Positive Operators on the $n$-Dimensional Ice Cream Cone*

RAPHAEL LOEWY†

Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706

AND

HANS SCHNEIDER

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Submitted by Garrett Birkhoff

Let $K_n = \{ x \in \mathbb{R}^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq x_n \}$ be the $n$-dimensional ice cream cone, and let $\Gamma(K_n)$ be the cone of all matrices in $\mathbb{R}^{n\times n}$ mapping $K_n$ into itself. We determine the structure of $\Gamma(K_n)$, and in particular characterize the extreme matrices in $\Gamma(K_n)$.

INTRODUCTION

In the vector space $\mathbb{R}^n$ of real $n$-tuples we consider the cone

$$K_n = \{ x \in \mathbb{R}^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq x_n \}.$$

A matrix $A \in \mathbb{R}^{n\times n}$ is a positive operator on $K_n$ if $AK_n \subseteq K_n$. In this paper, we determine the structure of the cone $\Gamma(K_n)$ of positive operators on $K_n$. In Section 2, we show that $AK_n = K_n$ or $AK_n = -K_n$ if and only if $A^t J_n A = \mu J_n$, for some $0 < \mu \in \mathbb{R}$, where $J_n = \text{diag}(-1, \ldots, -1, 1)$. Further, if rank $A = 1$, then $AK_n \subseteq K_n$ or $AK_n \subseteq -K_n$, if and only if, for some $\mu \geq 0$, $A^t J_n A - \mu J_n$ is a positive semidefinite symmetric matrix. These are related to the notion of copositivity for $K_n$. Section 3 contains some basic lemmas. Lemmas 3.5 and 3.7 exhibit interesting properties of matrices mapping $K_n$ onto itself. In Section 4 we prove our main result: The extreme matrices in the cone $\Gamma(K_n)$ are precisely the matrices $A$ which map

* The research of the authors was supported in part by NSF Grant No. GP-17815, and in part by the United States Army under Contract No. DA-31-124-ARO-D-462.
† Current address: Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916.

Copyright © 1975 by Academic Press, Inc. All rights of reproduction in any form reserved.
onto $K_n$ together with the matrices of the form $uv'$, where $u$ and $v$ belong to the boundary of $K_n$. As a corollary, we show that $\Gamma(K_n)$ is the closure of the convex hull of all matrices mapping $K_n$ onto $K_n$.

We have the following conjecture. Let $C$ be a cone in $\mathbb{R}^n$ (see Definition 1.2), and let $\Gamma(C)$ be the cone of positive operators on $C$. If $A$ is an extreme matrix of $\Gamma(C)$, then $A$ maps the set of extreme vectors of $C$ into itself. The converse of this conjecture is proved in Loewy and Schneider [5] for a nonsingular $A$ and an indecomposable cone $C$. The converse is in general false for a singular matrix $A$ or a decomposable cone $C$.

1. Preliminaries

Let $\mathbb{R}$ be the real field, and let $\mathbb{R}^n$ be the vector space of all real column vectors $x = (x_1, \ldots, x_n)^t$. (We use $x^t$ for the transpose of $x$ and we shall always assume that $n \geq 2$.) By $e_i$ we denote the $i$th standard unit vector in $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, let the vectors $\bar{x}$ and $\bar{x}'$ in $\mathbb{R}^{n-1}$ be given by

$$\bar{x} = (x_1, x_2, \ldots, x_{n-1}), \quad \bar{x}' = (x_2, x_3, \ldots, x_n).$$

**Definition 1.2.** A cone $C$ in $\mathbb{R}^n$ is a nonempty subset of $\mathbb{R}^n$ such that

1. $C + C \subseteq C$,
2. $\lambda C \subseteq C$, for $0 \leq \lambda \in \mathbb{R}$,
3. $C \cap (-C) = \{0\}$,
4. $C - C = \mathbb{R}^n$,
5. $C$ is closed.

We write $x \geq y$ if $x - y \in C$.

If $C$ is a cone, then so is

$$C' = \{y \in \mathbb{R}^n : y'x \geq 0, \forall x \in C\},$$

and $C'$ is called the cone dual to $C$. For the sake of precision, we define:

**Definition 1.3.** Let $C$ be a cone in $\mathbb{R}^n$. If $x \in C$, then $x$ is an extreme vector in $C$ if $0 \leq y \leq x$ implies that $y = \alpha x$, for some $\alpha \in \mathbb{R}$.

**Definition 1.4.** Let $S$ be a nonempty subset of $\mathbb{R}^n$. Then $x \in \text{hull } S$ if and only if there exist $x_i \in S$, $0 \leq \lambda_i \in \mathbb{R}$, $i = 1, \ldots, r$, such that

$$x = \sum_{i=1}^{r} \lambda_i x_i.$$

After the completion of this paper R. C. O'Brien has shown in an interesting counterexample (private communication) that the conjecture is false in general.
We denote the set of extreme vectors in \( C \) by \( E(C) \). It is known that \( E(C) \subseteq \partial C \), the boundary of \( C \), and that \( C = \text{hull}(E(C)) \) (cf. [6, p. 167]). Note that hull \( S \) satisfies (i) and (ii) of 1.2, but not necessarily the others. The interior of \( C \) is denoted by \( \text{int} \ C \).

**Definition 1.5.** A cone \( C \) in \( \mathbb{R}^n \) is called *decomposable* if there exist nonzero subcones \( C_1 \) and \( C_2 \) such that

(i) \( C_1 + C_2 = C \), and

(ii) \( \text{span} \ C_1 \cap \text{span} \ C_2 = \{0\} \).

(Here \( \text{span} \ X \) is the linear span of a subset \( X \) of \( \mathbb{R}^n \).) A cone that is not decomposable is called *indecomposable.*

**Notation 1.6.** Let \( C \) be a cone in \( \mathbb{R}^n \). Then

(i) \( \Gamma(C) = \{ A \in \mathbb{R}^{n \times n} : AC \subseteq C \} \),

(ii) \( \Theta(C) = \{ A \in \mathbb{R}^{n \times n} : AC = C \} \),

(iii) \( \Delta(C) = \{ A \in E(\Gamma(C)) : \text{rank} \ A \leq 1 \} \).

It is known that \( \Gamma(C) \) is a cone in \( \mathbb{R}^{n \times n} \) (Schneider and Vidyasagar [8]). For \( A, B \in \mathbb{R}^{n \times n} \) we write \( A \succcurlyeq B \) if \( A - B \in \Gamma(C) \).

