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Inequalities for Determinants and Permanents

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We give an upper bound for the determinant of a matrix with positive dominant diagonal and nonpositive off-diagonal elements. A lower bound is derived for the permanent of a nonnegative matrix with dominant diagonal. Conditions for equality are investigated. There are applications to stochastic, nonnegative, and M-matrices.

1. INTRODUCTION

Lower bounds for the determinant of a matrix with positive dominant diagonal and nonpositive off-diagonal elements are classical; upper bounds have not been considered to any great extent. It is an easy consequence of a theorem due to Ostrowski [4] that the determinant of such a matrix is less than or equal to the product of diagonal elements. In Theorem 1, we improve this bound and we investigate conditions for equality of our bound. In Theorem 2, we show that any upper bound for the determinant of this type of matrix yields a lower bound for the permanent of the corresponding matrix of absolute values. Thus, in Theorem 3, we state a lower bound for the permanent of a nonnegative matrix with dominant diagonal and immediately derive conditions of equality from corresponding parts of Theorems 1 and 2. We give some applications to nonnegative matrices and to M-matrices, and consider stochastic matrices in particular. In the last section we state a similar result for the determinant of a positive semi-definite Hermitian matrix H with known minimal eigenvalue. This result sharpens the Hadamard inequality.

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The conditions for equality in our theorems and corollaries describe the patterns of zeros in the various matrices. At Gatlinburg V, we learned that our Theorem 5 was anticipated in a slightly weaker form by Keilson-Styan [3].

2. CONDITIONS FOR TRIANGULABILITY

Let μ be a nonempty subset of $\{1, \ldots, n\}$. By Γ_μ we denote the symmetric group on μ. The identity in Γ_μ will be denoted by ε. A *cycle* α is a permutation in Γ_μ, if, for some $r > 1$, and distinct integers $i_1, \ldots, i_r$, the permutation α satisfies $\alpha(i_p) = i_{p+1}$, $p = 1, \ldots, r-1$, $\alpha(i_r) = i_1$, and $\alpha(j) = j$ for all other j in μ. (Thus ε is *not* a cycle.) We write $\alpha = (i_1, \ldots, i_r)$ and call $\{i_1, \ldots, i_r\}$ the support of α. The *support* of α will be denoted by $\bar{\alpha}$.

A cycle $\alpha \in \Gamma_\mu$ is called a *full cycle* in Γ_μ, if $\bar{\alpha} = \mu$. Let ν be a nonempty subset of μ. If $\alpha \in \Gamma_\mu$ and $\bar{\alpha} \subseteq \nu$, then α restricts to a permutation $\alpha' \in \Gamma_\nu$. Conversely, if $\alpha' \in \Gamma_\nu$, then we can extend α' to a permutation $\alpha \in \Gamma_\mu$ by putting $\alpha(i) = i$ for $i \in \mu \backslash \nu$. In the sequel, we shall write $\alpha' = \alpha$.

If F is a set, F_μ is the set of all square matrices over μ. Let $M \in F_\mu$. If $\phi \subset \beta \subseteq \mu$ (we use $\subset$ for proper set inclusion), $M[\beta]$ will denote the submatrix of M lying in the rows and columns indexed by elements of β in their natural order. Similarly, $M(\beta)$ will denote the principal submatrix complementary to $M[\beta]$ in M. If $M \in F_\mu$, where F is a field, and $\alpha \in \Gamma_\mu$, then $\prod_\alpha(M) = \prod_{i \in \mu} m_{i\alpha(i)}$. A matrix $M \in F_\mu$ is *triangulable* if there exists a permutation matrix $P \in F_\mu$ such that $P^{-1}MP$ is upper triangular.

By $\mathbf{R}$, $\mathbf{C}$, $\mathbf{R}^+$ we denote respectively the real field, complex field, and the nonnegative numbers. We put $\langle n \rangle = \{1, \ldots, n\}$.

The following lemma is essentially Harary–Norman–Carwright [2], Theorem (10.1), equivalence of conditions (1) and (6).

LEMMA 1 *Let F be a field, and let $M \in F_{\langle n \rangle}$. Then the following are equivalent:*

$$\text{M is triangulable,} \tag{2.1}$$

$$\text{For all cycles } \alpha \in \Gamma_{\langle n \rangle},\ \prod_\alpha(M[\bar{\alpha}]) = 0. \tag{2.2}$$

To show the power of Lemma 1, we prove a lemma closely related to a result due to R. A. Brualdi [1], Theorem 2.1, who gives a different proof.

LEMMA 2 *Let $M \in \mathbf{R}^+_{\langle n \rangle}$ and let $m_{ii} > 0$, $i = 1, 2, \ldots, n$. Then the following are equivalent:*

$$\text{M is triangulable,} \tag{2.3}$$

$$\operatorname{per} M = \prod_{i=1}^{n} m_{ii}. \tag{2.4}$$

Proof (2.3) ⇒ (2.4). This is immediate.

(2.4) ⇒ (2.3). Suppose that (2.4) holds. Let α be a cycle in $\Gamma_{\langle n\rangle}$. Then $0 = \prod_\alpha(M) = \prod_\alpha(M[\alpha]) \prod_{i \notin \bar{\alpha}} m_{ii}$. Thus $\prod_\alpha(M[\bar{\alpha}]) = 0$, since $m_{ii} > 0$, for $i \in \langle n\rangle$. Hence by Lemma 1, M is triangulable.

If $\alpha \in \Gamma_\mu$, then $P_\alpha = P$ is the permutation matrix given by $p_{ij} = \delta_{\alpha(i)j}$, where δ_{ij} is the Kronecker delta.

