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Matrices Hermitian for an Absolute Norm

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Let ν be a (standardized) absolute norm on C_n. A matrix H in C_{nn} is called norm-Hermitian if the numerical range $V(H)$ determined by ν is real. Let $\mathscr{H}$ be the set of all norm-Hermitians in C_{nn}. We determine an equivalence relation $\sim$ on $\{1, \ldots, n\}$ with the following property: Let $H \in C_{nn}$. Then $H \in \mathscr{H}$ if and only if H is Hermitian and $h_{ij} = 0$ if $i \nsim j$. Let $\mathscr{J} = \mathscr{H} + i\mathscr{H}$. Then $\mathscr{J}$ is a subalgebra of C_{nn} and, for $A \in \mathscr{J}$, $V(A)$ equals the Euclidean numerical range and hence is convex. Let $\mathscr{V}$ be the group of isometries for ν, and let $\mathscr{U} = \{\exp(iH)\colon H \in \mathscr{H}\}$. Then $\mathscr{U}$ is a normal subgroup of $\mathscr{V}$ and $\mathscr{V} = \mathscr{U}\mathscr{P}$, where $\mathscr{P}$ is a group of permutation matrices.

For an operator, the concept of the numerical range (field of values) with respect to a norm on the underlying space was introduced independently by Lumer [5] and Bauer [1]. By now there are many interesting applications (cf. Bonsall and Duncan [3]); some of the most fascinating concern norm-Hermitian operators—operators whose numerical range is real. In this paper we consider a special but not unimportant case: (1) Our space will be C_n, the complex n-tuples—concretely given; and (2) We shall consider a norm ν which depends only on the absolute values of the coordinates of $x \in C_n$. Such norms are called absolute (cf. Bauer, Stoer, Witzgall [2], and Bauer [1]). For the sake of convenience we shall also standardize ν so that $\nu(e^i) = 1$, for all canonical basis vectors e^i in C_n.

Our main results are these:

We show that it is possible to determine an equivalence relation $\sim$ on $\{1, \ldots, n\}$ such that a matrix H in C_{nn} is norm-Hermitian if and only if H is Hermitian,‡ and $h_{ij} = 0$ if $i \nsim j$ (theorem (6.2)). If $\mathscr{H}$ is the set of

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‡ We shall always use the term Hermitian matrix H in the traditional sense: $h_{ij} = \bar{h}_{ji}$, $i, j = 1, \ldots, n$. A matrix with real numerical range will be called norm-Hermitian or ν-Hermitian.

norm-Hermitian matrices and $\mathscr{J} = \mathscr{H} + i\mathscr{H}$, then $\mathscr{J}$ is a subalgebra of C_{nn}, and for each $A \in \mathscr{J}$, the numerical range $V(A)$ equals the Euclidean numerical range. Hence $V(A)$ is convex for all $A \in \mathscr{J}$ (theorems (6.3) and (6.4)). Let ν^0 be the operator norm associated with ν. We also show that $\nu^0(A) = \chi^0(A)$, for all $A \in \mathscr{J}$, where χ is the Euclidean norm on C_n (6.5). It is well known that, for $H \in \mathscr{H}$, $\exp(iH)$ is a ν-isometry on C_n. By use of our characterization of norm-Hermitian matrices, we show that the set $\mathscr{U}$ of all ν-isometries of the form $\exp(iH)$, $H \in \mathscr{H}$, forms a normal subgroup of the group $\mathscr{V}$ of all ν-isometries, and that $\mathscr{V}/\mathscr{U}$ is finite. More precisely, there is a group $\mathscr{P}$ of ν-isometries which are also permutation matrices such that for each $V \in \mathscr{V}$ there exist unique $U \in \mathscr{U}$, $P \in \mathscr{P}$ such that $V = UP$ (7.3, 7.7, 7.8).

While the absoluteness of the norm ν plays an essential role in our results, the standardization $\nu(e^i) = 1$, $i = 1, \ldots, n$ is a matter of convenience. Thus, a simple modification of our results will make them applicable to all absolute norms. In the case of our main theorems, we give them also in this more general form.

NOTATIONS AND DEFINITIONS

1.1 Coordinate subspaces

Let C be the complex field, R the real field, R^+ the set of nonnegative numbers. We put

$$C_n = \{x = (x_1, \ldots, x_n) \colon x_i \in C\}$$

and define R_n, R_n^+ analogously. By e^i, $i = 1, \ldots, n$ we denote the vector in C_n (or R_n) defined by $e^i_i = 1$, $e^i_j = 0$ otherwise. We call e^i a unit vector. A coordinate subspace of C_n (or R_n) is the space spanned by a set of unit vectors.

1.2 Norms

On C_n (or on any coordinate subspace of C_n), χ will denote the Euclidean norm

$$\chi(x) = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}.$$

If $x \in C_n$, then $|x| = (|x_1|, \ldots, |x_n|) \in R_n^+$.

A norm on C_n is (as usual) a function ν of C_n into R^+ such that

i) $\nu(x) = 0$ if and only if $x = 0$,

ii) $\nu(x + z) \leqslant \nu(x) + \nu(z)$

iii) $\nu(\alpha x) = |\alpha| \nu(x)$, for $\alpha \in C$.

A norm ν is called *absolute* if, in addition,

iv) $\nu(x) = \nu(|x|)$, all $x \in C_n$,

and *standardized* if

v) $\nu(e^i) = 1$, $i = 1, \ldots, n$.

Unless otherwise stated, ν will always denote a *standardized absolute* norm.

We also make the following conventions. If $x \in C_n$, then x_i is the ith coordinate of x. On the other hand, if C_n is split as the direct sum of coordinate subspaces: $C_n = E_1 \oplus \cdots \oplus E_r$, $x_{(i)}$ will denote the component of x in E_i. If dim $E_i = m$, then $x_{(i)} \in C_m$, and we therefore write

$$x = x_{(1)} \oplus \cdots \oplus x_{(r)}$$

(rather than $x = x_{(1)} + \cdots + x_{(r)}$).

1.3 Dual norms and numerical ranges of matrices

For $x, y \in C_n$, we put

$$\langle y, x \rangle = \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n.$$

If ν is a norm on C_n, so is ν^D:

$$\nu^D(y) = \sup_{x \neq 0} \frac{|\langle y, x \rangle|}{\nu(x)}.$$

If ν is absolute, so is ν^D (cf. [2]), and it is easy to see that if ν is standardized absolute, so is ν^D.

If $x, y \in C_n$, and $1 = \langle y, x \rangle = \nu^D(y)\nu(x)$, then y is called dual to x; we write $y \| x$. It is well known that for each $x \in C_n$, $x \neq 0$, there is at least one $y \in C_n$ such that $y \| x$, and for each $y \in C_n$, $y \neq 0$, there is an $x \in C_n$ such that $y \| x$.

By C_{nn} we denote the set of all $(n \times n)$ matrices over C. The *numerical range* $V(A)$ for $A \in C_{nn}$ is defined by

$$V(A) = \{\langle y, Ax \rangle : x, y \in C_n \text{ and } y \| x\}.$$

If $V(A)$ is real, then A is called *norm-Hermitian* or *ν-Hermitian*.

2

2.1

DEFINITION *Let ν be a standardized absolute norm C_n. On $\{1, 2, \ldots, n\}$ we define a relation $\sim$ thus: $i \sim j$ if for all $x, y \in C_n$ such that*

$$|x_i|^2 + |x_j|^2 = |y_i|^2 + |y_j|^2, \quad \text{and} \quad |x_k| = |y_k|, \quad \text{for } k \neq i, j,$$

we have $\nu(x) = \nu(y)$.

2.2

LEMMA *The relation $\sim$ is an equivalence relation on $\{1, 2, \ldots, n\}$.*

Proof Since ν is absolute, $i \sim i$ for all $i = 1, 2, \ldots, n$. Clearly, $i \sim j$ implies that $j \sim i$. Suppose that h, i, j are distinct integers with $h \sim i$ and $i \sim j$. For $x \in C_n$, define $\tilde{x}$ by $\tilde{x}_h = \tilde{x}_j = 0$, $\tilde{x}_i = (|x_h|^2 + |x_i|^2 + |x_j|^2)^{\frac{1}{2}}$ and $\tilde{x}_k = |x_k|$ for $k \neq h, i, j$. It is easy to see that $\nu(x) = \nu(\tilde{x})$. Now suppose that $x, y \in C_n$, and $|x_h|^2 + |x_j|^2 = |y_h|^2 + |y_j|^2$, and $|x_k| = |y_k|$ for $k \neq h, j$. Then $|x_h|^2 + |x_i|^2 + |x_j|^2 = |y_h|^2 + |y_i|^2 + |y_k|^2$ whence $\tilde{x} = \tilde{y}$. Thus $\nu(x) = \nu(\tilde{x}) = \nu(\tilde{y}) = \nu(y)$. It follows that $h \sim j$.