## 2. Copositivity and the Ice Cream Cone

For \( x \in \mathbb{R}^n \), we shall always use the Euclidean norm:

\[
\| x \| = (x_1^2 + \cdots + x_n^2)^{1/2}.
\]

The \( n \)-dimensional *ice cream cone* \( K_n \) is defined by

\[
K_n = \{ x \in \mathbb{R}^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq x_n \}.
\]

Equivalently, \( x \in K_n \) if and only if \( \| x \| \leq \sqrt{2} x_n \). It is well known that \( K_n \) is a cone. Further, \( K_n \) is self-dual, viz. \( K_n^* = K_n \). Since the norm \( \| \cdot \| \) is strictly convex (i.e., if \( \| x \| = \| y \| = 1 \), then \( \| x + y \| = 2 \) if and only if \( x = y \)), it follows that \( \partial K_n = E(K_n) \) and that the cone \( K_n \) is indecomposable for \( n \geq 3 \) (Barker and Schneider [1], Loewy and Schneider [5]).

Let \( \Sigma_n \) be the set of symmetric matrices in \( \mathbb{R}^{n \times n} \). If \( H_1, H_2 \in \Sigma_n \), we shall write

\[
H_1 \succcurlyeq H_2,
\]

to mean that \( H_1 - H_2 \) is positive semidefinite (p.s.d.) and

\[
H_1 \succ H_2,
\]

to mean that \( H_1 - H_2 \) is positive definite (p.d.).
Let $J_n = \text{diag}(-1, -1, \ldots, -1, 1) \in \Sigma_n$. We have

$$K_n = \{x \in \mathbb{R}^n : x^t J_n x \geq 0 \text{ and } x_n \geq 0\}.$$

**Definition 2.1.** Let $C$ be a cone in $\mathbb{R}^n$, and let $M \in \Sigma_n$. Then $M$ is called **copositive** for $C$ if $x^t M x \geq 0$ for all $x \in C$.

**Lemma 2.2.** Let $K_n$ be the ice cream cone in $\mathbb{R}^n$, and let $M \in \Sigma_n$. Then $M$ is copositive for $K_n$ if and only if there is a $\mu \in \mathbb{R}$, $\mu \geq 0$, such that $M \succeq \mu J_n$.

**Proof.** Let $\mu \geq 0$ and let $M \succeq \mu J_n$. Then $x^t J_n x \geq 0$ implies that $x^t M x \geq \mu x^t J_n x \geq 0$.

Conversely, suppose that $M$ is copositive for $K_n$. Let $x^t J_n x \geq 0$. Then $x \in K_n \cup (-K_n)$, whence $x^t M x \geq 0$.

**Case 1.** Let $n \geq 3$. We first consider the special case when

$$x^t J_n x \geq 0 \quad \text{and} \quad x \neq 0 \quad \text{imply that} \quad x^t M x > 0. \quad (2.2.1)$$

Then, by a result due to Greub–Milnor [4, p. 256] it follows that there exists a nonsingular $T$ in $\mathbb{R}^{nn}$ such that $T^t J_n T$ and $T^t M T$ are diagonal matrices. By Sylvester’s law of inertia (cf. [3, Vol. I, p. 297]) we may suppose that $T^t J_n T = J_n$. Let

$$T^t M T = D = \text{diag}(d_1, d_2, \ldots, d_n).$$

It follows from (2.2.1) that

$$x^t J_n x \geq 0 \quad \text{and} \quad x \neq 0 \quad \text{imply that} \quad x^t D x > 0. \quad (2.2.2)$$

Let $x = x_i e^i + e^n$, where $1 \leq i < n$ and $|x_i| \leq 1$. Then

$$x^t J_n x = 1 - x_i^2 \geq 0.$$ 

Hence

$$x^t D x = d_n + d_i x_i^2 > 0.$$ 

Thus $d_n > 0$ and $d_n + d_i > 0$. Hence

$$D = d_n J_n + \text{diag}(d_n + d_1, \ldots, d_n + d_{n-1}, 0) \succeq d_n J_n.$$ 

Thus,

$$M = (T^{-1})^t D T^{-1} \succeq d_n (T^{-1})^t J_n T^{-1} =: d_n J_n.$$
We now turn to the general case. Let \( P^{(r)} = M + (1/r) I \), for \( r = 1, 2, \ldots \). Then \( x^t J_n x \geq 0, x \neq 0 \) imply that \( x^t P^{(r)} x > 0 \). Hence there exists a \( \mu_r > 0 \) and a p.s.d. \( H^{(r)} \in \Sigma_n \) such that \( P^{(r)} = \mu_r J_n + H^{(r)} \). Then

\[
m_{nn} + 1 \geq p^{(r)}_{nn} = \mu_r + h^{(r)}_{nn} \geq \mu_r.
\]

Hence \( \mu_1, \mu_2, \ldots \), are bounded and we may select a subsequence that converges to a nonnegative \( \mu \). Since the limit of the same subsequence of \( P^{(1)}, P^{(2)}, \ldots \), is \( M \), it follows that \( M \geq \mu J_n \).

Case 2. Let \( n = 2 \). In this case the Greub–Milnor theorem cannot be applied. Let

\[
M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix},
\]

be copositive and let \( x^t = (x_1, 1) \), where \( |x_1| \leq 1 \). We have

\[
f(x_1) = x^t M x = m_{11} x_1^2 + 2m_{12} x_1 + m_{22} \geq 0.
\]

Let

\[
f(c) = \min \{ f(x_1): -1 \leq x_1 \leq 1 \}.
\]

Case 2a. \( |c| < 1 \). In this case we must have \( m_{11} > 0 \) and \( |m_{12}| < m_{11} \). Thus \( c = -m_{12}/m_{11} \) and \( f(c) = m_{11}^{-1} \det M \geq 0 \). Hence \( M \geq 0 = \mu J_2 \), where \( \mu = 0 \).

Case 2b. \( |c| = 1 \). In this case either \( m_{11} \leq 0 \) or \( 0 \leq m_{11} \leq |m_{12}| \). Then \( f(c) = m_{11} - 2 |m_{12}| + m_{22} \geq 0 \). Choose \( \mu = \frac{1}{2}(m_{22} - m_{11}) \). Then

\[
\mu \geq |m_{12}| - m_{11} \geq 0
\]

and

\[
M - \mu J_2 = \begin{bmatrix} m_{11} + \mu & m_{12} \\ m_{12} & m_{22} - \mu \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(m_{11} + m_{22}) & m_{12} \\ m_{12} & \frac{1}{2}(m_{11} + m_{22}) \end{bmatrix} \geq 0.
\]

The next theorem essentially characterizes them atrices which map \( K_n \) into \( K_n \) or \( K_n^{*} \) into \( -K_n^{*} \).