LEMMA 3 *Let $A \in F_{\langle n\rangle}$ and let $\phi \subset \mu \subseteq \langle n\rangle$. If A is triangulable, then $A[\mu]$ is triangulable.*

Proof Let α be a cycle in Γ_μ. Then α is also a cycle in $\Gamma_{\langle n\rangle}$. By Lemma 1, $\prod_\alpha(A[\bar{\alpha}]) = 0$. Hence, again by Lemma 1, $A[\mu]$ is triangulable.

LEMMA 4 *Let $A \in F_{\langle n\rangle}$. Then the following are equivalent:*

$A(i)$ is triangulable for all i, $i = 1, \ldots, n$,

Either a) *A is triangulable,* (2.5)

or b) *$A = D_1 + P_\gamma D_2$, where D_1, D_2 are diagonal matrices* (2.6)
and γ is full cycle in $\Gamma_{\langle n\rangle}$.

(*To accommodate the case $n = 1$, we adopt the following conventions: If $A \in F_{\langle 1\rangle}$, then $A[1]$ is triangulable, and also* (2.6b) *holds.*)

Proof (2.6) ⇒ (2.5).

a) Suppose A is triangulable and let $i \in \langle n\rangle$. By Lemma 3, $A(i)$ is triangulable.

b) Suppose (b) holds.

Let $i \in \langle n\rangle$, and let $B = I + P_\gamma D_2$. It is enough to prove $B(i)$ triangulable. Let $\langle n\rangle' = \langle n\rangle \backslash \{i\}$. Let α be a cycle in $\Gamma_{\langle n\rangle'}$. Then, extending α to permutation in $\Gamma_{\langle n\rangle}$, we have $\prod_\alpha(B) = \prod_\alpha(B(i))$. Since $\alpha \neq \gamma$ and $\alpha \neq \varepsilon$, we have $\prod_\alpha(B) = 0$. Since $b_{kk} = 1$, for $k \in \langle n\rangle$, also $\prod_\alpha(B[\bar{\alpha}]) = 0$. Hence, Lemma 1, $B(i)$ is triangulable.

(2.5) ⇒ (2.6). Let (2.5) hold and assume A is not triangulable. Then, by Lemma 1, there exists a cycle γ in $\Gamma_{\langle n\rangle}$ such that $\prod_\gamma(A[\bar{\gamma}]) \neq 0$. Suppose γ is not a full cycle. Then there exists an i such that $\gamma(i) = i$. Restrict γ to $\Gamma_{\langle n\rangle'}$, where $\langle n\rangle' = \langle n\rangle \backslash \{i\}$. Then $\prod_\gamma(A(i)[\bar{\gamma}]) \neq 0$, whence $A(i)$ is not triangulable, by Lemma 1. It follows that γ is a full cycle. To complete the proof we must show that $i \neq j$ and $a_{ij} \neq 0$ imply that $j = \gamma(i)$. So suppose that $i \neq j$, $a_{ij} \neq 0$ and that $j \neq \gamma(i)$. We may write $\gamma = (i_1, i_2, \ldots, i_n)$, where $i = i_1$ and $j \neq i_2$. There is an r, $2 < r \leq n$ such that $j = i_r$. Let $\alpha = (i_1, i_r, i_{r+1}, \ldots, i_n)$. Then $\alpha \neq \varepsilon$ and $\prod_\alpha A[\bar{\alpha}] \neq 0$. By Lemma 1, $A[\bar{\alpha}]$ is not triangulable and thus contrary to (2.5), $A(i_2)$ is not triangulable. Hence $a_{ij} \neq 0$ implies $j = \gamma(i)$, and (b) follows.

3. UPPER BOUNDS FOR DETERMINANTS

For $M \in \mathbf{R}_\mu$, we put

$$m_i' = (\sum_{j \in \mu} |m_{ij}|) - |m_{ii}|, \quad i \in \mu.$$

Define

$$\begin{aligned}
\mathscr{Z}_\mu &= \{M \in \mathbf{R}_\mu : m_{ii} \geqslant 0,\ i \in \mu \text{ and } m_{ij} \leqslant 0,\ i \neq j,\ i, j \in \mu\},\\
\mathscr{Z}_\mu^d &= \{M \in \mathscr{Z}_\mu : m_{ii} > 0\},\\
\mathscr{M}_\mu &= \{M \in \mathscr{Z}_\mu : m_{ii} \geqslant m_i',\ i \in \mu\},\\
\mathscr{M}_\mu^d &= \{M \in \mathscr{M}_\mu : m_{ii} > 0,\ i \in \mu\},\\
\mathscr{M}_\mu^s &= \{M \in \mathscr{M}_\mu : m_{ii} > m_i',\ i \in \mu\}.
\end{aligned}$$

For $A \in F_\mu$, we put

$$g(A) = \det A - \prod_{i \in \mu} a_{ii}. \tag{3.1}$$

We adopt the following conventions:

$$\prod_{i \in \phi} d_i = 1,$$

$$\det A[\phi] = 1,$$

but

$$g(A[\phi]) = 0,$$

LEMMA 5 *Let* $A \in F_{\langle n \rangle}$, *and let* $D = \operatorname{diag}(d_1, \ldots, d_n) \in F_{\langle n \rangle}$. *Then*

$$g(A + D) = \sum_{\mu \subset \langle n \rangle} (\prod_{i \in \mu} d_i) g(A(\mu)).$$

Proof By a well-known result, e.g. Schneider-Barker [5], p. 249,

$$\det (A + D) = \sum_{\mu \subseteq \langle n \rangle} (\prod_{i \in \mu} d_i) \det A(\mu).$$

Hence (with $\mu' = \langle n \rangle \backslash \mu$)

$$\begin{aligned}
\det (A + D) &= \sum_{\mu \subseteq \langle n \rangle} (\prod_{i \in \mu} d_i)(g(A(\mu)) + \prod_{i \in \mu'} a_{ii})\\
&= \sum_{\mu \subseteq \langle n \rangle} (\prod_{i \in \mu} d_i) g(A(\mu)) + \sum_{\mu \subseteq \langle n \rangle} (\prod_{i \in \mu} d_i) \prod_{i \in \mu'} a_{ii}\\
&= \sum_{\mu \subset \langle n \rangle} (\prod_{i \in \mu} d_i) g(A(\mu)) + \prod_{i \in \langle n \rangle} (a_{ii} + d_i),
\end{aligned}$$

and the results follows.