2.3

LEMMA *Let ν be a standardized absolute norm on C_n. Let $N_1, \ldots, N_r$ be the equivalence classes in $\{1, \ldots, n\}$ given by $\sim$. Let E_k be the coordinate subspace spanned by the vectors e^i with $i \in N_k$, and write $x \in C_n$ as $x = x_{(1)} \oplus \cdots \oplus x_{(r)}$, where $x_{(k)} \in E_k$. Then there is a standardized absolute norm μ on C_r such that $\nu(x) = \mu(\chi(x_{(1)}), \ldots, \chi(x_{(r)}))$.*

Proof Let us suppose that $N_1 = \{1, \ldots, s)$ (to save writing). Put $x^{(1)} = 0 \oplus x_{(2)} \oplus \cdots \oplus x_{(r)}$. Then

$$\begin{aligned}\nu(x) &= \nu\left(\sum_{i=1}^{s} x_i e^i + x^{(1)}\right) \\ &= \nu\left((|x_1|^2 + |x_2|^2)^{\frac{1}{2}} e^1 + \sum_{i=3}^{s} x_i e^i + x^{(1)}\right) \cdots \\ &= \nu(\chi(x_{(1)}) e^1 + x^{(1)}).\end{aligned}$$

After repetitions of this argument, we have

$$\nu(x) = \nu(\chi(x_{(1)}) e^{j_1} + \cdots + \chi(x_{(r)}) e^{j_r}),$$

where $j_i \in N_i$, $i = 1, \ldots, r$. So, for $\alpha \in C_r$, we define

$$\mu(\alpha) = \nu\left(\sum_{k=1}^{r} \alpha_k e^{j_k}\right).$$

Then $\nu(x) = \mu(\chi(x_{(1)}), \ldots, \chi(x_{(r)}))$. It is easily verified that μ is a standardized absolute norm on C_r.

2.4

COROLLARY *Let $U \in C_{nn}$ be a unitary matrix such that $u_{ij} = 0$ if $i \nsim j$. Then U is a ν-isometry* (i.e., $\nu(Ux) = \nu(x)$ *for all* $x \in C_n$).

Proof We may write

$U = U_1 \oplus \cdots \oplus U_r$, where U_i is a unitary matrix on E_i, $i = 1, \ldots, r$. Since $\chi(U_i x_{(i)}) = \chi(x_{(i)})$, $i = 1, \ldots, r$, we have

$$\nu(x) = \mu(\chi(x_{(1)}), \ldots, \chi(x_{(r)})) = \mu(\chi(U_1 x_{(1)}), \ldots, \chi(U_r x_{(r)})) = \nu(Ux).$$

2.5

COROLLARY *If $i \sim j$ for all $i, j \in (1, \ldots, n)$, then $\nu = \chi$.*

Proof In this case $x_{(1)} = x \in C_n$, and so by (2.3) and since μ is standardized, $\nu(x) = \mu(\chi(x)) = \chi(x)$.

3

In this section we shall explore the geometric significance of the equivalence relation introduced in Section 2. We begin with a simple geometric lemma on real 2-space.

If K is a convex body in R_2, denote its boundary by ∂K, and put $K^+ = K \cap R_2^+$.

3.1

LEMMA *Let K be a convex body in R_2 such that $0 \in K$, $(1, 0) \in \partial K$ and $(0, 1) \in \partial K$. Then there exists a $P = (x_1, x_2) \in \partial K$ with $x_1 > 0$, $x_2 > 0$ such that the perpendicular l to OP through P is a support line to K.*

Proof For each θ, $0 \leqslant \theta \leqslant \pi/2$, let $r(\theta) \in (\cos\theta, \sin\theta) \in \partial K$. Then r is nonzero and continuous in $[0, \pi/2]$. Hence r attains its maximum M and its minimum m in that interval, and $0 < m \leqslant 1 \leqslant M$.

We consider three cases (which overlap).

Case 1 $m = 1 = M$.

In this case K^+ is a quarter circle. If P is any point on the boundary, the perpendicular l through P is a support line to K.

Case 2 $1 < M$, say $r(\theta_0) = M$. Clearly $0 < \theta_0 < \pi/2$. Thus K^+ is contained in the circle center O, radius M. If $P = r(\theta_0)(\cos\theta_0, \sin\theta_0)$, then l is a support line.

Case 3 $m < 1$, say $r(\theta_1) = m$. Again $0 < \theta_1 < \pi/2$. Let

$$P = r(\theta_1)(\cos\theta_1, \sin\theta_1),$$

and l the perpendicular to OP at P. We claim that l is the (only) support line to K at P. Suppose l is not a support line to K at P. Then there exists a support line l' at P, and l' is not perpendicular to OP. Since $(1, 0) \in K$,

$(0, 1) \in K$, and $|OP| = m < 1$, the slope of l' is negative. Hence the perpendicular to l' from O meets l' is a point Q in the first quadrant. Clearly $|OQ| < |OP|$. Since Q is either on the boundary of K or in the exterior of K, there is a point $R = r(\theta_2)(\cos\theta_2, \sin\theta_2)$, $0 < \theta_2 < \pi/2$ on OQ which is on the boundary of K. Thus

$$r(\theta_2) = |OR| \leqslant |OQ| < |OP| = m,$$

a contradiction.

The lemma is proved.

3.2

COROLLARY *Let κ be a standardized absolute norm on C_2. Then there exists an $x \in R_2^+$, with $x_1 > 0$ and $x_2 > 0$ such that $\langle x, x\rangle^{-1}x \| x$.*

Proof Let $K = \{x \in R_2 : \kappa(x) \leqslant 1\}$. Then K is convex and satisfies the conditions of (3.1). Let $P = x$, where $x_1 > 0$, $x_2 > 0$, be a point such that the perpendicular l through P to OP is a support line to K. Then for all $z \in R_2^+$, we have $\langle x, z\rangle \leqslant \langle x, x\rangle\kappa(z)$. Since κ is absolute, it follows that $|\langle x, z\rangle| \leqslant \langle x, x\rangle\kappa(z)$ for all $z \in C_2$. Hence $\kappa^D(x) = \langle x, x\rangle$, and $\langle x, x\rangle^{-1}x \| x$.

3.3

DEFINITIONS 1) *Let $1 \leqslant i, j \leqslant n$; $i \neq j$. In the rest of this section, we shall write $E' = \operatorname{span}\{e^i, e^j\}$, $E'' = \operatorname{span}\{e^k : k \neq i, j\}$. For $x \in C_n$, we shall put $x = x' \oplus x''$, where $x' \in E'$, $x'' \in E''$. Also $x' = (x_i, x_j)$, and we shall identify x'' and $0 \oplus x''$, where $0 \in E'$.*

2) *Let $K = \{x \in C_n : \nu(x) \leqslant 1\}$. If $x'' \in E''$ and $\nu(x'') \leqslant 1$, we put*

$$K_{x''} = \{x' \in E' : \nu(x' \oplus x'') \leqslant 1\}.$$

We call $K_{x''}$ a section *of K. Suppose that $x'' \in E''$ and $\nu(x'') \leqslant 1$. Let $\kappa_{x''}$ be the mapping of E' into $R^+ \cup \{\infty\}$ defined by*

$$\kappa_{x''}(x') = \inf\left(\alpha > 0 : \frac{1}{\alpha}x' \in K_{x''}\right).$$

(Thus $\kappa_{x''}(x') = \infty$ if $\beta x' \in K_{x''}$ and $\beta \geqslant 0$ imply that $\beta = 0$.)

3) *We shall call $K_{x''}$* circular *if there is a nonnegative r such that*

$$K_{x''} = \{(x_i, x_j) : |x_i|^2 + |x_j|^2 \leqslant r^2\}.$$

4) *Let $x = x' \oplus x''$, $y = y' \oplus y''$ be elements of C_n. We shall write $y|x$ if*

a) $y \| x$,

and

b) *There is a positive d such that $y' = dx'$.*

3.4

LEMMA *Let $x'' \in E''$ with $\nu(x'') \leqslant 1$. Then*

1) *$K_{x''}$ is a convex body in E' with $0 \in K_{x''}$.*
2) *$x' \in K_{x''}$ if and only if $|x'| \in K_{x''}$.*
3) *If $\nu(x'') < 1$, then $0 \in \operatorname{int} K_{x''}$.*
4) *If $\nu(x'') < 1$, then $\kappa_{x''}$ is an absolute norm on E' and is standardized if $x'' = 0$.*

Proof 1) Clearly $0 \in K_{x''}$ since $\nu(0 \oplus x'') \leqslant 1$. If $x', y' \in K_{x''}$ and $0 \leqslant \alpha \leqslant 1$, then

$$\nu(\alpha x' + (1-\alpha)y' \oplus x'') \leqslant \alpha\nu(x' \oplus x'') + (1-\alpha)\nu(y' \oplus x'') \leqslant 1,$$

when $\alpha x' + (1-\alpha)y' \in K_{x''}$. Thus $K_{x''}$ is convex.

