q_1 A Q_2$, and so $Q_1 A Q_2 \in E(\Gamma(K_n))$. The proof of the converse is similar.

Recall that e^n is the nth unit vector in $\mathbb{R}^n$.

LEMMA 3.5. *Let* $x \in \operatorname{int} K_n$. *Then there exists* $Q \in \Theta(K_n)$ *such that* $Qx = e^n$.

Proof, Case 1. $n = 2$. Let $x \in \operatorname{int} K_2$. Without loss of generality suppose that $x^t = (\alpha, 1)$, where $|\alpha| < 1$. Let

$$A = (1 - \alpha^2)^{-1} \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix}.$$

Then $Ax = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $A^t J_2 A = (1 - \alpha^2)^{-1} J_2$, whence by Theorem 2.4 A maps K_2 onto itself.

Proof, Case 2. $n > 2$. Let $x \in \operatorname{int} K_n$. Without loss of generality assume that $x^t = (x_1, x_2, \dots, x_{n-1}, 1)$, where $\|\tilde{x}\|^2 = \sum_{i=1}^{n-1} x_i^2 < 1$. [$\tilde{x}$ is defined in (1.1)]. There exists an orthogonal matrix $\tilde{Q} \in \mathbb{R}^{n-1,n-1}$ such that

$$\tilde{Q}\tilde{x} = \|\tilde{x}\| (0, 0, \dots, 0, 1)^t \in \mathbb{R}^{n-1}.$$

Let

$$Q_1 = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{nn}.$$

Then

$$y = Q_1 x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \|\tilde{x}\| \\ 1 \end{bmatrix} \in \operatorname{int} K_n,$$

and $Q_1^t J_n Q_1 = J_n$, so that $Q_1 \in \Theta(K_n)$.

By the case $n = 2$, there exists a $T \in \mathbb{R}^{22}$ such that $T^t J_2 T = \mu J_2$ for some $\mu > 0$, and

$$T \begin{bmatrix} \|\tilde{x}\| \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let

$$Q_2 = \begin{bmatrix} (\mu)^{1/2} I_{n-2} & 0 \\ 0 & T \end{bmatrix} \in \mathbb{R}^{nn}.$$

Then $Q_2 y = e^n$ and $Q_2{}^t J_n Q_2 = \mu J_n$, whence $Q_2 \in \Theta(K_n)$ by Theorem 2.4. Let $Q = Q_2 Q_1$. Then $Qx = e^n$ and $Q \in \Theta(K_n)$.

Thus we have shown that there are matriees mapping K_n onto itself which are far from rotations around the axis of the ice cream cone.

LEMMA 3.6. *Let*

$$\Pi = \{x \in \mathbb{R}^n : x_1 = 0\}. \tag{3.6.1}$$

If L is any $n-1$ dimensional subspace of $\mathbb{R}^n$ such that $L \cap \operatorname{int} K_n \neq \varnothing$ then there exists $Q \in \Theta(K_n)$ such that $QL = \Pi$.

Proof. Let $x \in L \cap \operatorname{int} K_n$. By Lemma 3.5, there exists a $Q_1 \in \Theta(K_n)$ such that $Q_1 x = e^n$. Let $L_1 = Q_1 L$. There exists a vector $b \in \mathbb{R}^n$ such that

$$\| b \| = 1 \qquad \text{and} \qquad L_1 = \{y \in \mathbb{R}^n : y^t b = 0\}.$$

Since $e^n \in L_1$, we have $b_n = 0$. Let $\tilde{b}$ be defined by (1.1). Then there exists an orthogonal matrix $\tilde{Q}_2 \in \mathbb{R}^{n-1,n-1}$ such that $\tilde{Q}_2 \tilde{b} = (1, 0,..., 0)^t \in \mathbb{R}^{n-1}$. Let

$$Q_2 = \begin{bmatrix} \tilde{Q}_2 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{nn}.$$