THEOREM 1 i) *Let* $M \in \mathscr{M}_{\langle n \rangle}$. *Then*

$$\det M \leqslant \prod_{i=1}^{n} m_{ii} - \prod_{i=1}^{n} m_i'. \tag{3.2}$$

†ii) *Let* $M \in \mathcal{M}^d_{\langle n\rangle}$. *Then*

$$\det M = \prod_{i=1}^{n} m_{ii} \tag{3.3}$$

if and only if

$$M \text{ is triangulable.} \tag{3.4}$$

iii) *Let* $M \in \mathcal{M}^d_{\langle n\rangle}$. *Then the equality holds in* (3.2) *if and only if*

$$M(i) \text{ is triangulable for each } i \text{ such that } m_{ii} > m'_i. \tag{3.5}$$

iv) *Let* $M \in \mathcal{M}^s_{\langle n\rangle}$. *Then the equality holds in* (3.2) *if and only if M satisfies* (2.6).

Proof i) Let g be defined by (3.1). Evidently (3.2) is equivalent to

$$g(M) \leqslant -\prod_{i\in\langle n\rangle} m'_i. \tag{3.6}$$

Let $d_i = m_{ii}-m'_i$, $i = 1, \ldots, n$, and let $D = \operatorname{diag}(d_1, \ldots, d_n)$, and $A = M-D \in \mathrm{R}_{\langle n\rangle}$. Since $Ae = 0$ where $e^T = (1, 1, \ldots, 1)$, follows that $\det A = 0$, and so

$$g(A) = -\prod_{i\in\langle n\rangle} a_{ii} = -\prod_{i\in\langle n\rangle} m'_i. \tag{3.7}$$

By Lemma 5,

$$\begin{aligned} g(M) = g(A+D) &= \sum_{\mu\subset\langle n\rangle} \Big(\prod_{i\in\mu} d_i\Big) g(A(\mu)) \\ &= g(A) + \sum_{\phi\subset\mu\subset\langle n\rangle} \Big(\prod_{i\in\mu} d_i\Big) g(A(\mu)). \end{aligned} \tag{3.8}$$

It now follows from (3.6) and (3.7) that (3.2) is equivalent to

$$\sum_{\phi\subset\mu\subset\langle n\rangle} \Big(\prod_{i\in\mu} d_i\Big) g(A(\mu)) \leqslant 0. \tag{3.9}$$

(For future reference, we observe that the equality holds in one of (3.2), (3.6), (3.9) if and only if the equality holds in all of these.)

We now proceed to prove (3.6) by induction on the order n. Clearly (3.6) holds by convention if $n = 1$. Suppose $n \geqslant 2$ and (3.6) holds for matrices of smaller order. Let $\phi \subset \mu \subset \langle n\rangle$. Since $A \in \mathcal{M}_{\langle n\rangle}$, we have $A[\mu'] \in \mathcal{M}_{\mu'}$, where $\mu' = \langle n\rangle\backslash\mu$. Hence, by inductive assumption

$$g(A(\mu)) = g(A[\mu']) \leqslant -\prod_{i\in\mu'} a_{ii} \leqslant 0. \tag{3.10}$$

But $d_i \geqslant 0$ for $i \in \langle n\rangle$. Hence

$$\Big(\prod_{i\in\mu} d_i\Big) g(A(\mu)) \leqslant 0,$$

† The equivalence of (3.3) and (3.4) is also an easy consequence of Ostrowski [4], Zusatz zu Satz I. We require (ii) in the proof of (iii) and hence have included a proof. The inequality in (i) of Theorem 1 is also contained in [4] formula (18) where the symbol $\Pi^n_{\mu=1}$ should be deleted.

and (3.9) (and hence (3.6)) follows. The inductive step is proved, and (3.2) follows by induction. This completes the proof of (i).

ii) Clearly (3.4) implies (3.3). To prove that (3.3) implies (3.4), we again proceed by induction. The result is true if $n = 1$. Suppose the result holds with n replaced by $(n-1)$. Let $M \in \mathscr{M}^d_{\langle n \rangle}$, and $\det M = \prod_{i=1}^{n} m_{ii}$. Hence by (3.2), $\prod_{i=1}^{n} m'_i = 0$. Thus there exists an $i \in \langle n \rangle$ such that $m'_i = 0$. For this i,

$$m_{ij} = 0 \text{ if } j \neq 1, j \in \langle n \rangle. \tag{3.11}$$

Hence

$$\prod_{j=1}^{n} m_{jj} = \det M = m_{ii} \det M(i).$$

Since $m_{ii} > 0$, it follows that $\det M(i) = \prod_{i \in \langle n \rangle'} m_{jj}$, where $\langle n \rangle' = \langle n \rangle \backslash \{i\}$. Hence, by inductive assumption, $M(i)$ is triangulable. But then, by (3.11), M is triangulable. We now obtain (3.4) by induction.

iii) Suppose (3.5) holds. Let $\phi \subset \mu \subset \langle n \rangle$.