2) Since $\nu(x' \oplus x'') = \nu(|x'| \oplus x'')$, 2) follows.

3) Suppose $\nu(x'') < 1$. Then for all $x' \in E'$ with $\nu(x') < 1 - \nu(x'')$, we have $\nu(x' \oplus x'') \leqslant \nu(x') + \nu(x'') < 1$, whence $x' \in K_{x''}$. Hence $0 \in \operatorname{int} K_{x''}$.

4) Follows immediately from 1), 2), and 3).

3.5

THEOREM *Let ν be a standardized absolute norm on C_n, and let $1 \leqslant i, j \leqslant n$. Then $i \sim j$ if and only if for all $x'' \in E''$ with $\nu(x'') \leqslant 1$, the section $K_{x''}$ is circular.*

Proof Suppose that $i \sim j$ and let $x'' \in E''$ with $\nu(x'') \leqslant 1$. Let $x', y' \in E'$ and assume that $\chi(x') = \chi(y')$. Then $\nu(y' \oplus x'') = \nu(x' \oplus x'')$. Hence $x' \in K_{x''}$ if and only if $y' \in K_{x''}$. Thus $K_{x''}$ is circular.

Conversely, suppose that $K_{x''}$ is circular for all $x'' \in E''$ with $\nu(x'') \leqslant 1$. Let $x, y \in E$ and assume that $x = x' \oplus x''$, $y = y' \oplus y''$ where $\chi(x') = \chi(y')$ and $|x''| = |y''|$. If $x' = 0$, then $y' = 0$ or $\nu(x) = \nu(y)$. So suppose that $x' \neq 0$. Thus $y' \neq 0$. Put $u = x/\nu(x)$, $v = y/\nu(x)$ and observe that $\nu(u) = 1$. If $u = u' \oplus u''$, then $\nu(u'') \leqslant 1$, since ν is absolute (cf. [2]). Thus $u' \in K_{u''}$. But if $v = v' \oplus v''$, then $\chi(v') = \chi(u')$, and since $K_{u''}$ is circular, we also have that $v' \in K_{u''}$. Further $|v''| = |u''|$, whence $\nu(v) = \nu(v' \oplus u'') \leqslant 1$. It follows that $\nu(x) \geqslant \nu(y)$. Reversing the roles of x and y, we obtain $\nu(y) \geqslant \nu(x)$, whence $\nu(x) = \nu(y)$. Thus $i \sim j$.

3.6

LEMMA *If for all $x'' \in E''$ with $\nu(x'') < 1$, the section $K_{x''}$ is circular then $K_{x''}$ is also circular if $x'' \in E''$ and $\nu(x'') = 1$.*

Proof Let $x'' \in E''$ with $\nu(x'') = 1$. Let $x', y' \in E', x' \in K_{x''}$ and $\chi(y') = \chi(x')$. We must show that $y' \in K_{x''}$.

Let $0 < \varepsilon < 1$. Since $\nu(x' \oplus x'') \leqslant 1$ and ν is absolute, $\nu(x' \oplus (1-\varepsilon)x'') \leqslant 1$. But $K_{(1-\varepsilon)x''}$ is circular, whence $\nu(y' \oplus (1-\varepsilon)x'') \leqslant 1$. Hence also

$$\nu(y' \oplus x'') \leqslant 1,$$

and the desired result $y' \in K_{x''}$ follows.

3.7

LEMMA *Let $x = x' \oplus x'' \in R_n^+$, where $\nu(x'') < \nu(x) = 1$. If $y = y' \oplus y'' \in R_n^+$ and $y \| x$ then $x' \neq 0$, $c^{-1} = 1 - \langle y'', x'' \rangle > 0$ and $cy' \| x'$ with respect to the norm $\kappa_{x''}$.*

Proof Clearly $x' \neq 0$. Since $\nu^D(y'') \leqslant \nu^D(y) \leqslant 1$, $1 - \langle y'', x'' \rangle > 0$. Hence $c = (1 - \langle y'', x'' \rangle)^{-1} > 0$. Clearly $\langle y', x' \rangle = 1 - \langle y'', x'' \rangle$. Let $\kappa_{x''}(z') = 1$. Then also $\kappa_{x''}(|z'|) = 1$, whence $\nu(|z'| \oplus x'') = 1$. Hence

$$\langle y', |z'| \rangle + \langle y'', x'' \rangle \leqslant 1$$

whence

$$|\langle y', z' \rangle| \leqslant \langle y', |z'| \rangle \leqslant 1 - \langle y'', x'' \rangle$$

Hence $(\kappa_{x''})^D(y') = 1 - \langle y'', x'' \rangle = c^{-1}$ and $cy' \| x'$, with respect to $\kappa_{x''}$.

3.8

LEMMA *Let $x'' \in E'' \cap R_{n-2}^+$, where $\nu(x'') < 1$. Suppose that for all $x' \in E' \cap R_2^+$ such that $\nu(x) = 1$, $x = x' \oplus x''$, there is a $y \in R_n^+$ such that $y | x$. Then the the section $K_{x''}$ is circular.*

Proof Let x' satisfy the hypotheses of the lemma. Let $y \in R_n^+$, $y \| x$, $\nu^D(y) = 1$ and $y' = dx'$, where $d > 0$. By (3.7) there is a positive c such that $cy' \| x'$ and hence $cdx' \| x'$ with respect to the norm $\kappa_{x''}$. Applying Lemma (3.1) of Gries [5], we see that the corresponding norm body $K_{x''} \cap R_2$ is circular. But by (2) of (3.4), it now follows that $K_{x''}$ is circular in the complex space E'.

3.9

THEOREM *Let ν be a standardized absolute norm and let E', E'' be defined as in (3.3). Suppose for all $x \in R_n^+$ with $\nu(x'') < \nu(x) = 1$ there is a $y \in R_n^+$ such that $y | x$. Then $i \sim j$.*

Proof Let $x'' \in E'' \cap R_{n-2}^+$, where $\nu(x'') < 1$. By (3.8), $K_{x''}$ is circular. It follows from (2) of (3.4) that $K_{x''}$ is circular if $x'' \in E''$ and $\nu(x'') < 1$. But now it follows from (3.6) that $K_{x''}$ is circular for all $x'' \in E''$ such that $\nu(x'') \leqslant 1$. By (3.5), $i \sim j$.

4

4.1

LEMMA *Let E_i, $i = 1, \ldots, r$ be coordinate subspaces of C_n such that $C_n = E_1 \oplus \cdots \oplus E_r$. Let λ_i be a standardized absolute norm on E_i, let μ be a standardized absolute norm on C_r, and suppose*

$$\nu(x_{(1)} \oplus \cdots \oplus x_{(r)}) = \mu(\lambda_1(x_{(1)}), \ldots, \lambda_r(x_{(r)})) \qquad (4.1.1)$$

where $x_{(i)} \in E_i$, $i = 1, \ldots, r$. Then

1) *ν is a standardized absolute norm on C_n,*

2) $\nu^D(y_{(1)} \oplus \cdots \oplus y_{(r)}) = \mu^D(\lambda_1^D(y_{(1)}), \ldots, \lambda_r^D(y_{(r)}))$

for $y_{(i)} \in E_i$, $i = 1, \ldots, r$.