**Theorem 2.3.** Let \( A \in \mathbb{R}^{mn} \). If \( A \in \Gamma(K_n) \cup (-\Gamma(K_n)) \), then

\[
A^t J_n A \geq \mu J_n \quad \text{for some } \mu \geq 0.
\]

(2.3.1)

Conversely, if rank \( A \neq 1 \) and there is a \( \mu \geq 0 \) such that (2.3.1) holds, then

\[
A \in \Gamma(K_n) \cup (-\Gamma(K_n)).
\]
Proof. Let \( A \in \Gamma(K_n) \cup (-\Gamma(K_n)) \). Then \( x^tJ_n x \geq 0 \) implies that

\[ x^tA^tJ_nAx = (Ax)^tJ_n(Ax) \geq 0, \]

whence \( A^tJ_nA \) is copositive for \( K_n \). Thus (2.3.1) holds, by Lemma 2.2.

Conversely, suppose (2.3.1) holds and that rank \( A \geq 1 \). Let \( x \in K_n \).
Then \( (Ax)^tJ_nAx \geq \mu x^tJ_nx \geq 0 \), whence \( Ax \in K_n \cup -K_n \). We must show that either \( Ax \in K_n \) for all \( x \in K_n \) or \( Ax \in -K_n \) for all \( x \in K_n \). So suppose these exist \( x^1, x^2 \in K_n \) such that \( y^1 = Ax^1 \in K_n \), \( y^2 = Ax^2 \in -K_n \) and \( y^1, y^2 \neq 0 \). Let \( x^o = \alpha x^1 + (1 - \alpha)x^2 \), where \( 0 \leq \alpha \leq 1 \), and let

\[
F_1 = \{ \alpha \in [0, 1] : Ax^o \in K_n \},
F_2 = \{ \alpha \in [0, 1] : Ax^o \in -K_n \}.
\]

Since \( F_1 \) and \( F_2 \) are closed nonempty intervals in \([0, 1]\) and \( F_1 \cup F_2 = [0, 1] \), there exists a \( \beta \), \( 0 < \beta < 1 \), such that \( \beta \in F_1 \cap F_2 \). Thus \( Ax^o \in K_n \cap (-K_n) \), whence \( Ax^o = 0 \). Thus, \( \beta y^1 + (1 - \beta)y^2 = 0 \), and \( y^1 \) and \( y^2 \) are linearly dependent. It follows that rank \( A = 1 \), contrary to assumption.

In our next theorem we characterize the matrices that map \( K_n \) onto itself.

**Theorem 2.4.** Let \( A \in \mathbb{R}^{n \times n} \). Then \( A \in \Theta(K_n) \cup (-\Theta(K_n)) \) if and only if

\[ A^tJ_nA = \mu J_n \quad \text{for some } \mu > 0. \tag{2.4.1} \]

**Proof.** Suppose (2.4.1) holds. Then \( A \) is nonsingular and

\[ (A^{-1})^tJ_nA^{-1} = \mu^{-1}J_n. \]

Hence, by Theorem 2.3, both \( A \) and \( A^{-1} \) belong to \( \Gamma(K_n) \cup (-\Gamma(K_n)) \). It follows easily that either both \( A \) and \( A^{-1} \) belong to \( \Gamma(K_n) \) or both \( A \) and \( A^{-1} \) belong to \( -\Gamma(K_n) \). Thus, either \( A \in \Theta(K_n) \) or \( A \in -\Theta(K_n) \).

Conversely, suppose that \( A \in \Theta(K_n) \cup (-\Theta(K_n)) \). Then \( A \) and \( A^{-1} \) belong to \( \Gamma(K_n) \cup (-\Gamma(K_n)) \). By Theorem 2.3, there exist \( H \geq 0, P \geq 0, \mu \geq 0 \) and \( v \geq 0 \) such that

\[ A^tJ_nA = \mu J_n + H \quad \text{and} \quad (A^{-1})^tJ_nA^{-1} = vJ_n + P. \]

Hence

\[
J_n = \mu(A^{-1})^tJ_nA^{-1} + (A^{-1})^tHA^{-1} = \mu vJ_n + \mu P + (A^{-1})^tHA^{-1}.
\]

So \( (1 - \mu v) J_n \geq 0 \). Thus \( \mu v = 1 \), whence \( \mu P + (A^{-1})^tHA^{-1} = 0 \). Hence \( P = 0 \) and \( H = 0 \), and (2.4.1) holds.
LEMMA 3.1. Let \( \text{rank } A = 1 \). Then the following are equivalent:

1. There exist \( u, v \in K_n \) such that \( A = uv^t \),

2. \( A \in \Gamma(K_n) \).

Proof. (1) \( \Rightarrow \) (2). If \( A = uv^t \), where \( u, v \in K_n \), then for every \( x \in K_n \) we have \( Ax = (v^tx)u \in K_n \), since \( K_n \cap K_n' = K_n \).

(2) \( \Rightarrow \) (1). Let \( A \in \Gamma(K_n) \). Then \( A = uv^t \) for some \( u, v \in \mathbb{R}^n \). There exists \( x \in K_n \) such that \( Ax \neq 0 \). Then \( Ax = (v^tx)u \in K_n \), whence \( u \in K_n \cap (-K_n) \). If \( u \in -K_n \), we replace \( u \) by \(-u\), \( v \) by \(-v\), so we can assume that \( u \in K_n \). Then for all \( x \in K_n \) \( Ax = (v^tx)u \in K_n \), whence \( v \in K_n' = K_n \).

LEMMA 3.2. Let \( A \in \mathbb{R}^{n \times n} \). Then the following are equivalent:

1. There exist \( u, v \in \partial K_n \) such that \( A = uv^t \),

2. \( A \in \Delta(K_n) \).

Proof. (1) \( \Rightarrow \) (2). Let \( 0 \leq B \leq uv^t \) in the ordering of \( \Gamma(K_n) \). Let \( x \in K_n \). Then \( 0 \leq Bx \leq (v^tx)u \). Hence, as \( u \in B(K_n) \), we have \( Bx = \alpha_x u \) for some \( \alpha_x \geq 0 \). Thus \( B = uv^t \) for some \( w \in \mathbb{R}^n \). But by Lemma 3.1 \( w \in K_n \). Thus

\[
0 \leq (w^tx)u \leq (v^tx)u \quad \text{for all } x \in K_n ,
\]

so \( w^tx \leq v^tx \) for all \( x \in K_n \). Hence \( v - w \in K_n' = K_n \), or, equivalently, \( 0 : w < v \). Since \( v \in B(K_n) \), we have \( w = \beta v \) for some \( \beta \geq 0 \). Thus \( B = \beta A \), and so \( A \in \Delta(K_n) \).