Case I $d_i = m_{ii} - m'_i = 0$, for all $i \in \mu$.
Then

$$\prod_{i \in \mu} d_i = 0.$$

Case II There exists an $i \in \mu$ such that $d_i > 0$. Then, by (3.5), $M(i)$ is triangulable, and so is $A(i)$. But $\langle n \rangle \backslash \mu \subseteq \langle n \rangle \backslash \{i\}$, whence $A(i)$ is also triangulable. It follows that $g(A(\mu)) = \det A(\mu) - \prod_{i \in \mu} a_{ii} = 0$.

Thus, in either case,

$$\left(\prod_{i \in \mu} d_i\right) g(A(\mu)) = 0. \tag{3.12}$$

It follows that the equality holds in (3.9) and hence in (3.2).

Conversely, suppose that the equality holds in (3.2) and hence in (3.9). Since $d_i \geqslant 0$, $i \in \langle n \rangle$, and $g(A(\mu)) \leqslant 0$, $\phi \subset \mu \subset \langle n \rangle$, (since (3.2) holds for $\mathscr{M}_\mu$).

Now let $i \in \langle n \rangle$ be such that $d_i = m_{ii} - m'_i > 0$ and put $\mu = \{i\}$. Then $d_i g(A(i)) = 0$ and so

$$g(A(i)) = 0. \tag{3.12}$$

It follows by (ii) that $A(i)$ is triangulable.

iv) This follows from (iii) and lemma 4.

Remark In the proof of theorem 1, we did not use the full force of the inductive hypothesis, but rather the weaker inequality $g(A) \leqslant 0$ (cf. (3.10)). Suppose that for $\phi \subset \mu \subset \langle n \rangle$ we have the bound $g(A(\mu)) \leqslant -b(\mu) \leqslant 0$.

Then we obtain from (3.8) for $M \in \mathscr{M}_{\langle n \rangle}$,

$$\det M \leqslant \prod_{i=1}^{n} m_{ii} \prod_{i=1}^{n} m_i' - \sum_{\phi \subset \mu \subset \langle n \rangle} \prod_{i \in \mu} (m_{ii} - m_i') b(\mu).$$

Define $m(\mu)_i' = (\sum_{j \notin \mu} |m_{ij}|) - |m_{ii}|$ for $i \notin \mu$. Since $M \in \mathscr{M}_{\langle n \rangle}$ implies $A(\mu) \in \mathscr{M}_{\langle n \rangle \setminus \mu}$ we may use the bound of theorem 1 for $b(\mu)$ and obtain

$$\det M \leqslant \prod_{i=1}^{n} m_{ii} - \prod_{i=1}^{n} m_i' - \sum_{\phi \subset \mu \subset \langle n \rangle} (m_{ii} - m_i') \prod_{i \notin \mu} m(\mu)_i'. \tag{3.14}$$

If $n = 1$, the classical bound $\det M \leqslant \prod_{i=1}^{n} m_{ii}$ is sharp. If $n \leqslant 2$, then the equality always holds in (3.2). Hence, by (3.8) for $n \leqslant 3$, equality always holds in (3.14). In general, given $r > 0$, starting with the classical $\det M \leqslant \prod_{i=1}^{n} m_{ii}$ and iterating our procedure $r - 1$ times, we obtain a bound which is an identity if $n \leqslant r$. Similar remarks apply to some of our subsequent theorems. The same arguments can also be used to improve the classical Hadamard bound for the determinant of a nonnegative definite Hermitian matrix (cf. section 7).

4. Per A + det M

Our next lemma is a corollary to theorem 1(iv), but a direct proof is easy.

LEMMA 6 *Let $M \in \mathscr{Z}_{\langle n \rangle}$, and let α be a cycle in $\Gamma_{\langle n \rangle}$. Then* $(\operatorname{sgn} \alpha) \prod_{\alpha} (M) \leqslant 0$.

Proof Let $\alpha = (i_1, \ldots, i_r)$ be a cycle. Then $\alpha = (i_1 i_2)(i_2 i_3) \ldots (i_{r-1} i_r)$, whence $\operatorname{sgn} \alpha = (-1)^{r-1}$. Hence $(\operatorname{sgn} \alpha) \prod_{\alpha} (M) = (-1)^{r-1} \prod_{i \in \langle n \rangle} m_{i\alpha(i)}$ $= (-1)^{2r-1} \prod_{i \in \langle n \rangle} |m_{i\alpha(i)}| \leqslant 0$.

If $M \in \mathbf{R}_\mu$, we shall define $|M| \in \mathbf{R}_\mu^+$ by $A = |M|$ where $a_{ij} = |m_{ij}|$.

THEOREM 2 i) *Let $M \in \mathscr{Z}_{\langle n \rangle}$. Then*

$$\det M + \operatorname{per} |M| \geqslant 2 \prod_{i=1}^{n} m_{ii}. \tag{4.1}$$

ii) *Let $M \in \mathscr{Z}^d_{\langle n \rangle}$. The equality holds in* (4.1) *if and only if*

For all sets μ such that $\phi \subset \mu \subset \langle n \rangle$ either (4.2)

$M[\mu]$ or $M(\mu)$ is triangulable.

Proof i) For $\alpha \in \Gamma_{\langle n \rangle}$, let

$$f_\alpha(M) = (\operatorname{sgn} \alpha) \prod_{\alpha} (M) + \prod_{\alpha} (|M|). \tag{4.3}$$

Then $f_\varepsilon(M) = 2 \prod_\varepsilon (M)$ and for $\alpha \in \Gamma_{\langle n \rangle}$,

$$f_\alpha(M) = (\pm \operatorname{sgn} \alpha + 1) \prod_\alpha (|M|) \geqslant 0. \tag{4.4}$$

But

$$\det M + \operatorname{per} |M| = \sum_{\alpha \in \Gamma_{\langle n \rangle}} f_\alpha(M), \tag{4.5}$$

and it follows from (4.4) and (4.5) that

$$\det M + \operatorname{per} |M| \geqslant 2 \prod_\varepsilon (M) = 2 \prod_{i=1}^{n} m_{ii}.$$

iii) Let $M \in \mathscr{X}^d_{\langle n \rangle}$. Suppose that (4.2) holds, and let $\alpha \in \Gamma_{\langle n \rangle}$, $\alpha \neq \varepsilon$. Then $\alpha = \alpha_1 \alpha_2 \ldots \alpha_r$, where the α_i are cycles with disjoint support. We shall prove that $f_\alpha(M) = 0$.