Further, let $x = x_{(1)} \oplus \cdots \oplus x_{(r)}$, $x_{(i)} \in E_i$, $y = y_{(1)} \oplus \cdots \oplus y_{(r)}$, $y_{(i)} \in E_i$, and suppose that $\lambda_i(x_{(i)}) = \alpha_i$, $\lambda_i^D(y_{(i)}) = \beta_i$, $i = 1, \ldots, j$. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_r)$. Then

3) *$y \| x$ with respect to ν if and only if*

a) *$\beta \| \alpha$ with respect to μ, and*

b) *$\beta_i^{-1} y_{(i)} \| \alpha_i^{-1} x_{(i)}$ with respect to λ_i, whenever $\beta_i \alpha_i > 0$, $i = 1, \ldots, r$.*

Proof 1) Let

$$x = x_{(1)} \oplus \cdots \oplus x_{(r)},$$
$$z = z_{(1)} \oplus \cdots \oplus z_{(r)}.$$

Then†

$$\begin{aligned}\nu(x + z) &= \mu(\lambda_1(x_{(1)} + z_{(1)}), \ldots, \lambda_r(x_{(r)} + z_{(r)})) \\ &\leqslant \mu(\lambda_1(x_{(1)}) + \lambda_1(z_{(1)}), \ldots, \lambda_r(x_{(r)}) + \lambda_r(z_{(r)})) \\ &\leqslant \mu(\lambda_1(x_{(1)}), \ldots, \lambda_r(x_{(r)})) + \mu(\lambda_1(z_{(1)}), \ldots, \lambda_r(z_{(r)})) \\ &= \nu(x) + \nu(z).\end{aligned}$$

Here the first inequality follows from the absoluteness of μ. Similarly,

$$\nu(\alpha x) = |\alpha| \nu(x), \quad \text{and} \quad \nu(|x|) = \nu(x),$$

since all of μ and λ_i are absolute. Clearly $\nu(e^i) = \lambda_j(f)$, for some j, $1 \leqslant j \leqslant r$, and some unit vector f of E_j, whence $\nu(e^i) = 1$. This proves 1).

2) and 3) Suppose that α and β are defined as in the statement of the lemma. Let $y \in C_n$. Then for any x with $\nu(x) = 1$ we have $\mu(\alpha) = 1$ and

$$\begin{aligned}|\langle y, x\rangle| &\leqslant |\langle y_{(1)}, x_{(1)}\rangle| + \cdots + |\langle y_{(r)}, x_{(r)}\rangle| \\ &\leqslant \beta_1\alpha_1 + \cdots + \beta_r\alpha_r \\ &\leqslant \mu^D(\beta)\mu(\alpha) = \mu^D(\beta).\end{aligned}$$

Hence $\nu^D(y) \leqslant \mu^D(\beta)$.

† This argument also occurs in Ostrowski [13] p. 12, where it is merely assumed that μ is monotonic in R_n^+.

Suppose further that x is so chosen that $\beta \| \alpha$ and that for $i = 1, \ldots, r$, $\beta_i^{-1} y_{(i)} \| \alpha_i^{-1} x_{(i)}$, whenever $\beta_i \alpha_i > 0$. Since $\langle y_{(i)}, x_{(i)} \rangle = 0$ whenever $\beta_i \alpha_i = 0$, it follows that

$$\begin{aligned}\langle y, x \rangle &= \langle y_{(1)}, x_{(1)} \rangle + \cdots + \langle y_{(r)}, x_{(r)} \rangle \\ &= \beta_1 \alpha_1 + \cdots + \beta_r \alpha_r \\ &= \mu^D(\beta)\mu(\alpha) = 1.\end{aligned}$$

Hence $\nu^D(y)$ is given by 2), and if x satisfies the conditions of 3), then $y \| x$.

We must still prove that for all pairs, $x, y \in C_n$ with $y \| x$, the conditions of 3) are satisfied. So suppose that $y \| x$. Then

$$1 = \langle y, x \rangle = \nu^D(y)\nu(x) = \mu^D(\beta)\mu(\alpha)$$

and

$$(*) \qquad \begin{aligned}\langle y, x \rangle &= \langle y_{(1)}, x_{(1)} \rangle + \cdots + \langle y_{(r)}, x_{(r)} \rangle \\ &\leqslant |\langle y_{(1)}, x_{(1)} \rangle| + \cdots + |\langle y_{(r)}, x_{(r)} \rangle| \\ &\leqslant \beta_1 \alpha_1 + \cdots + \beta_r \alpha_r \\ &\leqslant \mu^D(\beta)\mu(\alpha).\end{aligned}$$

Hence all inequalities in (*) are equalities, and

$$1 = \mu^D(\beta)\mu(\alpha) = \beta_1 \alpha_1 + \cdots + \beta_r \alpha_r.$$

Thus $\beta \| \alpha$ follows.

Finally, suppose that $\beta_i \alpha_i > 0$. Since $|\langle y_{(i)}, x_{(i)} \rangle| \leqslant \beta_i \alpha_i$ and since we have equalities in (*), we may deduce that

$$\langle y_{(i)}, x_{(i)} \rangle = |\langle y_{(i)}, x_{(i)} \rangle| = \beta_i \alpha_i.$$

Hence $1 = \langle \beta_i^{-1} y_{(i)}, \alpha_i^{-1} x_{(i)} \rangle = \lambda_i^D(\beta_i^{-1} y_{(i)}) \lambda_i(\alpha_i^{-1} x_{(i)})$. Thus $\beta_i^{-1} y_{(i)} \| \alpha_i^{-1} x_{(i)}$.

4.2

Counterexample If we drop the condition that the λ_i are absolute, then ν will still be a norm on C_n. But the condition that μ is absolute cannot be omitted. Consider the following counterexample. Let E_1, E_2 be the two one-dimensional coordinate subspaces of C_2. Let $\lambda_1(x_1) = |x_1|$, $\lambda_2(x_2) = |x_2|$, and let $\mu(\alpha_1, \alpha_2) = \max\{|\alpha_1 - \alpha_2|, |\alpha_2|\}$, and let $\nu(x_1, x_2) = \mu(\lambda_1(x_1), \lambda_2(x_2))$. If $x = (2, 1)$, $z = (1, -1)$, then $x + z = (3, 0)$. Hence

$$\begin{aligned}\nu(x) &= \mu(2, 1) = 1, \\ \nu(z) &= \mu(1, 1) = 1,\end{aligned}$$

but

$$\nu(x + z) = \mu(3, 0) = 3.$$

Thus $\nu(x + z) > \nu(x) + \nu(z)$.

We next slightly extend an important result due to Zenger† [12], (2.26).

† See also Stoer and Witzgall [14], Theorem 1.

4.3

LEMMA *Let μ be an absolute norm on C_r. Let $\gamma_i \geqslant 0$, $\gamma_1 + \cdots + \gamma_r = 1$. Then there exist $\alpha, \beta \in C_r$, such that $\beta \| \alpha$* and $\beta_i \alpha_i = \gamma_i$.

Proof If all $\gamma_i > 0$, then the existence of such α, β is guaranteed by Zenger's Lemma [12]. So suppose that, after reordering coordinates, $\gamma_i > 0$, $i = 1, \ldots, s$, $\gamma_i = 0$, $i = s + 1, \ldots, r$, where $s < r$. There exist $\alpha', \beta' \in C_s$ such that $\beta' \| \alpha'$ (with respect to the restriction of μ to C_s), and $\beta_i' \alpha_i' = \gamma_i$, $i = 1, \ldots, r$. Let $\beta = \beta' \oplus 0$, $\alpha = \alpha' \oplus 0$, where 0 is zero vector of C_{r-s}. Since μ is absolute, $\beta \| \alpha$ and clearly $\beta_i \alpha_i = \gamma_i$, $i = 1, \ldots, r$.

Remark Since $\beta \| \alpha$ implies that $\lambda^{-1}\beta \| \lambda\alpha$, for $\lambda > 0$, we may normalize $\mu(\alpha) = \mu^D(\beta) = 1$, in the above lemma.

4.4

DEFINITION *Let $\Sigma_1, \ldots, \Sigma_r$ be subsets of the complex plane. We define the convex sum of $\Sigma_1, \ldots, \Sigma_r$ to be the set of all sums $\alpha_1\sigma_1 + \cdots + \alpha_r\sigma_r$, where $\sigma_i \in \Sigma_i$, $0 \leqslant \alpha_i \leqslant 1$, $i = 1, \ldots, r$ and $\sum_{i=1}^{r} \alpha_i = 1$.*

Observe that the convex sum of sets need not be a convex set.

4.5

LEMMA *Let $E_1, \ldots, E_r$ be coordinate subspaces of C_n, and let ν be given as in* (4.1). *Let $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix acting on E_i, $i = 1, \ldots, r$. Then the numerical range of A is the convex sum of $V_1(A_1), \ldots, V_r(A_r)$, where $V_i(A_i)$ is the numerical range of A_i with respect to the norm λ_i.*

Proof Let $\gamma \in R_r^+$, $\sigma_i \in V_i(A_i)$, $i = 1, \ldots, n$ and $\sigma = \sum_{i=1}^{r} \gamma_i \sigma_i$ where $\sum_{i=1}^{r} \gamma_i = 1$.