(2) \( \Rightarrow \) (1). Let \( A \in \Delta(K_n) \). If \( A = 0 \), choose \( u = 0 \) and any \( v \in \partial K_n \). So we may assume that \( \text{rank } A = 1 \). By Lemma 3.1, there exist \( u, v \in K_n \) such that \( A = uv^t \). We prove that \( u, v \in \partial K_n \). Suppose \( u \notin \partial K_n \). Then there exists a \( w, 0 \leq w < u \), such that for any \( \beta \geq 0 \) \( w \neq \beta u \). Hence \( 0 : uv^t \leq uv^t \) and \( uv^t \neq \beta A \) for any \( \beta \geq 0 \). It follows that \( A \) is not an extreme matrix of \( \Gamma(K_n) \), contrary to assumption. Thus \( u \in \partial K_n \). Similarly one proves that \( v \in \partial K_n \).

LEMMA 3.3. Let \( A \in \Gamma(K_n) \). If there exists \( x \in \text{int } K_n \) such that \( Ax \in \partial K_n \), then \( \text{rank } A = 1 \).

Proof. Let \( Ax = y \). Let \( z \in \mathbb{R}^n \). Then there exists \( \epsilon > 0 \) such that \( x \pm \epsilon z \in K_n \). Hence \( A(x \pm \epsilon z) = y \pm \epsilon Az \in K_n \). But,

\[
y = \frac{1}{2}(y + \epsilon Az) + \frac{1}{2}(y - \epsilon Az),
\]
whence \( \frac{1}{2}(y + \varepsilon Az) \leq y \). Thus \( \varepsilon Az \leq y \), and since \( y \in E(K_n) \), it follows that \( Az = \alpha_x y \) for some \( \alpha_x \geq 0 \). Hence \( \text{rank } A :< 1 \).

**Lemma 3.4.** Let \( Q_1, Q_2 \in \Theta(K_n) \). Then \( A \in E(\Gamma(K_n)) \) if and only if \( Q_1 A Q_2 \in E(\Gamma(K_n)) \).

**Proof.** Suppose \( A \in E(\Gamma(K_n)) \) and let \( 0 \leq B \leq Q_1 A Q_2 \). Since \( Q_1^{-1}, Q_2^{-1} \in \Gamma(K_n) \), it follows that \( 0 \leq Q_1^{-1} B Q_2^{-1} \leq A \), whence \( Q_1^{-1} B Q_2^{-1} = \beta A \), where \( 0 \leq \beta \leq 1 \). Hence \( B = \beta Q_1 A Q_2 \), and so \( Q_1 A Q_2 \in E(\Gamma(K_n)) \). The proof of the converse is similar.

Recall that \( e^n \) is the \( n \)-th unit vector in \( \mathbb{R}^n \).

**Lemma 3.5.** Let \( x \in \text{int } K_n \). Then there exists \( Q \in \Theta(K_n) \) such that \( Qx = e^n \).

**Proof.** Case 1. \( n = 2 \). Let \( x \in \text{int } K_2 \). Without loss of generality suppose that \( x^l = (\alpha, 1) \), where \( |\alpha| < 1 \). Let

\[
A = (1 - \alpha^2)^{-1} \begin{bmatrix} 1 & -\alpha \\ -\alpha & 1 \end{bmatrix}.
\]

Then \( Ax = [1] \) and \( A^t J_2 A = (1 - \alpha^2)^{-1} J_2 \), whence by Theorem 2.4 \( A \) maps \( K_2 \) onto itself.

**Proof.** Case 2. \( n > 2 \). Let \( x \in \text{int } K_n \). Without loss of generality assume that \( x^l = (x_1, x_2, \ldots, x_{n-1}, 1) \), where \( \|\tilde{x}\|^2 = \sum_{i=1}^{n-1} x_i^2 < 1 \). \([\tilde{x}] \) is defined in \((1.1)\). There exists an orthogonal matrix \( Q \in \mathbb{R}^{n-1,n-1} \) such that

\[
Q\tilde{x} = \|\tilde{x}\|(0, 0, \ldots, 0, 1)^t \in \mathbb{R}^{n-1}.
\]

Let

\[
Q_1 = \begin{bmatrix} \tilde{Q} & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n,n}.
\]

Then

\[
y = Q_1 x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \|\tilde{x}\| \\ 1 \end{bmatrix} \in \text{int } K_n,
\]

and \( Q_1^t J_n Q_1 = J_n \), so that \( Q_1 \in \Theta(K_n) \).

By the case \( n = 2 \), there exists a \( T \in \mathbb{R}^{2,2} \) such that \( T^t J_2 T = \mu J_2 \) for some \( \mu > 0 \), and

\[
T \begin{bmatrix} \|\tilde{x}\| \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
n-DIMENSIONAL ICE CREAM CONE

Let
\[ Q_2 = \begin{bmatrix} (\mu)^{1/3} I_{n-2} & 0 \\ 0 & T \end{bmatrix} \in \mathbb{R}^{nn}. \]

Then \( Q_2 y = e^n \) and \( Q_2^{1/3} J_n Q_2 = \mu J_n \), whence \( Q_2 \in \Theta(K_n) \) by Theorem 2.4.

Let \( Q = Q_2 Q_1 \). Then \( Qx = e^n \) and \( Q \in \Theta(K_n) \).

Thus we have shown that there are matrices mapping \( K_n \) onto itself which are far from rotations around the axis of the ice cream cone.

**Lemma 3.6.** Let
\[ \Pi = \{ x \in \mathbb{R}^n : x_1 = 0 \}. \tag{3.6.1} \]

If \( L \) is any \( n - 1 \) dimensional subspace of \( \mathbb{R}^n \) such that \( L \cap \text{int} K_n \neq \emptyset \) then there exists \( Q \in \Theta(K_n) \) such that \( QL = \Pi \).

**Proof.** Let \( x \in L \cap \text{int} K_n \). By Lemma 3.5, there exists a \( Q_1 \in \Theta(K_n) \) such that \( Q_1 x = e^n \). Let \( L_1 = Q_1 L \). There exists a vector \( b \in \mathbb{R}^n \) such that \( ||b|| = 1 \) and \( L_1 = \{ y \in \mathbb{R}^n : y'^t b = 0 \} \).