Case I Suppose that $r = 1$. Then, by Lemma 6, $(\operatorname{sgn} \alpha) \prod_\alpha (M) \leqslant 0$ whence $(\operatorname{sgn} \alpha) \prod_\alpha (M) = -\prod_\alpha (|M|)$ and so $f_\alpha(M) = 0$.

Case II Suppose that $r > 1$. Let $\mu = \bar{\alpha}_1$. Then $\phi \subset \mu \subset \langle n \rangle$, and, by (4.2), either $M[\mu]$ or $M(\mu)$ is triangulable. Assume that $M[\mu]$ (and hence $|M[\mu]|$) is triangulable. By Lemma 1, $\prod_{\alpha_1} (|M[\mu]|) = 0$ (where we interpret α_1 as a permutation on μ). Next, suppose that $M(\mu)$ is triangulable. Then, since $\bar{\alpha}_2 \subseteq \langle n \rangle \backslash \mu$, $M[\bar{\alpha}_2]$ is triangulable. By Lemma 1, $\prod_{\alpha_2} (|M[\bar{\alpha}_2]|) = 0$. Hence, in either case,

$$\begin{aligned}\prod_\alpha (|M|) &= \prod_{\alpha_1} (|M[\bar{\alpha}_1]|) \prod_{\alpha_2} (|M[\bar{\alpha}_2]|) \ldots \prod_{\alpha_r} (|M[\bar{\alpha}_r]|) \prod_{i \in \mu \backslash \bar{\alpha}} m_{ii} \\ &= 0,\end{aligned}$$

whence $f_\alpha(M) = 0$. Thus, by (4.5),

$$\det M + \operatorname{per} |M| = 2 \prod_\varepsilon (M) = 2 \prod_{i=1}^{n} m_{ii}.$$

To prove the converse, suppose that μ_1 is a set, $\phi \subset \mu \subset \langle n \rangle$ such that neither $M[\mu_1]$ nor $M(\mu_1)$ is triangulable. Let $\mu_2 = \langle n \rangle \backslash \mu_1$. By Lemma 1, these exist cycles $\alpha_i \in \Gamma_{\mu_i}$ such that $\prod_{\alpha_i} (M[\mu_i]) \neq 0$, $i = 1, 2$. Hence, by Lemma 6,

$$(\operatorname{sgn} \alpha_i) \prod_{\alpha_i} (M[\mu_i]) < 0, \quad i = 1, 2. \tag{4.6}$$

Consider α_i as a permutation in $\Gamma_{\langle n \rangle}$, and put $\alpha = \alpha_1 \alpha_2 \in \Gamma_{\langle n \rangle}$. Then $(\operatorname{sgn} \alpha) \prod_\alpha (M) = (\operatorname{sgn} \alpha_1)(\operatorname{sgn} \alpha_2) \prod_\alpha (M[\mu_1]) \prod_{\alpha_2} (M[\mu_2]) \prod_\varepsilon (M[\mu_3])$, where

$\mu_3 = \langle n \rangle \backslash (\mu_1 \bigcup \mu_2)$ and $\varepsilon \in \Gamma_{\mu_3}$. Since $\prod_{\varepsilon} M(\mu_3) > 0$, it follows from (4.6) that $(\operatorname{sgn} \alpha) \prod_{\alpha} (M) > 0$. Hence $f_\alpha(M) > 0$. We now deduce from (4.4) and (4.5) that $\det M + \operatorname{per} |M| > 2 \prod_{i=1}^{n} m_{ii}$.

COROLLARY 2 *Let M be a matrix in $\mathscr{L}_{\langle n \rangle}$ where $n \leqslant 3$. Then*

$$\det M + \operatorname{per} |M| = 2 \prod_{i=1}^{n} m_{ii}.$$

Example:

We indicate (without proof) how condition (4.2) is satisfied when $n = 4$. Let

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ b_2 & a_2 & d_2 & c_2 \\ c_3 & d_3 & a_3 & b_3 \\ d_4 & c_4 & b_4 & a_4 \end{bmatrix}$$

be in $F_{\langle 4 \rangle}$, where F is a field. Then (4.2) holds if and only if there exist i, j, k, $1 \leqslant i, j, k \leqslant 4$, such that $b_i = c_j = d_k = 0$.

5 LOWER BOUNDS FOR PERMANENTS

THEOREM 3 i) *Let $M \in \mathscr{M}_{\langle n \rangle}$, and $A = |M|$. Then*

$$\operatorname{per} A \geqslant \prod_{i=1}^{n} m_{ii} + \prod_{i=1}^{n} m_i'. \tag{5.1}$$

ii) *Let $M \in \mathscr{M}^d_{\langle n \rangle}$. Then the equality holds in* (5.1) *if and only if* (4.2) *and* (3.5) *hold.*

iii) *If $M \in \mathscr{M}^s_{\langle n \rangle}$, then the equality holds if* (5.1) *if and only if* (2.6) holds.