Then there exist $y_{(i)}, x_{(i)} \in E_i$, such that $y_i \| x_i$ with respect to λ_i,

$$\lambda_i(x_i) = \lambda_i^D(y_i) = 1, \quad \text{and} \quad \langle y_{(i)}, A_i x_{(i)} \rangle = \sigma_i.$$

By (4.3) there exists $\alpha, \beta \in R_1^+$ such that $\beta \| \alpha$ with respect to μ, and $\beta_i \alpha_i = \gamma_i$. Let

$$x = \alpha_1 x_{(1)} \oplus \cdots \oplus \alpha_r x_{(r)},$$
$$y = \beta_1 y_{(1)} \oplus \cdots \oplus \beta_r y_{(r)}.$$

By (4.1), $y \| x$ with respect to ν. But

$$\langle y, Ax \rangle = \sum_{i=1}^{r} \beta_i \alpha_i \langle y_{(i)}, x_{(i)} \rangle.$$

$$= \sum_{i=1}^{r} \lambda_i \sigma_i = \sigma,$$

whence $\sigma \in V(A)$.

Conversely, let $\sigma \in V(A)$, say $\sigma = \langle y, Ax \rangle$ where $y \| x$ with respect to ν. Let us now write

$$x = x'_{(1)} \oplus \cdots \oplus x'_{(r)}$$

$$y = y'_{(1)} \oplus \cdots \oplus y'_{(r)},$$

where we put $\alpha_i = \lambda_i(x'_{(i)})$, $\beta_i = \lambda_i^D(y'_{(i)})$. Let us suppose that $\gamma_i = \beta_i \alpha_i > 0$, $i = 1, \ldots, s \leqslant r$, and $\gamma_i = \beta_i \alpha_i = 0$, $i = s + 1, \ldots, r$. Then putting $x_{(i)} = \alpha_i^{-1} x'_{(i)}$, $y_i = \beta_{(i)}^{-1} y'$, $i = 1, \ldots, s$, we have by (5.1) that $\beta \| \alpha$ with respect to μ, and $y_{(i)} \| x_{(i)}$ with respect to λ_i, $i = 1, \ldots, s$. Hence

$$\sigma_i = \langle y_{(i)}, A_i x_{(i)} \rangle \in V_i(A_i), \qquad i = 1, \ldots, s.$$

But $\gamma_i \geqslant 0$ and $\sum_{i=1}^{r} \gamma_i = 1$, and so

$$\langle y, Ax \rangle = \sum_{i=1}^{r} \langle y_{(i)}, A_i x_{(i)} \rangle = \sum_{i=1}^{s} \gamma_i \sigma_i = \sum_{i=1}^{r} \gamma_i \sigma_i.$$

The lemma is proved.

Comment Thus, for a norm ν satisfying $\nu(x) = \mu(\lambda_1(x_{(1)}), \ldots, \lambda_r(x_{(r)}))$, as in (4.1.1), and $A = A_1 \oplus \cdots \oplus A_r$, the numerical range $V(A)$ does not depend on μ. In particular, if ν is any (standardized) absolute norm on C_n, and $D = \operatorname{diag}(d_1, \ldots, d_n)$, then $V(D)$ is the convex hull of $d_1, \ldots, d_n$, cf. Gries [5].

If ν is any norm on C_n, then the corresponding operator norm ν^0 on C_n is defined

$$\nu^0(A) = \sup\{\nu(Ax) : \nu(x) = 1\}.$$

It is well known, and easy to prove, that

$$\nu^0(A) = \sup\{|\langle y, Ax \rangle| : \nu(x) = 1, \nu^D(y) = 1\}.$$

4.6

LEMMA *Let E_i, $i = 1, \ldots, r$ be coordinate subspaces of $|V$ such that $E_1 \oplus \cdots \oplus E_r = C_n$. Let λ_i be a standardized absolute norm on E_i, $i = 1, \ldots, r$ and μ a standardized absolute norm on C_r, and let ν be given by* (4.1.1). *Let $A \in C_{nn}$ and suppose that $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix on E_i. Then*

$$\nu^0(A) = \max\{\lambda_i^0(A_i) : \qquad i = 1, \ldots, r\}.$$

Proof Let $\max\{\lambda_i^0(A) : i = 1, \ldots, r\} = \lambda_k^0(A_k)$, where $1 \leqslant k \leqslant n$. Then using (4.1), we obtain

$$\begin{aligned}
\nu^0(A) &= \sup\{|\langle y, Ax\rangle| : \nu(x) = 1, \nu^D(y) = 1\}\\
&\leqslant \sup\left\{\sum_{i=1}^{r} |\langle y_{(i)}, A_i x_{(i)}\rangle| : \lambda_i(x_{(i)}) = \alpha_i, \lambda_i^D(y_{(i)}) = \beta_i, \mu(\alpha) = \mu^D(\beta) = 1\right\}\\
&\leqslant \sup\left\{\sum_{i=1}^{r} |\beta_i \lambda_i^0(A_i)\alpha_i| : \alpha, \beta \in E_n^+, \mu(\alpha) = \mu^D(\beta) = 1\right\}\\
&\leqslant \lambda_k^0(A_k)\left(\sum_{i=1}^{r} \beta_i\alpha_i : \alpha, \beta \in R_r^+, \mu(\alpha) = \mu^D(\beta) = 1\right)\\
&\leqslant \lambda_k^0(A_k).
\end{aligned}$$

On the other hand, let $x_{(k)}, y_{(k)} \in E_k$ such that $\lambda_k(x_{(k)}) = \lambda_k^D(y_{(k)}) = 1$, and $\lambda_k^0(A_k) = \langle y_{(k)}, A_k x_{(k)}\rangle$. If $x_{(i)} = y_{(i)} = 0$, for $i \neq k$, then $\nu(x) = \mu(e^k) = 1$, $\nu^D(y) = \mu^D(e^k) = 1$, and $\langle y, Ax\rangle = \langle y_{(k)}, A_k x_{(k)}\rangle = \lambda_k^0(A_k)$.

The lemma is proved.

We comment that it is almost as easy to prove (4.6) directly from the definition $\nu^0(x) = \sup\{\nu(Ax): \nu(x) = 1\}$, without use of (4.1). When $\dim E_i = 1$, $i = 1, \ldots, n$, (4.6) reduces to the well known theorem that $\nu^0(D) = \max\{|d_{ii}|, i = 1, \ldots, n\}$ for a diagonal matrix D, cf. [2].

5

5.1

LEMMA *Let $\Omega = \{u \in C_n : |u_i| = 1\}$. Let $K \in C_{nn}$ be a Hermitian matrix such that $k_{ii} = 0$, $i = 1, \ldots, n$. If $\langle u, Ku\rangle = 0$ for all $u \in \Omega$, then $K = 0$.*

Proof The proof is by induction on n. Evidently the result is true for $n = 1$. Suppose that it holds for $n = r - 1$, and let $n = r$.

Setting $u_i = e^{i\theta_i}$, we have

$$\begin{aligned}
\langle u, Ku\rangle &= \sum_{1 \leqslant i,j \leqslant r} k_{ij} e^{-i(\theta_j - \theta_i)}\\
&= 2 \sum_{1 \leqslant i \leqslant j \leqslant r} \mathrm{Re}(k_{ij} e^{i(\theta_i - \theta_j)})
\end{aligned}$$

whence

$$-\mathrm{Re}((\sum_{1 \leqslant i \leqslant r-1} k_{ir} e^{-i\theta_i}) e^{i\theta_r}) = \mathrm{Re}(\sum_{1 \leqslant i < j < r-1} k_{ij} e^{-i(\theta_i - \theta_j)})$$

Since this holds for all θ_r, it follows that

$$\sum_{1 \leqslant i \leqslant r-1} k_{ir} e^{-i\theta_i} = 0.$$

Again, this holds for all $\theta_1, \ldots, \theta_{r-1}$.

We can choose $(r - 1)$ linearly independent vectors

$$(e^{-i\theta_1}, \ldots, e^{-i\theta_{(r-1)}}), \quad \text{e.g.,} \quad v^s = (\omega^s, \omega^{2s}, \ldots, \omega^{(r-1)s})$$

where ω is a primitive rth root of 1. Hence

$$k_{ri} = k_{ir} = 0, \qquad i = 1, \ldots, r-1.$$

Now we obtain that for all θ_i

$$\sum_{1 \leq i,j \leq r-1} k_{ij} e^{-i(\theta_j - \theta_i)} = 0,$$

whence by inductive assumption $k_{ij} = 0$, $i, j = 1, \ldots, r-1$. Thus $K = 0$.

The lemma follows by induction.

5.2

LEMMA *Let $\Omega = \{u \in C_n : |u_i| = 1,\ i = 1, \ldots, n\}$. Let $A \in C_{nn}$, where a_{ii} is real, $i = 1, \ldots, n$. If for all $u \in \Omega$, $\langle u, Au\rangle$ is real, then A is Hermitian.*

Proof Let $A = H + iK$, where H, K are Hermitian. Then $k_{ii} = 0$, $i = 1, \ldots, n$. Since $\langle u, Ku\rangle = 0$, for all $u \in \Omega$, we obtain $K = 0$ by (5.1). Hence $A = H$.