Since \( e^n \in L_1 \), we have \( b_n = 0 \). Let \( \delta \) be defined by (1.1). Then there exists an orthogonal matrix \( Q_2 \in \mathbb{R}^{n-1,n-1} \) such that \( Q_2 \delta = (1, 0, ..., 0)^t \in \mathbb{R}^{n-1} \). Let
\[ Q_2 = \begin{bmatrix} Q_2 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{nn}. \]

Then \( Q_2 b = e^1 \) and \( Q_2 \in \Theta(K_n) \). We shall show that \( Q_2 L_1 = \Pi \). So let \( x \in L_1 \).

Then \( x'^t b = 0 \), and hence \( (Q_2 x)^t e^1 = x'^t Q_2^t Q_2 b = x'^t b = 0 \). Thus \( Q_2 x \in \Pi \), and \( Q_2 L_1 \subseteq \Pi \). But dimension \( (Q_2 L_1) = n - 1 = \text{dimension} (\Pi) \), so \( Q_2 L_1 = \Pi \).

If \( Q = Q_2 Q_1 \), then \( Q \in \Theta(K_n) \) and \( QL = \Pi \).

**Lemma 3.7.** Let
\[ u^n = (0, 0, ..., 1, 1)^t \in \mathbb{R}^n, \]
\[ v^n = (0, 0, ..., -1, 1)^t \in \mathbb{R}^n. \tag{3.7.1} \]

If \( x, y \) are linearly independent vectors belonging to \( \partial K_n \), then there exists a matrix \( Q \in \Theta(K_n) \) such that \( Qx = u^n \) and \( Qy = v^n \).

**Proof.** The proof is by induction on \( n \). The case \( n = 2 \) is easy and the proof is omitted. So assume the result holds for \( K_{n-1} \), and let \( x, y \) be linearly independent vectors belonging to \( \partial K_n \). Let \( L \) be some \( n - 1 \) dimensional subspace containing \( x \) and \( y \). Since \( x \perp y \in \text{int} K_n \), it follows that \( L \cap \text{int} K_n \neq \emptyset \). By Lemma 3.6, there exists \( Q_1 \in \Theta(K_n) \) such that \( Q_1 L = \Pi \).
Let $Q_1 x = z$ and $Q_1 y = w$. Observe that $z$ and $w$ belong to $\partial K_n$, and $z_1 = w_1 = 0$. Hence $z$ (see (1.1)), $\hat{w}$ belong to $\partial K_{n-1}$. Also, $\hat{z}$ and $\hat{w}$ are linearly independent in $\mathbb{R}^{n-1}$. Hence, by the induction hypothesis, there exists $\hat{Q} \in \Theta(K_{n-1})$ such that $\hat{Q}\hat{z} = u^{n-1}$ and $\hat{Q}\hat{w} = v^{n-1}$. By Theorem 2.4, $Q^*_n J_{n-1} \hat{Q} = \mu J_{n-1}$ for some $\mu > 0$. Let $Q_x = ((\mu)^{1/2}) \hat{Q}$. Then $Q_x z = u^n$, $Q_x w = v^n$ and $Q_x^* J_n Q_x = \mu J_n$, whence $\hat{Q}\hat{Q} \in \Theta(K_n)$. If $Q = Q_x Q_1$, then $Q$ has the required properties.

Thus, given any two linearly independent vectors $x, y$ in $\partial K_n$, however close they are, there is a matrix $A$ mapping $K_n$ onto itself such that $Ax$ and $Ay$ are the given orthogonal vectors $u^n$ and $v^n$.

4. THE EXTREME MATRICES IN $\Gamma(K_n)$

In this section we shall use the following result concerning the singular values of a matrix. For a proof in the complex case see [7, p. 349], the proof in the real case is similar.

**Proposition 4.1.** Let $A \in \mathbb{R}^{n \times n}$. Then there exist orthogonal matrices $P, Q \in \mathbb{R}^{n \times n}$ such that

$$PAQ = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_r),$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_r > 0 = \lambda_{r+1} = ... = \lambda_n$ are the singular values of $A$ and $r = \text{rank } A$.

**Lemma 4.2.** Let $A \in \mathbb{R}^{n \times n}$. If $A^t J_n A \succeq 0$, then $\text{rank } A \leq 1$.

**Proof.** (due to J. Bognár). Let rank $A = r$. Since $A^t J_n A \succeq 0$, then $Ax \in K_n \cup -K_n$ for every $x \in \mathbb{R}^n$. Suppose that $r \geq 2$. Then $K_n \cup -K_n$ contains a 2-dimensional subspace $L$. But $L$ meets the $(n-1)$-dimensional subspace

$$\{x \in \mathbb{R}^n : x_n = 0\}$$

in a nonzero subspace. This contradicts the fact that if $x \in K_n$ and $x_n = 0$ then $x = 0$. Hence $r \leq 1$.

**Corollary 4.3.** Let $A \in \mathbb{R}^{n \times n}$. If $A \in E(\Gamma(K_n))$ and $A^t J_n A \succeq 0$, then $A \in \Delta(K_n)$.

**Proof.** Follows immediately from the previous lemma and the definition of $\Delta(K_n)$.
LEMMA 4.4. Let $G \geq 0$. Suppose that $Gu^n = 0$ and $Gz = 0$, where $u^n$ is given by (3.7.1) and $z^i = (z_1, z_2, \ldots, z_{n-2}, -1, 1)$. Then there exists $y \in \mathbb{R}^n$ such that

$$y, u^n \text{ are linearly independent},$$

$$y \in \partial K_n,$$

$$Gy = 0.$$  \hfill (4.4.1) \hfill (4.4.2) \hfill (4.4.3)

Proof. Let $y = \alpha u^n + z$, where $\alpha = \frac{1}{n} \sum_{i=1}^{n-2} z_i^2$. Then $y$ has the required properties.

Our main theorem is Theorem 4.5.

THEOREM 4.5. Let $n \geq 3$. Then

$$E(\Gamma(K_n)) = \Theta(K_n) \cup \Delta(K_n).$$

Proof. (a) We first prove the inclusion

$$E(\Gamma(K_n)) \subseteq \Theta(K_n) \cup \Delta(K_n).$$

Let $0 \neq A \in E(\Gamma(K_n))$. Let

$$F = \{ x \in \mathbb{R}^n : x \in \partial K_n \text{ and } Ax \in \partial K_n \},$$

and let $L$ be the linear space spanned by $F$. Let $q = \text{dimension } L$. We want to show that $q = n$. We shall show this by assuming that $q < n$ and obtaining a contradiction.