Then $Q_2 b = e^1$ and $Q_2 \in \Theta(K_n)$. We shall show that $Q_2 L_1 = \Pi$. So let $z \in L_1$. Then $z^t b = 0$, and hence $(Q_2 z)^t e^1 = z^t Q_2{}^t Q_2 b = z^t b = 0$. Thus $Q_2 z \in \Pi$, and $Q_2 L_1 \subseteq \Pi$. But dimension $(Q_2 L_1) = n - 1 =$ dimension (Π), so $Q_2 L_1 = \Pi$. If $Q = Q_2 Q_1$, then $Q \in \Theta(K_n)$ and $QL = \Pi$.

LEMMA 3.7. *Let*

$$\begin{aligned} u^n &= (0, 0,..., 1, 1)^t \in \mathbb{R}^n, \\ v^n &= (0, 0,..., -1, 1)^t \in \mathbb{R}^n. \end{aligned} \tag{3.7.1}$$

If x, y are linearly independent vectors belonging to ∂K_n, then there exists a matrix $Q \in \Theta(K_n)$ such that $Qx = u^n$ and $Qy = v^n$.

Proof. The proof is by induction on n. The case $n = 2$ is easy and the proof is omitted. So assume the result holds for K_{n-1}, and let x, y be linearly independent vectors belonging to ∂K_n. Let L be some $n-1$ dimensional subspace containing x and y. Since $x + y \in \operatorname{int} K_n$, it follows that $L \cap \operatorname{int} K_n \neq \varnothing$. By Lemma 3.6, there exists $Q_1 \in \Theta(K_n)$ such that $Q_1 L = \Pi$.

Let $Q_1 x = z$ and $Q_1 y = w$. Observe that z and w belong to ∂K_n, and $z_1 = w_1 = 0$. Hence $\hat{z}$ (see (1.1)), $\hat{w}$ belong to ∂K_{n-1}. Also, $\hat{z}$ and $\hat{w}$ are linearly independent in $\mathbb{R}^{n-1}$. Hence, by the induction hypothesis, there exists $\hat{Q} \in \Theta(K_{n-1})$ such that $\hat{Q}\hat{z} = u^{n-1}$ and $\hat{Q}\hat{w} = v^{n-1}$. By Theorem 2.4, $\hat{Q}^t J_{n-1} \hat{Q} = \mu J_{n-1}$ for some $\mu > 0$. Let $Q_2 = ((\mu)^{1/2}) \oplus \hat{Q}$. Then $Q_2 z = u^n$, $Q_2 w = v^n$ and $Q_2{}^t J_n Q_2 = \mu J_n$, whence $Q_2 \in \Theta(K_n)$. If $Q = Q_2 Q_1$, then Q has the required properties.

Thus, given any two linearly independent vectors x, y in ∂K_n, however close they are, there is a matrix A mapping K_n onto itself such that Ax and Ay are the given orthogonal vectors u^n and v^n.

4. The Extreme Matrices in $\Gamma(K_n)$

In this section we shall use the following result concerning the singular values of a matrix. For a proof in the complex case see [7, p. 349], the proof in the real case is similar.

Proposition 4.1. *Let $A \in \mathbb{R}^{nn}$. Then there exist orthogonal matrices P, $Q \in \mathbb{R}^{nn}$ such that*

$$PAQ = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1 \geqslant \lambda_2, \ldots, \geqslant \lambda_r > 0 = \lambda_{r+1} = \cdots = \lambda_n$ are the singular values of A and $r = \operatorname{rank} A$.

Lemma 4.2. *Let $A \in \mathbb{R}^{nn}$. If $A^t J_n A \geqq 0$, then* $\operatorname{rank} A \leqslant 1$.