6

6.1

LEMMA *Let ν be a standardized absolute norm. Let N_i, E_i be as in* (2.3). *If $A \in C_{nn}$ is such that $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix on E_i, then $V(A) = V_\chi(A)$, where $V_\chi(A)$ is the Euclidean numerical range.*

Proof Let $x = x_{(1)} \oplus \cdots \oplus x_{(r)}$, where $x_{(i)} \in E_i$. By (2.3)

$$\nu(x) = \mu(\chi(x_{(1)}), \ldots, \chi(x_{(r)})),$$

where μ is a standardized absolute norm on C_r. By (4.5), therefore $V(A)$ is the convex sum of the $V_i(A_i)$, $i = 1, \ldots, r$. But $V_i(A_i) = V_\chi(A_i)$ since $\lambda_i = \chi$.

Next, note that

$$\chi(x) = \check{\chi}(\chi(x_{(1)}), \ldots, \chi(x_{(r)}))$$

and recall the comment after (4.5) that $V(A)$ does not depend on μ. Thus $V_\chi(A)$ is also the convex sum of the $V_\chi(A_i)$, $i = 1, \ldots, n$. Thus $V(A) = V_\chi(A)$.

6.2

THEOREM *Let ν be a standardized absolute norm on C_n. Let the equivalence $\sim$ be defined as in* (2.1). *Then $H \in C_{nn}$ is norm-Hermitian if and only if*

a) $h_{ij} = \bar{h}_{ji}$ for $i \sim j$,

and

b) $h_{ij} = 0$ for $i \nsim j$.

Proof Suppose H satisfies a) and b). If N_i, E_i, $i = 1, \ldots, r$ are defined as in (2.3), then $H = H_1 \oplus \cdots \oplus H_r$, where H_i is a Hermitian matrix on E_i. Hence, by (6.1), $V(H) = V_\chi(H)$, which is real. Thus H is norm-Hermitian.

Conversely, suppose that H is norm-Hermitian. Since $e^i \| e^i$, $i = 1, \ldots, n$, it follows that $h_{ii} = \langle e^i, He^i \rangle$ is real. Suppose that $x, y \in R_n^+$, $y \| x$. Let $K \in C_{nn}$ be given by $k_{ij} = y_i h_{ij} x_j$, $i, j = 1, \ldots, n$. Let $\Omega = \{u \in C_n : |u_i| = 1, i = 1, \ldots, n\}$. If we define $v, w \in C_n$, by $v_i = u_i x_i$, $w_i = u_i y_i$, then $w \| v$. Hence $\langle w, Hv \rangle = \langle u, Ku \rangle$ is real. But k_{ii} is real, $i = 1, \ldots, n$. Hence, by (5.2), K is Hermitian. It follows that

c) $y_j \bar{h}_{ji} x_i = y_i h_{ij} x_j$, $\quad i, j = 1, \ldots, n$ for all $y, x \in R_n^+$ with $y \| x$.

Now let i, j be two fixed integers in $\{1, 2, \ldots, n\}$ such that $h_{ij} \neq 0$. We must prove that $\bar{h}_{ji} = h_{ij}$ and that $i \sim j$. We shall use the notation of (3.3). Thus for $x' \in E'$, $\kappa_0(x') = \nu(x' \oplus 0)$, where, by (3.4), κ_0 is a standardized absolute norm on E'.

Hence, by (3.2) we can find an $x' \in E' \cap R_2^+$ such that both coordinates of x' are positive, $\kappa_0(x') = 1$ and there is a positive c for which $cx' \| x'$ with respect to κ_0. If $x = x' \oplus 0 \in C_n$, $y = cx' \oplus 0$, then $x_i > 0$, $x_j > 0$, $y_i > 0$, $y_j > 0$. Further, $y \| x$ with respect to ν, since $\langle y, x \rangle = \langle y', x' \rangle = 1$ and for any $z = z' \oplus z''$, $\nu(z) = 1$. Since for this particular x and y, we have $y_j x_i = y_i x_j \neq 0$, it follows from c) that $\bar{h}_{ji} = h_{ij} \neq 0$.

We may now deduce from c) that

d) $y_j x_i = y_i x_j$, $\quad$ for *all* $y, x \in R_n^+$ with $y \| x$.

Suppose that $x \in R_n^+$ and that $\nu(x'') < \nu(x) = 1$. Since ν is absolute, there is a $y \in R_n^+$ such that $y \| x$. Then by d), $y' = d_x x'$ where $d_x \geqslant 0$. But by (3.7), $y' \neq 0$, whence $d_x > 0$. Thus $y | x$. It now follows by (3.9) that $i \sim j$, and the theorem is proved.

6.3

THEOREM *Let ν be a standardized absolute norm on C_n. Let*

$$\mathscr{J} = \{H + iK : H, K \text{ are norm-Hermitian}\}.$$

Then $\mathscr{J} = \mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_r$ where $\mathscr{M}_i$ is the complete matrix algebra on E_i. Further $\mathscr{J}$ is a subalgebra of C_{nn}.

Proof Since any matrix A_i on E_i is of form $A_i = H_i + iK_i$, where H_i, K_i are Hermitian, it follows that $A \in \mathscr{J}$ if and only if $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix on E_i. The result follows immediately.

6.4

THEOREM *If $A \in \mathscr{J}$, then $V(A) = V_\chi(A)$ and is convex.*

Proof By (6.3), $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix on E_i. Hence, by (6.1), $V(A) = V_\chi(A)$, which is convex.

An important theorem due to Vidav [10] and Palmer [9] (cf. Bonsall and Duncan [3], p. 65) is now stated in a slightly special case. Let V be a Banach space and let $\mathscr{A}$ be an algebra of operators on V (normed by the operator norm) such that for each $A \in \mathscr{A}$, $A = H + iK$ where H, K are norm-Hermitian. Define $A^* = H - iK$. Then there exists a Hilbert space V' and an isomorphism of $\mathscr{A}$ onto an algebra of operators $\mathscr{A}'$ on V' which preserves both the norm and the star operation.

Given the dimension of V, the Vidav–Palmer theorem by itself gives no information on the dimension of V'. In our special case, the impact of our next theorem is that one may choose $V' = V$.

6.5

THEOREM *Let $A \in \mathscr{J}$. Then $\nu^0(A) = \chi^0(A)$.*

Proof By (6.4), $A = A_1 \oplus \cdots \oplus A_r$, where A_i is a matrix on E_i. Hence, by (2.3) and (4.6), $\nu^0(A) = \max\{\chi^0(A_i): i = 1, \ldots, r\}$. But

$$\chi(x) = \chi(\chi(x_{(1)}), \ldots, \chi(x_r)),$$

whence again, by (4.6), $\chi^0(A) = \max\{\chi^0(A_i): i = 1, \ldots, r\}$. The theorem follows.

There are, of course, many obvious corollaries to (6.2) and (6.5). We shall give some immediate applications to ν-normal matrices.

6.6

DEFINITION *A matrix $A \in C_{nn}$ is called ν-normal if $A = H + iK$, where H, K are ν-Hermitian and $HK = KH$.*

If $\nu = \chi$, a ν-normal matrix is normal in the traditional sense.

6.7

THEOREM *Let ν be a standardized absolute norm and let A be in C_{nn}. Then*

1) *A is ν-normal if and only if A is normal and $a_{ij} = 0$ for $i \nsim j$. If A is ν-normal, then*

2) $V(A) = \mathrm{co}(\mathrm{sp}\, A)$

3) $v(A) = \rho(A)$.

4) $\nu^0(A) = \rho(A)$.

(*Here* co(sp A) *is the convex hull of the spectrum of A*, $v(A) = \sup\{|\lambda| : \lambda \in V(A)\}$, *the numerical radius of A, and $\rho(A)$ is the spectral radius of A.*)

Proof 1) The matrix A is ν-normal if and only if $A = H + iK$, where H, K are Hermitian and $h_{ij} = k_{ij} = 0$ for $i \nsim j$.

2) By (6.3), $V(A) = V_\chi(A)$, and $V_\chi(A) = \operatorname{co}(\operatorname{sp} A)$.

3) Immediate.

4) Since $A \in \mathscr{J}$, $\nu^0(A) = \chi^0(A)$, by (6.6) and $\chi^0(A) = \rho(A)$.

For operators on a Banach space, 3) is known to be true (Palmer [8]), and his proof is much less elementary. By Sinclair's theorem [3, p. 54], $\rho(A) = \nu^0(A)$, where A is a norm-Hermitian operator. However, Crabb [4] has given a counterexample to 4) for a non-absolute norm on C_4.