Case 1. $q = 0$. Let

$$S = \partial K_n \cap \{ x \in \mathbb{R}^n : x_n = 1 \}.$$  

Then $S$ is a compact subset of $\mathbb{R}^n$, in fact $\| x \| = \sqrt{2}$ for all $x \in S$. It follows that $AS$ is also compact. By the assumption that $q = 0$, we have $AS \cap \partial K_n = \emptyset$. Since $\partial K_n$ is closed, it follows that there is an $\epsilon > 0$ such that $\| Ax - y \| \geq \epsilon$, for all $x \in S$ and $y \in \partial K_n$. We deduce that if $x \in S$ and $\| Ax - z \| \leq \epsilon$, then $z \in K_n$. Now, let $B = A - (e/2) I$. If $x \in S$, then $\| Ax - Bx \| = (e/2) \| x \| \leq \epsilon$, whence $Bx \in K_n$. Thus $0 \leq B \leq A$. Suppose it were true that $B = \alpha A$, where $0 \leq \alpha \leq 1$. Then $(e/2) I = (1 - \alpha) A$, contradicting $q = 0$. Hence $A$ is not extreme. It follows that $q \neq 0$.

Case 2. $q = 1$. Suppose that $Ae^n \in \partial K_n$. Then, by Lemma 3.3, rank $A \leq 1$, and $A \in \Delta(K_n)$. This implies $q = n$, by Lemma 3.2, a contradiction. Hence $Ae^n \in \text{int } K_n$. By Lemmas 3.4 and 3.5 we may assume without loss of generality that $Ae^n = e^n$. 


Since \( q = 1 \), there exists \( x \in \partial K_n \) such that \( x_n = 1 \) and \( Ax \in \partial K_n \). There exist orthogonal matrices
\[
Q_i := \begin{bmatrix} Q_i & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad i = 1, 2,
\]
such that \( Q_1 x := u^n, Q_2 (Ax) = \alpha u^n \) (where \( \alpha := (Ax)_n \)) and \( Q_i e^n := e^n, i = 1, 2 \). Hence, by replacing \( A \) by \( Q_2AQ_1^{-1} \), we may assume that \( Ae^n := e^n \) and \( Au^n := \alpha u^n \), where \( \alpha \geqslant 0 \). It follows that \( Ae^{n-1} = \alpha u^n - e^n \), and so \( A \) can be partitioned
\[
A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},
\]
where
\[
A_{22} = \begin{bmatrix} \alpha \\ \alpha - 1 & 1 \end{bmatrix}.
\]

We observe that \( \alpha \neq 1 \), for otherwise \( Av^n = v^n \), contradicting \( q = 1 \). Let \( B = A^t J_n A \). If we partition \( B \) conformably with \( A \) we obtain
\[
B = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^t & B_{22} \end{bmatrix},
\]
where
\[
B_{11} = -A_{11}^t d_{11} + A_{21}^t J_n A_{21},
\]
\[
B_{12} = \begin{bmatrix} (\alpha(a_{n1} - a_{n-1,1}) - a_{n1}) \\ (\alpha(a_{n2} - a_{n-1,2}) - a_{n2}) \\ \vdots \\ (\alpha(a_{n,n-2} - a_{n-1,n-2}) - a_{n,n-2}) \end{bmatrix} \in \mathbb{R}^{n-2,2}
\]
and
\[
B_{22} = \begin{bmatrix} 1 - 2\alpha & \alpha - 1 \\ \alpha - 1 & 1 \end{bmatrix}.
\]

By Theorem 2.3, \( B = \mu f_n + H \) where \( \mu \geqslant 0 \) and \( H \geqslant 0 \). Since \( (u^n)^t f_n u^n = 0 \) and \( Au^n = \alpha u^n \), we also have \( (u^n)^t B w^n = 0 \), and so \( (u^n)^t H u^n = 0 \). Since \( H \geqslant 0 \), it follows that \( H u^n = 0 \) (cf. [3, p. 319]).

By computing the last component of \( Hu^n \), we conclude that \( \alpha = \mu \). Since \( h_{nn} = 1 - \mu \) and since \( \mu \geqslant \alpha \), we have \( 0 \leqslant \mu < 1 \). But if \( \mu = 0 \) then \( A^t f_n A \geqslant 0 \), whence by Corollary 4.3 and Lemma 3.2 \( A = zw^t \) for some vectors \( z, w \in \partial K_n \). Thus \( q = n \), which is impossible. Hence \( 0 < \mu < 1 \). Computing the first \((n - 2)\) components of \( Hu^n = 0 \) we obtain
\[
a_{n-1,i} = a_{ni}, \quad i = 1, 2, \ldots, n - 2.
\]
Hence, if we partition $H$ conformably with $A$ we obtain

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix},$$

where

$$H_{11} = B_{11} + \mu I_{n-2},$$

and

$$H_{12} = B_{12} = \begin{bmatrix} -a_{n1} & a_{n1} \\ -a_{n2} & a_{n2} \\ \vdots & \vdots \\ -a_{n,n-2} & a_{n,n-2} \end{bmatrix},$$

and

$$H_{22} = \begin{bmatrix} 1 - \mu & \mu - 1 \\ \mu - 1 & -\mu \end{bmatrix}.$$
all $\epsilon$. Hence it is enough to prove that $\det P[i_1, i_2, \ldots, i_m] > 0$, where $1 \leq i_1 < i_2 < \cdots < i_m < n - 1$, and $P[i_1, i_2, \ldots, i_m]$ denotes the principal submatrix of $P$ based on rows and columns $i_1, i_2, \ldots, i_m$. If $i_m < n - 1$, then $\det P[i_1, i_2, \ldots, i_m] = \det H[i_1, i_2, \ldots, i_m] = 0$. So suppose that $i_m = n - 1$. If $\det H[i_1, i_2, \ldots, i_{m-1}, n - 1] > 0$, then

$$\det P[i_1, i_2, \ldots, i_{m-1}, n - 1] > 0$$

for $\epsilon$ sufficiently small and positive. So suppose that

$$\det H[i_1, i_2, \ldots, i_{m-1}, n - 1] = 0.$$

As can be shown by taking the $m$th compound of $H$ (cf. [3, p. 19]), it follows that $h^{(i_1)}, \ldots, h^{(i_{m-1})}, h^{(n-1)}$ are linearly dependent. Hence there exist $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, not all 0, such that

$$\alpha_1 h^{(i_1)} + \cdots + \alpha_{m-1} h^{(i_{m-1})} + \alpha_m h^{(n-1)} = 0. \tag{4.5.1}$$