Proof. (due to J. Bognár). Let rank $A = r$. Since $A^t J_n A \geqq 0$, then $Ax \in K_n \cup -K_n$ for every $x \in \mathbb{R}^n$. Suppose that $r \geqslant 2$. Then $K_n \cup -K_n$ contains a 2-dimensional subspace L. But L meets the $(n-1)$-dimensional subspace

$$\{x \in \mathbb{R}^n : x_n = 0\}$$

in a nonzero subspace. This contradicts the fact that if $x \in K_n$ and $x_n = 0$ then $x = 0$. Hence $r \leqslant 1$.

Corollary 4.3. *Let $A \in \mathbb{R}^{nn}$. If $A \in E(\Gamma(K_n))$ and $A^t J_n A \geqq 0$, then $A \in \Delta(K_n)$.*

Proof. Follows immediately from the previous Lemma and the definition of $\Delta(K_n)$.

LEMMA 4.4. *Let $G \geqslant 0$. Suppose that $Gu^n = 0$ and $Gz = 0$, where u^n is given by* (3.7.1) *and $z^t = (z_1, z_2, \dots, z_{n-2}, -1, 1)$. Then there exists $y \in \mathbb{R}^n$ such that*

$$y, u^n \text{ are linearly independent,} \tag{4.4.1}$$

$$y \in \partial K_n, \tag{4.4.2}$$

$$Gy = 0. \tag{4.4.3}$$

Proof. Let $y = \alpha u^n + z$, where $\alpha = \frac{1}{4} \sum_{i=1}^{n-2} z_i^2$. Then y has the required properties.

Our main theorem is Theorem 4.5.

THEOREM 4.5. *Let $n \geqslant 3$. Then*

$$E(\Gamma(K_n)) = \Theta(K_n) \cup \Delta(K_n).$$

Proof. (a) We first prove the inclusion

$$E(\Gamma(K_n)) \subseteq \Theta(K_n) \cup \Delta(K_n).$$

Let $0 \neq A \in E(\Gamma(K_n))$. Let

$$F = \{x \in \mathbb{R}^n : x \in \partial K_n \text{ and } Ax \in \partial K_n\},$$

and let L be the linear space spanned by F. Let $q =$ dimension L. We want to show that $q = n$. We shall show this by assuming that $q < n$ and obtaining a contradiction.

Case 1. $q = 0$. Let

$$S = \partial K_n \cap \{x \in \mathbb{R}^n : x_n = 1\}.$$

Then S is a compact subset of $\mathbb{R}^n$, in fact $\| x \| = \sqrt{2}$ for all $x \in S$. It follows that AS is also compact. By the assumption that $q = 0$, we have $AS \cap \partial K_n = \varnothing$. Since ∂K_n is closed, it follows that there is an $\epsilon > 0$ such that $\| Ax - y \| \geqslant \epsilon$, for all $x \in S$ and $y \in \partial K_n$. We deduce that if $x \in S$ and $\| Ax - z \| < \epsilon$, then $z \in K_n$. Now, let $B = A - (\epsilon/2) I$. If $x \in S$, then $\| Ax - Bx \| = (\epsilon/2) \| x \| < \epsilon$, whence $Bx \in K_n$. Thus $0 \leqslant B \leqslant A$. Suppose it were true that $B = \alpha A$, where $0 \leqslant \alpha \leqslant 1$. Then $(\epsilon/2) I = (1 - \alpha) A$, contradicting $q = 0$. Hence A is not extreme. It follows that $q \neq 0$.

Case 2. $q = 1$. Suppose that $Ae^n \in \partial K_n$. Then, by Lemma 3.3, rank $A \leqslant 1$, and $A \in \Delta(K_n)$. This implies $q = n$, by Lemma 3.2, a contradiction. Hence $Ae^n \in \operatorname{int} K_n$. By Lemmas 3.4 and 3.5 we may assume without loss of generality that $Ae^n = e^n$.