7

7.1

DEFINITIONS *Let ν be a standardized absolute norm.*

i) *The set of all norm-Hermitian H in C_{nn} will be denoted by $\mathscr{H}$.*

ii) *Let $\mathscr{U}$ be the set of all $U \in C_{nn}$ such that $U = \exp(iH)$ for some $H \in \mathscr{H}$.*

iii) *The set of all isometries $V \in C_{nn}$ will be denoted by $\mathscr{V}$.*

The following theorem is known (cf. Bonsall and Duncan [3, p. 46]) (where it is stated for Banach Algebras): A matrix $H \in C_{nn}$ is norm-Hermitian, if and only if $\exp(itH)$ is an isometry for all real t. Thus $\mathscr{U} \subseteq \mathscr{V}$. However, our special case is so simple that there is no need to appeal to the above theorem, and our conclusion is stronger than $\mathscr{U} \subseteq \mathscr{V}$. We first state a lemma.

7.2

LEMMA *Let ν be a norm on C_n. If $V \in \mathscr{V}$, and $H \in \mathscr{H}$, then also $V^{-1}HV \in \mathscr{H}$.*

Proof Let ν_V be defined by $\nu_V(x) = \nu(Vx)$, for all x in C_n. Since $V \in \mathscr{V}$, $\nu_V = \nu$. Let $y \| x$, and put $v = Vx$, $w = (V^{-1})^*y$. It follows from Lemma 1 of [7] that $w \| v$. Hence $\langle y, V^{-1}HVx\rangle = \langle w, Hv\rangle$ is real.

Remark If $\nu \neq \chi$, then there exists a nonsingular Z such that $Z^{-1}\mathscr{H}Z = \mathscr{H}$ where Z is not a scalar multiple of an isometry. For then we have r classes $N_1, \ldots, N_r$ for the equivalence relation $\sim$, where $r \geqslant 2$ (2.5). Let I_i be the identity on E_i, and put $Z = \alpha_1 I_1 \oplus \cdots \oplus \alpha_r I_r$, where $\alpha_i > 0$, $i = 1, \ldots, r$ and $\alpha_1 \neq \alpha_r$. If $\nu = \chi$, then $Z^{-1}\mathscr{H}Z = \mathscr{H}$ implies that Z is a scalar multiple of a unitary matrix. A simple proof uses the factorization $Z = UDV$, where U, V are unitary and $D = \operatorname{diag}(d_1, \ldots, d_n)$, $d_i > 0$, (essentially) the polar decomposition of Z.

7.3

THEOREM *Let ν be a standardized absolute norm on C_n. Let $\mathscr{V}$ and $\mathscr{U}$ be defined as in* (7.1).

1) *A matrix* $U \in \mathscr{U}$ *if and only if* U *is unitary and* $u_{ij} = 0$ *for* $i \nsim j$.

2) $\mathscr{V}$ *is a group and* $\mathscr{U}$ *is a normal subgroup of* $\mathscr{V}$.

Proof 1) Let $\mathscr{H}$ be the set of all norm-Hermitian matrices. Define E_k, $k = 1, \ldots, r$ as usual, and let $\mathscr{H}_k$ be the set of (traditional) Hermitian matrices on E_l. Then, by Theorem (6.2), $\mathscr{H} = \mathscr{H}_1 \oplus \cdots \oplus \mathscr{H}_r$. Thus $U \in \mathscr{U}$ if and only if $U = U_1 \oplus \cdots \oplus U_r$, where $U_l = \exp(iH_l)$, $H_l \in \mathscr{H}_l$. But $\exp(i\mathscr{H}_r)$ is well known (and easily seen) to be the set of all unitary matrices $\mathscr{U}_l$ on E_l. Hence $\mathscr{U} = \mathscr{U}_1 \oplus \cdots \oplus \mathscr{U}_r$, which is the assertion 1).

2) Since $\mathscr{U}_k$ is the group of all unitary matrices on E_k, $k = 1, \ldots, r$, it follows from 1): $\mathscr{U} = \mathscr{U}_1 \oplus \cdots \oplus \mathscr{U}_r$, that $\mathscr{U}$ is a group. By Lemma (2.4), $\mathscr{U} \subseteq \mathscr{V}$.

If $V_1, V_2 \in \mathscr{V}$, so is $V_1 V_2^{-1}$, whence $\mathscr{V}$ is a subgroup of the group of non-singular matrices.

Let $V \in \mathscr{V}$, $U \in \mathscr{U}$, say $U = \exp(iH)$, with $H \in \mathscr{H}$. By (7.2), $V^{-1}HV \in \mathscr{H}$, and $\exp(iV^{-1}HV) = V^{-1}\exp(iH)V = V^{-1}UV$. Thus $V^{-1}UV \in \mathscr{U}$, and so $\mathscr{U}$ is a normal subgroup of $\mathscr{V}$.

Remark For an arbitrary norm ν on C_n, we do not know if $\mathscr{U}$ is a group.

7.4

DEFINITIONS AND REMARKS 1) *Let* $\{N_1, \ldots, N_r\}$ *be the equivalence classes for* $\sim$ *in* $\{1, 2, \ldots, n\}$.

Denote the symmetric group on $\{1, \ldots, n\}$ *by* S_n. *Let* π *be a permutation in* S_n. *We call* π *a* block permutation *if*

a) *For each* k, $k = 1, \ldots, r$ *there is an* l *such that* $\pi(N_k) = N_l$,

b) *If* $i, j \in N_k$ *and* $i < j$, *then* $\pi(i) < \pi(j)$, *for* $k = 1, \ldots, r$.

2) *If* $\pi \in S_n$ *is a block permutation, then there is a unique permutation* $\rho \in S_r$ *such that* $\pi(N_k) = N_{\rho(k)}$, $k = 1, \ldots, r$. *Further,* $|N_{\rho(k)}| = |N_k|$ *where* $|N_k|$ *is the number of elements in* N_k.

3) *If* $\pi \in S_n$, *let* P_π *be the permutation matrix defined by* $P_\pi e^i = e^{\pi(i)}$, $i = 1, \ldots, n$. *If* π *is a block permutation, then* P_π *will be called a* block permutation matrix.

4) *The set of all block permutation matrices form a group* Q *under multiplication.*

5) *A block permutation which is also an isometry will be called a* block isometry.

6) *The set of block isometries form a subgroup* $\mathscr{P}$ *of* Q *under multiplication.*

7) *It is easy to prove that* $|Q| = \prod_{i=1}^{n} t_i \leq r!$ *where* t_i *is the maximum of* 1 *and the number of* N_k *with* $|N_k| = i$.

7.5

LEMMA *Let ν be an absolute norm, and let $P = P_\sigma$ be a permutation matrix which is also an isometry. If $i \sim j$, then $\sigma(i) \sim \sigma(j)$.*

Proof Let $E' = \operatorname{span}\{e^i, e^j\}$, $E'' = \operatorname{span}\{e^k : k \neq i, j\}$, $E'_\sigma = \operatorname{span}\{e^{\sigma(i)}, e^{\sigma(j)}\}$, $E''_\sigma = \operatorname{span}\{e^{\sigma(k)}: k \neq i, j\}$. Let $x = \sum_{i=1}^{n} x_i e^i$, $y = \sum_{i=1}^{n} y_i e^i$, where $|x_{\sigma(i)}|^2 + |x_{\sigma(j)}|^2 = |y_{\sigma(i)}|^2 + |y_{\sigma(j)}|^2$, and $|x_{\sigma(k)}| = |y_{\sigma(k)}|$, $k \neq i, j$. We wish to prove that $\nu(x) = \nu(y)$.

Let $\hat{x} = P^{-1}x$, $\hat{y} = P^{-1}y$. Then $\hat{x}_k = x_{\sigma(k)}$, $\hat{y}_k = y_{\sigma(k)}$, $k = 1, \ldots, n$. Hence $|\hat{x}_i|^2 + |\hat{x}_j|^2 = |\hat{y}_i|^2 + |\hat{y}_j|^2$, $|\hat{x}_k| = |\hat{y}_k|$, $k \neq i, j$. Since $i \sim j$, we have $\nu(\hat{x}) = \nu(\hat{y})$. But P is an isometry, and so $\nu(x) = \nu(y)$.