Proof i) We have by (3.2) and (4.1) that

$$\operatorname{per} A = (\det M + \operatorname{per} A) - \det M \geqslant 2 \prod_{i=1}^{n} m_{ii} - \left(\prod_{i=1}^{n} - \prod_{i=1}^{n} m_i' \right) \tag{5.2}$$

$$= \prod_{i=1}^{n} m_{ii} + \prod_{i=1}^{n} m_i',$$

and (5.1) is immediate.

ii) Let $M \in \mathscr{M}^d_{\langle n \rangle}$. The equality holds in (5.1) if and only if it holds in (5.2), and hence if and only if the equalities hold in (3.2) and (4.1). Hence, by Theorems 1 and 2, the equality holds in (5.1) if and only if (4.2) and (3.5) hold.

iii) Let $M \in \mathcal{M}^{\natural}_{\langle n\rangle}$. If (2.6) holds for M, then clearly the equality holds in (5.1). Suppose that the equality holds in (5.1). Then by (ii), $M(i)$ is triangulable for $i = 1, \ldots, n$, whence (2.6) holds, by Lemma 4.

LEMMA 7 i) *Let $a_i \geqslant b_i \geqslant 0$, $i = 1, \ldots, n$. Then*

$$\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i \geqslant 2 \prod_{i=1}^{n} \left(\frac{a_i + b_i}{2}\right). \tag{5.3}$$

ii) *Let $a_i > 0$, $i = 1, \ldots, n$. Then the equality holds in* (5.3) *if and only if*

There is a j, $1 \leqslant j \leqslant n$, such that $a_i = b_i$, for $i \neq j$ $i = 1, \ldots, n$. (5.4)

A proof is by induction, and the following identity: For $n \geqslant 2$,

$$\prod_{i=1}^{n} a_i + \prod_{i=1}^{n} b_i = \frac{a_n + b_n}{2}\left(\prod_{i=1}^{n-1} a_i + \prod_{i=1}^{n-1} b_i\right) + \frac{a_n - b_n}{2}\left(\prod_{i=1}^{n-1} a_i - \prod_{i=1}^{n-1} b_i\right).$$

A matrix $S \in \mathrm{R}_{\langle n\rangle}$ is a stochastic matrix if $s_{ij} \geqslant 0$, $i, j = 1, \ldots, n$ and $\sum_{j=1}^{n} s_{ij} = 1$, $i = 1, \ldots, n$. A matrix $S \in \mathrm{R}_{\langle n\rangle}$ is doubly stochastic if both S and the transpose S^T are stochastic.

COROLLARY 3 *Let $S \in \mathrm{R}_{\langle n\rangle}$ be a stochastic matrix such that $s_{ii} \geqslant \frac{1}{2}$, $i = 1, \ldots, n$.*

i) Then

$$\operatorname{per} S \geqslant (\tfrac{1}{2})^{n-1}. \tag{5.5}$$

ii) *The equality holds in* (5.5) *if and only if S satisfies* (3.5), (4.2) *and*

There is a j, $1 \leqslant j \leqslant n$ such that $s_{ii} = \frac{1}{2}$, for $i \neq j$, $1 \leqslant i \leqslant n$. (5.6)

iii) *If $n \geqslant 2$ and S is doubly stochastic, then the equality holds in* (5.5) *if and only if S satisfies* (4.2) *and $s_{ii} = \frac{1}{2}$, $i = 1, \ldots, n$.*

Proof i) Clearly there is an $M \in \mathcal{M}_{\langle n\rangle}$ such that $S = |M|$. Hence, by Theorem 3, and Lemma 7

$$\operatorname{per} S \geqslant \prod_{i=1}^{n} s_{ii} + \prod_{i=1}^{n} (1 - s_{ii}) \geqslant (\tfrac{1}{2})^{n-1}. \tag{5.7}$$

ii) The equality holds in (5.5) if and only if both equalities hold in (5.7); hence by Theorem 3 and Lemma 7, this equality holds if and only if (3.5), (4.2) and (5.6) hold for S.

iii) Suppose that S is doubly stochastic. To prove the last part of the corollary we need only prove that the equality in (5.5) implies $s_{ii} = \frac{1}{2}$, $i = 1, \ldots, n$. Suppose that the equality holds in (5.5). Then (3.5), (4.2) and (5.6) hold, by (ii). Suppose $s_{ii} > \frac{1}{2}$, for some i, $1 \leqslant i \leqslant n$, say $s_{nn} > \frac{1}{2}$. Then by (3.5), $S(n)$ is triangulable. Hence there exists a k, $1 \leqslant k \leqslant n-1$ such that $s_{kj} = 0$, if $j \neq k$, $1 \leqslant j \leqslant n-1$. By (5.6), $s_{kk} = \frac{1}{2}$, and so since S is doubly stochastic, $s_{kn} = \frac{1}{2}$. Hence $\sum_{l=1}^{n} s_{ln} \geqslant s_{kn} + s_{nn} > 1$, a contradiction. Hence $s_{ii} = \frac{1}{2}$, $i = 1, \ldots, n$.

An example of a stochastic matrix for which the equality holds in (5.5) is

$$S = \tfrac{1}{8}\begin{bmatrix} 6 & 1 & 0 & 1 \\ 0 & 4 & 2 & 2 \\ 2 & 0 & 4 & 2 \\ 4 & 0 & 0 & 4 \end{bmatrix}.$$

Observe that $s_{11} > \frac{1}{2}$.

An example of a doubly stochastic matrix for which the equality holds in (5.5) is

$$S = \tfrac{1}{4}\begin{bmatrix} 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

For $n \geqslant 3$, we observe that

$$(\tfrac{1}{2})^{n-1} > \frac{n!}{n^n}.$$

Thus Van der Waerden's conjecture holds for all (doubly) stochastic matrices with dominant diagonal.