Since $q = 1$, there exists $x \in \partial K_n$ such that $x_n = 1$ and $Ax \in \partial K_n$. There exist orthogonal matrices

$$Q_i = \begin{bmatrix} \tilde{Q}_i & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{nn}, \qquad i = 1, 2,$$

such that $Q_1 x = u^n, Q_2(Ax) = \alpha u^n$ (where $\alpha = (Ax)_n$) and $Q_i e^n = e^n, i = 1, 2$. Hence, by replacing A by $Q_2 A Q_1^{-1}$, we may assume that $Ae^n = e^n$ and $Au^n = \alpha u^n$, where $\alpha \geqslant 0$. It follows that $Ae^{n-1} = \alpha u^n - e^n$, and so A can be partitioned

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{22} = \begin{bmatrix} \alpha & 0 \\ \alpha - 1 & 1 \end{bmatrix}.$$

We observe that $\alpha \neq 1$, for otherwise $Av^n = v^n$, contradicting $q = 1$. Let $B = A^t J_n A$. If we partition B conformably with A we obtain

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{bmatrix},$$

where

$$B_{11} = -A_{11}^t A_{11} + A_{21}^t J_2 A_{21},$$

$$B_{12} = \begin{bmatrix} (\alpha(a_{n1} - a_{n-1,1}) - a_{n1}) & a_{n1} \\ (\alpha(a_{n2} - a_{n-1,2}) - a_{n2}) & a_{n2} \\ \vdots & \\ (\alpha(a_{n,n-2} - a_{n-1,n-2}) - a_{n,n-2}) & a_{n,n-2} \end{bmatrix} \in \mathbb{R}^{n-2,2}$$

and

$$B_{22} = \begin{bmatrix} 1 - 2\alpha & \alpha - 1 \\ \alpha - 1 & 1 \end{bmatrix}.$$

By Theorem 2.3, $B = \mu J_n + H$ where $\mu \geqslant 0$ and $H \geqslant 0$. Since $(u^n)^t J_n u^n = 0$ and $Au^n = \alpha u^n$, we also have $(u^n)^t Bu^n = 0$, and so $(u^n)^t Hu^n = 0$. Since $H \geqslant 0$, it follows that $Hu^n = 0$ (cf. [3, p. 319]).

By computing the last component of $Hu^n = 0$, we conclude that $\alpha = \mu$. Since $h_{nn} = 1 - \mu$ and since $\mu = \alpha \neq 1$, we have $0 \leqslant \mu < 1$. But if $\mu = 0$ then $A^t J_n A \geqslant 0$, whence by Corollary 4.3 and Lemma 3.2 $A = zw^t$ for some vectors $z, w \in \partial K_n$. Thus $q = n$, which is impossible. Hence $0 < \mu < 1$. Computing the first $(n - 2)$ components of $Hu^n = 0$ we obtain

$$a_{n-1,i} = a_{ni}, \qquad i = 1, 2,..., n - 2.$$

Hence, if we partition H conformably with A we obtain

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^t & H_{22} \end{bmatrix},$$

where

$$H_{11} = B_{11} + \mu I_{n-2},$$

$$H_{12} = B_{12} = \begin{bmatrix} -a_{n1} & a_{n1} \\ -a_{n2} & a_{n2} \\ \vdots & \vdots \\ -a_{n,n-2} & a_{n,n-2} \end{bmatrix},$$

and

$$H_{22} = \begin{bmatrix} 1-\mu & \mu - 1 \\ \mu - 1 & 1 - \mu \end{bmatrix}.$$

Let $D = A - \epsilon u^n(v^n)^t$, where $\epsilon > 0$. Since $\epsilon(u^n)(v^n)^t \geqslant 0$, we have $D \leqslant A$. We shall show that for sufficiently small ϵ, $D \geqslant 0$. We do this by showing that there exists $P \geqq 0$ such that $D^t J_n D = \mu J_n + P$. Since $Hu^n = 0$, we have