Suppose that $\alpha_m \neq 0$. Then, since $h^{(n)} = -h^{(n-1)}$,

$$\alpha_1 h^{(i_1)} + \cdots + \alpha_{m-1} h^{(i_{m-1})} + \frac{1}{2} \alpha_m h^{(n-1)} - \frac{1}{2} \alpha_m h^{(n)} = 0. $$

Let $z \in \mathbb{R}^n$ such that

$$z_{i_1} = -2\alpha_{m}^{-1} z_{i_2}, \ldots, z_{i_{m-1}} = -2\alpha_{m}^{-1} z_{i_{m-1}}, \quad z_{n-1} = z_n = 1 \quad \text{and} \quad z_j = 0 \text{ elsewhere.}$$

Then $Hz = 0$ and the conditions of Lemma 4.4 are satisfied. Hence there exists a vector $y \in \mathbb{R}^n$ satisfying (4.4.1), (4.4.2) and (4.4.3) (with respect to $H$). But $(Ay)^t f_n(Ay) = \mu y^t S_n y + y^t He = 0$. Hence $Ay \in \partial K_n$, contradicting $q = 1$. Thus $\alpha_m = 0$ in (4.5.1). It follows that $h^{(i_1)}, \ldots, h^{(i_{m-1})}$ are linearly dependent. Since $p[i_1, \ldots, i_{m-1}, n - 1, m - 1]$, we have

$$P[i_1, i_2, \ldots, i_{m-1}, n - 1] = 0.$$

Thus, for $\epsilon > 0$ sufficiently small $P \geq 0$. Since $Au^n = \mu u^n$ for some $0 < \mu < 1$ and $Ae^n = e^n$, we have rank $A \geq 2$. For sufficiently small and positive $\epsilon$ it follows that rank $D \geq 2$ and that $D - A - \epsilon u^n e^t$ is not a multiple of $A$. Since $D f_n D = \mu f_n + P$, it follows by Theorem 2.3 that $D \in E(K_n)$, contrary to $A \in E(G(K_n))$. Hence $q \neq 1$.

Case 3. $2 \leq q \leq n - 1$. There exist $x^1, x^2 \in F$ which are linearly independent and $Ax^1 = y^1 \in \partial K_n$, $Ax^2 = y^2 \in \partial K_n$. By Lemma 3.7 there
exists $Q_1 \in \Theta(K_n)$ such that $Q_1 x^1 = u^1$ and $Q_1 x^2 = v^2$. Then $AQ_1^{-1} u^1 = y^1$ and $AQ_1^{-1} v^2 = y^2$. By Lemma 3.4 we may assume without loss of generality that $Au^1 = y^1$ and $Av^2 = y^2$.

Case 3a. Suppose that $y^1$ and $y^2$ are linearly independent. By a similar argument we may assume that $Au^1 = u^1$ and $Av^2 = v^2$. Hence

$$Ae^{n-1} = e^{n-1}, \quad Ae^n = e^n.$$  \hspace{1cm} (4.5.2)

Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{n-2,n-2}$. It follows from (4.5.2) that $A_{12} = 0$ and $A_{22} = I_2$.

Since $A \in \Gamma(K_n)$, there exist $\mu \geq 0$ and $H \succeq 0$ such that $A^t J_n A = \mu J_n + H$.

Since $J_n := -I_{n-2} \oplus I_2$, we obtain

$$A^t J_n A = \left[ -A_{11}^t A_{11} + A_{21}^t J_2 A_{21} \begin{bmatrix} 0 & J_2 \\ J_2 & 0 \end{bmatrix} \right] = \mu \begin{bmatrix} -I_{n-2} & 0 \\ 0 & J_2 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^t & H_{22} \end{bmatrix},$$

where $H_{11} \in \mathbb{R}^{n-2,n-2}$. Thus $(1 - \mu) J_2 = H_{22} \succeq 0$, whence $\mu = 1$ and $H_{22} = 0$. It follows that $H_{12} = 0$. Hence $J_2 A_{21} = 0$, and so $A_{21} = 0$. Thus $A = A_{11} \oplus I_2$, and

$$-A_{11}^t A_{11} = -I_{n-2} + H_{11}. \hspace{1cm} (4.5.3)$$

We obtain $0 \leq A_{11}^t A_{11} \leq I_{n-2}$. If $A_{11}^t A_{11} = I_{n-2}$, then by (4.5.3) $H_{11} = 0$ and so $H = 0$. Thus $A^t J_n A = J_n$, whence by Theorem 2.4 and (4.5.2) $A \in \Theta(K_n)$. This contradicts $q < n$. Thus $A_{11}^t A_{11} \neq I_{n-2}$.

By Proposition 4.1 there exist $P, Q \in \mathbb{R}^{n-2,n-2}$ such that

$$PA_{11}Q = A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-2}).$$

It is easy to see that $0 \leq A^t A \leq I_{n-2}$ and $A^t A \neq I_{n-2}$. Thus $1 - \lambda_i^2 \geq 0$, $i = 1, 2, \ldots, n-2$ and $1 - \lambda_j^2 > 0$ for some $j$, $1 \leq j \leq n-2$, say $j = 1$. Choose $\epsilon > 0$ such that $|\lambda_1 \pm \epsilon| < 1$. Let

$$A(\pm \epsilon) = \text{diag}(\lambda_1 \pm \epsilon, \lambda_2, \ldots, \lambda_{n-2}),$$

$$A_{11}(\pm \epsilon) = P^t A(\pm \epsilon) Q^t \quad \text{and} \quad A(\pm \epsilon) = A_{11}(\pm \epsilon) \oplus I_2.$$  \hspace{1cm}

Since $0 \leq A^t(\pm \epsilon) A(\pm \epsilon) \leq I_{n-2}$, we have $A^t(\pm \epsilon) J_n A(\pm \epsilon) \succeq J_n$. Since $\text{rank } A \geq 2$ it follows from Theorem 2.3 and (4.5.2) that $A(\pm \epsilon) \in \Gamma(K_n)$. But $A = \frac{1}{2}[A(\epsilon) + A(-\epsilon)]$. Now suppose that $A(\epsilon) = \alpha A$ for some $\alpha \in \mathbb{R}$.
Then $I_{22} = 1I_{22}$, whence $\alpha = 1$. However, $A(\epsilon) \neq A$. Thus $A$ is not extreme, a contradiction.