Remark With the use of Lemma 3, it is easy to prove the following result: *Let $A \in F_{\langle n \rangle}$ and let $1 \leqslant i \leqslant n$. If $A(i)$ is triangulable, then A satisfies* (4.2). By means of this result, it is possible to restate the conditions for equality in Theorems 3 and 4 and Corollaries 3 and 4 in a more descriptive form. We shall restate the appropriate parts of Theorem 3 and Corollary 3, omitting the proofs.

THEOREM 3(ii), alternative form *Let $M \in \mathcal{M}^d_{\langle n \rangle}$. Then the equality holds in* (5.1) *if and only if either*

$$m_{ii} = m_i' \text{ for } i = 1, \ldots, n \text{ and } M \text{ satisfies (4.2)}, \tag{5.8}$$

or

$$\text{There is an } i,\ 1 \leqslant i \leqslant n \text{ such that } m_{ii} > m_i' \text{ and (3.5) holds.} \tag{5.9}$$

COROLLARY 3(ii), alternative form *The equality holds in* (5.5) *if and only if S satisfies either* (5.8) *or*

$$\begin{aligned} &\text{There is an } i \text{ such that } s_{jj} = \tfrac{1}{2} \text{ for } j \neq i,\ 1 \leqslant j \leqslant n, \\ &s_{ii} > \tfrac{1}{2} \text{ and } S(i) \text{ is triangulable.} \end{aligned} \tag{5.10}$$

6. NONNEGATIVES MATRICES AND M-MATRICES

For the definition of an irreducible matrix, see Varga [6], Definition 1.5, page 18; for a statement of the Perron–Frobenius Theorem, see [6], Theorem 2.1, page 30. For $S \in \mathbf{R}_{\langle n\rangle}$, we shall again use $s_i' = \sum_{j=1}^{n} |s_{ij}| - s_{ii}$, etc.

LEMMA 8 *Let $n \geq 2$ and let A be a nonnegative irreducible matrix in $\mathbf{R}_{\langle n\rangle}$ with spectral radius r. Let $a = \min\{a_{ii}: i = 1, \ldots, n\}$.*

i) *There exists a nonsingular diagonal matrix $X \in \mathbf{R}_{\langle n\rangle}$ such that, for all $t \geq r - 2a$,* (6.1)

a) $S = (r + t)^{-1}X^{-1}(tI + A)X$ *is stochastic,*

b) $s_{ii} = (r + t)^{-1}(t + a_{ii})$ and $s_i' = 1 - s_{ii} = (r + t)^{-1}(r - a_{ii})$, $i = 1, \ldots, n$.

c) $s_{ii} \geq \frac{1}{2}$, $i = 1, \ldots, n$.

ii) *Let $t \geq r - 2a$ and suppose S satisfies* (6.1). *Let* $1 \leq i \leq n$. *Then $s_{ii} = \frac{1}{2}$ if and only if*

$$t = r - 2a \text{ and } a_{ii} = a. \tag{6.2}$$

Proof i) By the Perron–Frobenius theorem there exists a strictly positive vector x such that $Ax = rx$. Let $X = \operatorname{diag}(x_1, \ldots, x_n)$. Then $B = X^{-1}AX$ satisfies $\sum_{j=1}^{n} b_{ij} = r$, $i = 1, \ldots, n$. Let $t \geq r - 2a$. Since A is irreducible, $r > a_{ii} \geq a$, $i = 1, \ldots, n$, whence $r + t \geq 2(r - a) > 0$. Hence $S = (r + t)^{-1}(tI + B)$ is stochastic. Suppose $1 \leq i \leq n$. Then $(r + t)s_{ii} = t + b_{ii} = t + a_{ii}$ and $(r + t)(1 - s_{ii}) = b_i' = r - a_{ii}$. This proves (6.1)(b). Finally, $t + 2a_{ii} \geq t + 2a \geq r$ whence $t + a_{ii} \geq r - a_{ii}$, and (6.1)(c) follows from (6.1)(b).

ii) Let $s_{ii} = \frac{1}{2}$. Then by (6.1)(b), $t + a_{ii} = r - a_{ii}$, whence

$$t = r - 2a_{ii} \leq r - 2a \leq t \tag{6.3}$$

and (6.2) follows. Conversely, if (6.2) holds then the equalities hold in (6.3). Thus $t + a_{ii} = r - 2a + a_{ii} = r - a_{ii}$, whence $s_{ii} = \frac{1}{2}$.

THEOREM 4 *Let A be a nonnegative matrix in $\mathbf{R}_{\langle n\rangle}$ with spectral radius r. Let $a = \min\{a_{ii}: i = 1, \ldots, n\}$.*

i) *If $t \geq r - 2a$, then*

$$\operatorname{per}(tI + A) \geq \prod_{i=1}^{n}(t + a_{ii}) + \prod_{i=1}^{n}(r - a_{ii}). \tag{6.4}$$

ii) *If A is irreducible and $t = r - 2a$, then the equality holds in* (6.4) *if and only if A satisfies* (4.2) *and*

$$\text{If } a_{ii} > a, \text{ then } A(i) \text{ is triangulable.} \tag{6.5}$$

iii) *If A is irreducible and $t > r - 2a$, then the equality holds in* (6.4) *if and only if A satisfies* (2.6)(b).