$$A^t J_n A u^n = \mu J_n u^n + H u^n = \mu J_n u^n.$$

But $Au^n = \mu u^n$ and $\mu > 0$, whence $A^t J_n u^n = J_n u^n$. Thus,

$$\begin{aligned} D^t J_n D &= A^t J_n A - \epsilon[v^n(u^n)^t J_n A + A^t J_n u^n(v^n)^t] + \epsilon^2 v^n(u^n)^t J_n u^n(v^n)^t \\ &= A^t J_n A - 2\epsilon v^n(v^n)^t, \end{aligned}$$

where from (3.7.1)

$$v^n(v^n)^t = 0_{n-2} \oplus \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let $D^t J_n D = \mu J_n + P$. Then, since $A^t J_n A = \mu J_n + H$, it follows that $P = H - 2\epsilon v^n(v^n)^t$. Thus, partitioning P in the same way as H,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^t & P_{22} \end{bmatrix},$$

we have $P_{11} = H_{11}$, $P_{12} = H_{12}$ and

$$P_{22} = \begin{bmatrix} 1 - \mu - 2\epsilon & \mu - 1 + 2\epsilon \\ \mu - 1 + 2\epsilon & 1 - \mu - 2\epsilon \end{bmatrix}.$$

We shall show that for sufficiently small and positive ϵ, $P \geqq 0$. Let $p^{(i)}$ [or $h^{(i)}$] be the ith column of P [or H]. Since $p^{(n)} = -p^{(n-1)}$, det $P = 0$ for all ϵ. Hence it is enough to prove that $\det P[i_1, i_2, ..., i_m] > 0$, where $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n - 1$ and $P[i_1, i_2, ..., i_m]$ denotes the principal submatrix of P based on rows and columns $i_1, i_2, ..., i_m$. If $i_m < n - 1$, then $\det P[i_1, i_2, ..., i_m] = \det H[i_1, i_2, ..., i_m] \geqslant 0$. So suppose that $i_m = n - 1$. If $\det H[i_1, i_2, ..., i_{m-1}, n - 1] > 0$, then

$$\det P[i_1, i_2, ..., i_{m-1}, n - 1] > 0$$

for ϵ sufficiently small and positive. So suppose that

$$\det H[i_1, i_2, ..., i_{m-1}, n - 1] = 0.$$

As can be shown by taking the mth compound of H (cf. [3, p. 19]), it follows that $h^{(i_1)}, ..., h^{(i_{m-1})}, h^{(n-1)}$ are linearly dependent. Hence there exist $\alpha_i \in \mathbb{R}$, $i = 1, 2, ..., m$, not all 0, such that

$$\alpha_1 h^{(i_1)} + \cdots + \alpha_{m-1} h^{(i_{m-1})} + \alpha_m h^{(n-1)} = 0. \tag{4.5.1}$$

Suppose that $\alpha_m \neq 0$. Then, since $h^{(n)} = -h^{(n-1)}$,

$$\alpha_1 h^{(i_1)} + \cdots + \alpha_{m-1} h^{(i_{m-1})} + \tfrac{1}{2}\alpha_m h^{(n-1)} - \tfrac{1}{2}\alpha_m h^{(n)} = 0.$$

Let $z \in \mathbb{R}^n$ such that

$$z_{i_1} = -2\alpha_m^{-1}\alpha_1, ..., z_{i_{m-1}} = -2\alpha_m^{-1}\alpha_{m-1},$$

$$-z_{n-1} = z_n = 1 \qquad \text{and} \qquad z_j = 0 \text{ elsewhere.}$$

Then $Hz = 0$ and the conditions of Lemma 4.4 are satisfied. Hence there exists a vector $y \in \mathbb{R}^n$ satisfying (4.4.1), (4.4.2) and (4.4.3) (with respect to H). But $(Ay)^t J_n(Ay) = \mu y^t J_n y + y^t H y = 0$. Hence $Ay \in \partial K_n$, contradicting $q = 1$. Thus $\alpha_m = 0$ in (4.5.1). It follows that $h^{(i_1)}, ..., h^{(i_{m-1})}$ are linearly dependent. Since $p^{(i_j)} = h^{(i_j)}$, $j = 1, 2, ..., m - 1$, we have