Case 3b. Suppose that $y^1 = Au^n$ and $y^2 = Av^n$ are linearly dependent, say $y^1 = \alpha y$ and $y^2 = \beta y$, where $y \in \partial K_n$ and $\alpha, \beta > 0$. By Lemmas 3.4 and 3.7 we may assume without loss of generality that $y = u^n$, so that $Au^n = \alpha u^n$, $Av^n = \beta u^n$. Since

$$Ae^n = \frac{1}{2} Au^n + \frac{1}{2} Av^n = \frac{1}{2} (\alpha + \beta) u^n$$

and

$$Ae^{n-1} = \frac{1}{2} Au^n - \frac{1}{2} Av^n = \frac{1}{2} (\alpha - \beta) u^n,$$

it follows that

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$A_{22} = \frac{1}{2} \begin{bmatrix} \alpha - \beta & \alpha + \beta \\ \alpha - \beta & \alpha + \beta \end{bmatrix}.$$

Let $B = A^T J_n A$. Then $B \gg \mu J_n$ for some $\mu > 0$, since $A \in \Gamma(K_n)$. If we partition $B$ conformably with $A$, we obtain $B_{22} = A_{22} A J_n A_{22} = 0$. Hence, $0 \gg \mu J_2$, whence $\mu = 0$. Thus $A^T J_n A \gg 0$, and since $A \in E(\Gamma(K_n))$, we conclude from Corollary 4.3 that $A \in \Delta(K_n)$. An immediate application of Lemma 3.2 yields $q = n$, a contradiction. This completes the investigation of Case 3.

We have proved that $q = n$. Hence there exists a basis $x^1, x^2, \ldots, x^n$ for $\mathbb{R}^n$, such that $x^i \in \partial K_n$ and $Ax^i \in \partial K_n$, $i = 1, 2, \ldots, n$. Since $A \in \Gamma(K_n)$, there exist $H \gg 0$ and $\mu > 0$ such that $A^T J_n A = \mu J_n + H$. Hence

$$0 = (Ax^i)^T J_n A x^i = \mu (x^i)^T J_n x^i + (x^i)^T H x^i.$$

It follows that $H x^i = 0$, $i = 1, 2, \ldots, n$, and so $H = 0$. Thus $A^T J_n A = \mu J_n$. If $\mu > 0$, then by Theorem 2.4 $A \in \Theta(K_n)$. If $\mu = 0$, then by Corollary 4.3 $A \in \Delta(K_n)$. We have established the proof that

$$E(\Gamma(K_n)) \subseteq \Theta(K_n) \cup \Delta(K_n).$$

(b) We now prove that $E(\Gamma(K_n)) \supseteq \Theta(K_n) \cup \Delta(K_n)$. So let $A \in \Theta(K_n) \cup \Delta(K_n)$. If $A \in \Delta(K_n)$, then $A \in E(\Gamma(K_n))$ by definition. So suppose that $A \in \Theta(K_n)$. Since the cone $K_n$ is indecomposable for $n \geq 3$ (see Barker and Schneider [1] and Loewy and Schneider [5]), it follows from Theorem 3.3 of Loewy and Schneider [5] that $A \in E(\Gamma(K_n))$. This completes the proof of the theorem.

Remark. Note that the condition $n \geq 3$ is required in the proof of Theorem 4.5. It is clear that $E(\Gamma(K_2)) = \Delta(K_2)$, since every nonsingular $A \in \Gamma(K_2)$ is the sum of two matrices in $\Gamma(K_2)$ of rank 1.
Theorem 4.5 characterizes the extreme matrices in \( \Gamma(K_n) \), for \( n \geq 3 \). We can now determine the structure of \( \Gamma(K_n) \):

**Corollary 4.6.** Let \( n \geq 2 \). Then \( \Gamma(K_n) = \text{hull}[\Theta(K_n) \cup \Delta(K_n)] \).

*Proof.* This is an immediate corollary of Theorem 4.5, the preceding remark and the well-known theorem that any cone in \( \mathbb{R}^k \) is the hull of its extremes (cf. [6, p. 167]).

Finally, we show that \( \Gamma(K_n) \) is determined by the matrices that map \( K_n \) onto itself.

**Theorem 4.7.** Let \( n \geq 2 \). Then \( \Gamma(K_n) = \text{hull}(\text{closure} \ \Theta(K_n)) \).

*Proof.* By Corollary 4.6 it is enough to show that any matrix \( A \) in \( \Delta(K_n) \) is a limit of matrices in \( \Theta(K_n) \). So let \( 0 \neq A \in \Delta(K_n) \). Then by Lemma 3.2 there exist nonzero vectors \( x, y \in \partial K_n \) such that \( A = xy^t \). By Lemma 3.7, there exist \( \Omega_1, \Omega_2 \in \Theta(K_n) \) such that \( \Omega_1 x = u^n \) and \( \Omega_2 y = u^n \). Let \( B = \frac{1}{2} \Omega_1 A \Omega_2^t \). Then \( B \in \Delta(K_n) \) and it is enough to show that \( B \) is a limit of matrices in \( \Theta(K_n) \). But

\[
B = \frac{1}{2} \Omega_1 A \Omega_2^t = \frac{1}{2} u^n u^n = 0_{n-2} \oplus \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

Let

\[
T_\varepsilon = (2\varepsilon)^{1/2} I_{n-2} \oplus \begin{bmatrix} \frac{1}{2} + \varepsilon & \frac{1}{2} - \varepsilon \\ \frac{1}{2} - \varepsilon & \frac{1}{2} + \varepsilon \end{bmatrix}.
\]

Clearly, \( \lim_{\varepsilon \to 0} T_\varepsilon = B \) and \( T_\varepsilon^t J_n T_\varepsilon = 2\varepsilon J_n \). Hence, by Theorem 2.4, \( T_\varepsilon \in \Theta(K_n) \), completing the proof.

**Addendum to**

**Positive Operators on the**

\( n \)-**Dimensional Ice Cream Cone**

After the completion of this paper J. Bognár informed us of a different and more elementary proof of Lemma (2.2). By modifying a proof found, for the complex case, in M. G. Kreén and Ju. L. Šmul'jan (Plus-operators in a space with indefinite metric, Amer. Math. Soc. Transl. 85 (1969), 93–113), one first proves: If \( x^t J_n x = 0 \) implies \( x^t M x > 0 \), then the relations \( y^t J_n y < 0 \) and \( x^t J_n x > 0 \) imply that \( y^t M y / y^t J_n y < x^t M z / x^t J_n z \). Lemma 2.2 then follows easily.
REFERENCES


Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.