Proof. If $n = 1$, the theorem is trivial, so let $n > 1$.

i) The functions $\operatorname{per}(A + tI)$ and r are continuous in the elements of A. Hence it is enough to prove the result for irreducible A. Let A be irreducible, and let $t > r - 2a$. Let S be the matrix satisfying (6.1). By Theorem 3(i),

$$\operatorname{per} S \geqslant \prod_{i=1}^{n} s_{ii} + \prod_{i=1}^{n} (1 - s_{ii}). \tag{6.6}$$

Since $\operatorname{per} A = (r + t)^n \operatorname{per}(tI + A)$, the inequality (6.4) follows from (6.6), and (6.1)(b).

ii) Let A be irreducible and let $t = r - 2a$. Let S be the matrix of (6.1). Suppose the equality holds in (6.4). Then the equality holds in (6.6), whence by Theorem 3(ii), S satisfies (3.5) and (4.2). For $1 \leqslant i \leqslant n$, by Lemma 8, $s_{ii} > \frac{1}{2}$ is equivalent to $a_{ii} > a$. Hence A satisfies (6.5) and (4.2).

The converse is obtained by reversing the above argument.

iii) Let A be irreducible and let $t > r - 2a$. Let S be the matrix of (6.1). Suppose the equality holds in (6.4). By Lemma 8(ii), $s_{ii} > \frac{1}{2}$, $i = 1, \ldots, n$ and hence (2.6) holds for S, and so for A. But A is irreducible, and therefore not triangulable. Thus A satisfies (2.6)(b).

Conversely, if (2.6)(b) holds, then by theorem 3(iii), the equality holds in (6.6) and therefore in (6.4).

COROLLARY 4 *Let A be a nonnegative matrix in $\mathrm{R}_{\langle n \rangle}$ with spectral radius r. Let $a = \min\{a_{ii}: i = 1, \ldots, n\}$.*

i) *If $t \geqslant r - 2a$, then*

$$\operatorname{per}(tI + A) \geqslant \tfrac{1}{2}\left(\frac{r + t}{2}\right)^n. \tag{6.7}$$

ii) *Let $n \geqslant 2$, and let A be irreducible. Then the equality holds in* (6.7) *if and only if $t = r - 2a$, A satisfies* (4.2), (6.5) *and*

There is a j, $1 \leqslant j \leqslant n$, such that $a_{ii} = a$, for $i \neq j$, $i = 1, \ldots, n$. (6.8)

Proof i) Since, $t + a_{ii} \geqslant r - a_{ii}$, $i = 1, \ldots, n$, the result follows from Theorem 4(i) and Lemma 7(i).

ii) Let S be the matrix of (6.1) and suppose the equality holds in (6.7). Then the equality holds in (5.5), whence S satisfies (4.2), (3.5) and (5.6). Since $n \geqslant 2$, there is a k, $1 \leqslant k \leqslant n$ such that $s_{kk} = \frac{1}{2}$. By applying Lemma 8(ii) to (3.5) and (5.6) for S, we see that $t = r - 2a$ and that A satisfies (4.2), (6.5) and (6.8). Conversely, if $t = r - 2a$, and A satisfies (4.2), (6.5) and (6.8), then S satisfies (4.2), (3.5) and (5.6), whence the equalities hold in (5.5) and (6.7).

A matrix $M \in \mathbf{R}_{\langle n \rangle}$ is called an M-matrix if there is a nonnegative $A \in \mathbf{R}_{\langle n \rangle}$ with spectral radius r such that $M = tI - A$ and $t \geqslant r$. (Our definition is equivalent to Ostrowski [4].) If $z = t - r$, then z is an eigenvalue of M, and we shall call z the *minimal eigenvalue* of M. The upper bound $\det M \leqslant t^n - r^n$ is due to J. Keilson and G. P. H. Styan [3], Theorem 2. We now improve this bound.

THEOREM 5 *Let M be an M-matrix in $\mathbf{R}_{\langle n \rangle}$ with minimal eigenvalue z.*
i) *Then*

$$\det M \leqslant \prod_{i=1}^{n} m_{ii} - \prod_{i=1}^{n} (m_{ii} - z). \tag{6.9}$$

ii) *If M is singular, then both sides of* (6.9) *are* 0.
iii) *If M is nonsingular and irreducible, then the equality holds in* (6.9) *if and only if M satisfies* (2.6)(b).

The proof of Theorem 5 is by use of Lemma 8 and Theorem 1 (in a manner similar to the use of Lemma 8 and Theorem 3 in the proof of Theorem 4). We give a short sketch of the proof. If M is an irreducible M-matrix, then M is diagonally similar to an irreducible matrix $P \in \mathscr{M}^{d}_{\langle n \rangle}$ such that $p_{ii} = m_{ii}$, and $p'_i = m_{ii} - z$, $i = 1, \ldots, n$. There are two cases in the discussion of equality in (6.9): either $z = 0$ and $p'_i = p_{ii}$, $i = 1, \ldots, n$, or $z = 0$ and $p'_i < p_{ii}$, $i = 1, \ldots, n$.

7 NONNEGATIVE DEFINITE HERMITIAN MATRICES

Analogues to Theorems 4 and 5 hold for nonnegative definite Hermitian matrices. Their proof involves techniques similar to those used in this paper. It appears possible to define a class of matrices which contain all M-matrices and all nonnegative definite Hermitian matrices in which similar theorems hold. We now state the analogue of Theorem 5 for a nonnegative definite Hermitian matrix, omitting the proof.

THEOREM 6 *Let H be a nonnegative definite Hermitian matrix in $\mathbf{C}_{\langle n \rangle}$. Let r denote the smallest eigenvalue of H.*
i) *Then*

$$\det H \leqslant \prod_{i=1}^{n} h_{ii} - \prod_{i=1}^{n} (h_{ii} - r). \tag{7.1}$$

ii) *If H is singular, then both sides of* (7.1) *are* 0.
iii) *If H is nonsingular, then equality holds in* (7.1) *if and only if either H is of dimension* 2 *or H is a diagonal matrix.*

Note added in proof: The inequality of Theorem 3(i) has been found independently by James J. Johnson: *Bounds for Certain Permanents and Determinants*, Lin. Alg. Apps., to appear.

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