$$\det P[i_1, i_2, ..., i_{m-1}, n - 1] = 0.$$

Thus, for $\epsilon > 0$ sufficiently small $P \geqslant 0$. Since $Au^n = \mu u^n$ for some $0 < \mu < 1$ and $Ae^n = e^n$, we have rank $A \geqslant 2$. For sufficiently small and positive ϵ it follows that rank $D \geqslant 2$ and that $D = A - \epsilon u^n(v^n)^t$ is not a multiple of A. Since $D^t J_n D = \mu J_n + P$, it follows by Theorem 2.3 that $D \in \Gamma(K_n)$, contrary to $A \in E(\Gamma(K_n))$. Hence $q \neq 1$.

Case 3. $2 \leqslant q \leqslant n - 1$. There exist $x^1, x^2 \in F$ which are linearly independent and $Ax^1 = y^1 \in \partial K_n$, $Ax^2 = y^2 \in \partial K_n$. By Lemma 3.7 there exists $Q_1 \in \Theta(K_n)$ such that $Q_1 x^1 = u^n$ and $Q_1 x^2 = v^n$. Then $AQ_1^{-1}u^n = y^1$ and $AQ_1^{-1}v^n = y^2$. By Lemma 3.4 we may assume without loss of generality that $Au^n = y^1$ and $Av^n = y^2$.

Case 3a. Suppose that y^1 and y^2 are linearly independent. By a similar argument we may assume that $Au^n = u^n$ and $Av^n = v^n$. Hence

$$Ae^{n-1} = e^{n-1}, \qquad Ae^n = e^n. \tag{4.5.2}$$

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{n-2,n-2}$. It follows from (4.5.2) that $A_{12} = 0$ and $A_{22} = I_2$. Since $A \in \Gamma(K_n)$, there exist $\mu \geqslant 0$ and $H \geqq 0$ such that $A^t J_n A = \mu J_n + H$. Since $J_n = -I_{n-2} \oplus J_2$, we obtain

$$A^t J_n A = \begin{bmatrix} -A_{11}^t A_{11} + A_{21}^t J_2 A_{21} & A_{21}^t J_2 \\ J_2 A_{21} & J_2 \end{bmatrix} = \mu \begin{bmatrix} -I_{n-2} & 0 \\ 0 & J_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^t & H_{22} \end{bmatrix},$$

where $H_{11} \in \mathbb{R}^{n-2,n-2}$. Thus $(1-\mu) J_2 = H_{22} \geqq 0$, whence $\mu = 1$ and $H_{22} = 0$. It follows that $H_{12} = 0$. Hence $J_2 A_{21} = 0$, and so $A_{21} = 0$. Thus $A = A_{11} \oplus I_2$, and

$$-A_{11}^t A_{11} = -I_{n-2} + H_{11}. \tag{4.5.3}$$

We obtain $0 \leqq A_{11}^t A_{11} \leqq I_{n-2}$. If $A_{11}^t A_{11} = I_{n-2}$, then by (4.5.3) $H_{11} = 0$ and so $H = 0$. Thus $A^t J_n A = J_n$, whence by Theorem 2.4 and (4.5.2) $A \in \Theta(K_n)$. This contradicts $q < n$. Thus $A_{11}^t A_{11} \neq I_{n-2}$.

By Proposition 4.1 there exist $P, Q \in \mathbb{R}^{n-2,n-2}$ such that

$$PA_{11}Q = \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-2}).$$

It is easy to see that $0 \leqq \Lambda^t \Lambda \leqq I_{n-2}$ and $\Lambda^t \Lambda \neq I_{n-2}$. Thus $1 - \lambda_i^2 \geqslant 0$, $i = 1, 2, \ldots, n-2$ and $1 - \lambda_j^2 > 0$ for some j, $1 \leqslant j \leqslant n-2$, say $j = 1$. Choose $\epsilon > 0$ such that $|\lambda_1 \pm \epsilon| < 1$. Let

$$\Lambda(\pm\epsilon) = \operatorname{diag}(\lambda_1 \pm \epsilon, \lambda_2, \ldots, \lambda_{n-2}),$$

$$A_{11}(\pm\epsilon) = P^t \Lambda(\pm\epsilon) Q^t \qquad \text{and} \qquad A(\pm\epsilon) = A_{11}(\pm\epsilon) \oplus I_2.$$

Since $0 \leqq \Lambda^t(\pm\epsilon)\Lambda(\pm\epsilon) \leqq I_{n-2}$, we have $A(\pm\epsilon)^t J_n A(\pm\epsilon) \geqq J_n$. Since rank $A \geqslant 2$ it follows from Theorem 2.3 and (4.5.2) that $A(\pm\epsilon) \in \Gamma(K_n)$. But $A = \frac{1}{2}[A(\epsilon) + A(-\epsilon)]$. Now suppose that $A(\epsilon) = \alpha A$ for some $\alpha \in \mathbb{R}$.

Then $I_{22} = \alpha I_{22}$, whence $\alpha = 1$. However, $A(\epsilon) \neq A$. Thus A is not extreme, a contradiction.

Case 3b. Suppose that $y^1 = Au^n$ and $y^2 = Av^n$ are linearly dependent, say $y^1 = \alpha y$ and $y^2 = \beta y$, where $y \in \partial K_n$ and $\alpha, \beta \geqslant 0$. By Lemmas 3.4 and and 3.7 we may assume without loss of generality that $y = u^n$, so that $Au^n = \alpha u^n$, $Av^n = \beta u^n$. Since

$$Ae^n = \tfrac{1}{2}Au^n + \tfrac{1}{2}Av^n = \tfrac{1}{2}(\alpha + \beta)\, u^n$$

and

$$Ae^{n-1} = \tfrac{1}{2}Au^n - \tfrac{1}{2}Av^n = \tfrac{1}{2}(\alpha - \beta)\, u^n,$$

it follows that

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{22} = \tfrac{1}{2}\begin{bmatrix} \alpha - \beta & \alpha + \beta \\ \alpha - \beta & \alpha + \beta \end{bmatrix}.$$

Let $B = A^t J_n A$. Then $B \gtrsim \mu J_n$ for some $\mu \geqslant 0$, since $A \in \Gamma(K_n)$. If we partition B conformably with A, we obtain $B_{22} = A_{22}^t J_2 A_{22} = 0$. Hence, $0 \gtrsim \mu J_2$, whence $\mu = 0$. Thus $A^t J_n A \gtrsim 0$, and since $A \in E(\Gamma(K_n))$ we conclude from Corollary 4.3 that $A \in \Delta(K_n)$. An immediate application of Lemma 3.2 yields $q = n$, a contradiction. This completes the investigation of Case 3.

We have proved that $q = n$. Hence there exists a basis $x^1, x^2, \ldots, x^n$ for $\mathbb{R}^n$, such that $x^i \in \partial K_n$ and $Ax^i \in \partial K_n$, $i = 1, 2, \ldots, n$. Since $A \in \Gamma(K_n)$, there exist $H \gtrsim 0$ and $\mu \geqslant 0$ such that $A^t J_n A = \mu J_n + H$. Hence

$$0 = (Ax^i)^t J_n A x^i = \mu (x^i)^t J_n x^i + (x^i)^t H x^i = (x^i)^t H x^i.$$

It follows that $Hx^i = 0$, $i = 1, 2, \ldots, n$, and so $H = 0$. Thus $A^t J_n A = \mu J_n$. If $\mu > 0$, then by Theorem 2.4 $A \in \Theta(K_n)$. If $\mu = 0$, then by Corollary 4.3 $A \in \Delta(K_n)$. We have established the proof that

$$E(\Gamma(K_n)) \subseteq \Theta(K_n) \cup \Delta(K_n).$$

(b) We now prove that $E(\Gamma(K_n)) \supseteq \Theta(K_n) \cup \Delta(K_n)$. So let $A \in \Theta(K_n) \cup \Delta(K_n)$. If $A \in \Delta(K_n)$, then $A \in E(\Gamma(K_n))$ by definition. So suppose that $A \in \Theta(K_n)$. Since the cone K_n is indecomposable for $n \geqslant 3$ (see Barker and Schneider [1] and Loewy and Schneider [5]), it follows from Theorem 3.3 of Loewy and Schneider [5] that $A \in E(\Gamma(K_n))$. This completes the proof of the theorem.

Remark. Note that the condition $n \geqslant 3$ is required in the proof of Theorem 4.5. It is clear that $E(\Gamma(K_2)) = \Delta(K_2)$, since every nonsingular $A \in \Gamma(K_2)$ is the sum of two matrices in $\Gamma(K_2)$ of rank 1.

Theorem 4.5 characterizes the extreme matrices in $\Gamma(K_n)$, for $n \geqslant 3$. We can now determine the structure of $\Gamma(K_n)$:

COROLLARY 4.6. *Let* $n \geqslant 2$. *Then* $\Gamma(K_n) = \mathrm{hull}[\Theta(K_n) \cup \Delta(K_n)]$.

Proof. This is an immediate corollary of Theorem 4.5, the preceding remark and the well-known theorem that any cone in $\mathbb{R}^k$ is the hull of its extremes (cf. [6, p. 167]).

Finally, we show that $\Gamma(K_n)$ is determined by the matrices that map K_n onto itself.

THEOREM 4.7. *Let* $n \geqslant 2$. *Then* $\Gamma(K_n) = \mathrm{hull}(\mathrm{closure}\ \Theta(K_n))$.

Proof. By Corollary 4.6 it is enough to show that any matrix A in $\Delta(K_n)$ is a limit of matrices in $\Theta(K_n)$. So let $0 \neq A \in \Delta(K_n)$. Then by Lemma 3.2 there exist nonzero vectors $x, y \in \partial K_n$ such that $A = xy^t$. By Lemma 3.7, there exist $Q_1, Q_2 \in \Theta(K_n)$ such that $Q_1 x = u^n$ and $Q_2 y = u^n$. Let $B = \frac{1}{2} Q_1 A Q_2^t$. Then $B \in \Delta(K_n)$ and it is enough to show that B is a limit of matrices in $\Theta(K_n)$. But

$$B = \tfrac{1}{2} Q_1 A Q_2^t = \tfrac{1}{2} u^n (u^n)^t = 0_{n-2} \oplus \tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let

$$T_\epsilon = (2\epsilon)^{1/2} I_{n-2} \oplus \begin{bmatrix} \frac{1}{2} + \epsilon & \frac{1}{2} - \epsilon \\ \frac{1}{2} - \epsilon & \frac{1}{2} + \epsilon \end{bmatrix}.$$

Clearly, $\lim_{\epsilon \to 0} T_\epsilon = B$ and $T_\epsilon^t J_n T_\epsilon = 2\epsilon J_n$. Hence, by Theorem 2.4, $T_\epsilon \in \Theta(K_n)$, completing the proof.

Addendum to Positive Operators on the n-Dimensional Ice Cream Cone

After the completion of this paper J. Bognár informed us of a different and more elementary proof of Lemma (2.2). By modifying a proof found, for the complex case, in M. G. Kreĭn and Ju. L. Šmul'jan (Plus-operators in a space with indefinite metric, *Amer. Math. Soc. Transl.* **85** (1969), 93–113), one first proves: If $x^t J_n x = 0$ implies $x^t M x \geqslant 0$, then the relations $y^t J_n y < 0$ and $z^t J_n z > 0$ imply that $y^t M y / y^t J_n y \leqslant z^t M z / z^t J_n z$. Lemma 2.2 then follows easily.

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