7.6

COROLLARY *If P_σ is both a permutation matrix and an isometry, then $P_\sigma = P_\rho P_\pi$, where $P_\rho \in \mathscr{U}$ and $P_\pi \in \mathscr{P}$.*

Proof Let $1 \leqslant k \leqslant r$. By (7.5), there is an l such that $P_\sigma(E_k) \subseteq E_l$. Hence $\sigma(N_k) \subseteq N_l$. But the sets N_k, $k = 1, \ldots, r$ are finite, and σ is $1 - 1$ and onto $\{1, \ldots, n\}$. Hence there is a permutation τ in S_r such that $\sigma(N_k) = N_{\tau(k)}$, $k = 1, \ldots, r$. Let π be the corresponding block permutation in S_n. Clearly $\sigma(N_k) = \pi(N_k)$. Then there is a permutation ρ in S_n such that $\rho(N_k) = N_k$, $k = 1, \ldots, n$ and $\sigma = \rho\pi$. It follows that $P_\sigma = P_\rho P_\pi$ where $P_\pi \in Q$. Further, P_ρ is a direct sum of permutation matrices on E_i, each of which is unitary on E_i. Hence $P_\rho \in \mathscr{U}$. Thus $P_\pi = P_\rho^{-1} P_\sigma$ is an isometry whence $P_\pi \in \mathscr{P}$.

7.7

THEOREM *Let ν be a standardized absolute norm on C_n, and let $V \in C_{nn}$ be an isometry on C_n. Then there exist unique $U \in \mathscr{U}$ and $P \in \mathscr{P}$ such that $V = UP$.*

Proof Let $D^{(i)}$, $i = 1, \ldots, n$ be the diagonal matrix with $d_{ii} = 1$, $d_{kk} = 0$ for $k \neq i$. Let $K^{(i)} = VD^{(i)}V^{-1}$, $i = 1, \ldots, n$. Since $D^{(i)} \in \mathscr{H}$, also $K^{(i)} \in \mathscr{H}$, $i = 1, \ldots, n$, by (7.2). Hence $K^{(i)} = K_1^{(i)} \oplus \cdots \oplus K_r^{(i)}$, where $K_k^{(i)}$ is Hermitian on E_k. But $D^{(1)}, \ldots, D^{(n)}$ commute in pairs, hence so do $K^{(1)}, \ldots, K^{(n)}$. Thus there exist unitary matrices W_k on E_k such that $W_k K_k^{(i)} W_k^{-1}$ is a real diagonal matrix. Set $W = W_1 \oplus \cdots \oplus W_k$. Then $W \in \mathscr{U}$, and

$$G^{(i)} = WK^{(i)}W^{-1} = WVD^{(i)}V^{-1}W^{-1}$$

is a real diagonal matrix for $i = 1, \ldots, n$. But the $G^{(i)}$, like the $D^{(i)}$, are projections summing to I, and $G^{(i)}G^{(j)} = 0$, for $i \neq j$. Hence $G^{(i)} = D^{(\sigma^{-1}(i))}$, $i = 1, \ldots, n$, for some permutation σ of $\{1, \ldots, n\}$, and so $G^{(i)} = P_\sigma^{-1} D^{(i)} P_\sigma$.

Put

$$X = WVP_\sigma.$$

Then $G^{(i)} = WVD^{(i)}V^{-1}W^{-1} = X^{-1}G^{(i)}X$, for $i = 1, \ldots, n$. We may now deduce that X is diagonal, say $X = \operatorname{diag}(x_1, \ldots, x_n)$. Hence

$$WVe^{\sigma(i)} = XP_\sigma^{-1}e^{\sigma(i)} = Xe^i = x_ie^i, \qquad i = 1, \ldots, n$$

Since WV is an isometry, it follows that

$$1 = \nu(WVe^{\sigma(i)}) = \nu(x_ie^i) = |x_i|\nu(e^i) = |x_i|, \qquad i = 1, \ldots, n.$$

Thus $X \in \mathcal{U}$. We now obtain that $V = W^{-1}XP_\sigma^{-1}$. By (7.6), $P_\sigma^{-1} = P_{\sigma^{-1}} = P_\rho P$, where $P_\rho \in \mathcal{U}$, $P \in \mathcal{P}$. Let $U = W^{-1}XP_\rho$. Then $U \in \mathcal{U}$, and $V = UP$.

To prove uniqueness, suppose that $V = UP = U'P'$, where also $U' \in \mathcal{U}$, $P' \in \mathcal{P}$, then $A = U'^{-1}U = P'P^{-1} \in \mathcal{U} \cap \mathcal{P}$. But A is then block permutation matrix, say $A = P_\varepsilon$, with $\varepsilon(N_k) = N_k$, $k = 1, \ldots, r$. Hence ε is the identity permutation and $A = I$. Thus $U' = U$, $P' = P$ and the decomposition is unique. The theorem is proved.

Let $\mathfrak{a}$ be a group, $\mathfrak{b}$ and $\mathfrak{n}$ subgroups of $\mathfrak{a}$ with $\mathfrak{n}$ normal in $\mathfrak{a}$. If $\mathfrak{n} \cap \mathfrak{b} = (1)$ and $\mathfrak{n}\mathfrak{b} = \mathfrak{a}$, then $\mathfrak{a}$ is called a semi-direct product of $\mathfrak{n}$ and $\mathfrak{b}$.

7.8

COROLLARY *$\mathcal{V}$ is a semi-direct product $\mathcal{U}$ and $\mathcal{P}$ and $\mathcal{V}/\mathcal{U} \cong \mathcal{P}$.*

Proof Since $\mathcal{U}\mathcal{P} = \mathcal{V}$, and $\mathcal{U}$ is normal in $\mathcal{V}$, the results are immediate by (7.7).

Comment It is also clear that the connected components of $\mathcal{V}$ are precisely the sets $\mathcal{U}P$, for $P \in \mathcal{P}$.

Remark Similarly, every $V \in \mathcal{V}$ can be represented uniquely as $V = P'U'$, where $P' \in \mathcal{P}$ and $U' \in \mathcal{U}$. Indeed, if $P'U' = V = UP$, then $P(P^{-1}UP) = V$, and $P^{-1}UP \in \mathcal{U}$. Hence $P' = P$ and $U' = P^{-1}UP$.

7.9

Examples 1) if ν is an l_p-norm, $\nu(x) = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $p \geqslant 1$ and $p \neq 2$, then the equivalence classes for $\sim$ are singletons. Hence $\mathcal{U}$ consists of all diagonal matrices $U = \operatorname{diag}(u_1, \ldots, u_n)$, with $|u_i| = 1$, $i = 1, \ldots, n$. The group $\mathcal{P}$ consists of all permutation matrices.

2) Let ν be any standardized absolute norm on C_2 and suppose there is a $z \in C_2$ with $\nu(z_1, z_2) \neq \nu(z_2, z_1)$. Then $\mathcal{U}$ consists of all diagonal matrices $U = \operatorname{diag}(u_1, u_2)$ with $|u_1| = |u_2| = 1$, and $\mathcal{P}$ of the identity matrix. Hence $\mathcal{V} = \mathcal{U}$.

8

In this section we shall restate our main results for an absolute norm ν_1 on C_n which is not necessarily standardized by $\nu_1(e^i) = 1, i = 1, \ldots, n$. Theorem (6.2)′ will correspond to Theorem (6.2), etc.

Let $\nu_1(e^i) = d_i$, $i = 1, \ldots, n$, and let $D = \operatorname{diag}(d_1, \ldots, d_n)$. Define $\nu(x) = \nu_1(D^{-1}x)$, for all $x \in C_n$. Then ν is a standardized absolute norm. Now define the equivalence relation $\sim$ in terms of ν, and let N_i and E_i be as before. Explicitly, we now have:

$$i \sim j \quad \text{if and only if for } x, y \in C_n, \tag{2.1$'$}$$

$$d_i^2|x_i|^2 + d_j^2|x_j|^2 = d_i^2|y_i|^2 + d_j^2|y_j|^2,$$

and

$$|x_k| = |y_k|, \qquad k \neq i, j$$

imply that $\nu_1(x) = \nu_1(y)$.

Define $V(A)$, $\mathscr{H}$, $\mathscr{J}$, $\mathscr{U}$, $\mathscr{V}$ as before for the standardized norm ν, and let $V_1(A)$, $\mathscr{H}_1$, $\mathscr{J}_1$, $\mathscr{U}_1$, $\mathscr{V}_1$ be defined correspondingly for ν_1. The basic results translating theorems for ν into theorems for ν_1 are that $\nu_1^0(A) = \nu^0(DAD^{-1})$ and $\mathscr{V}_1(A) = V(DAD^{-1})$ (Nirschl and Schneider [7]). Hence $K \in \mathscr{H}_1$ if and only if $DKD^{-1} \in \mathscr{H}$. Thus $\mathscr{H}_1 = D^{-1}\mathscr{H}D$. Explicitly:

6.2′

THEOREM *Let ν be an absolute norm on C_n, and suppose that $\nu(e^i) = d_i$, $i = 1, \ldots, n$. Let $D = \operatorname{diag}(d_1, \ldots, d_n)$. Then $K \in C_{nn}$ is norm-Hermitian if and only if DKD^{-1} is Hermitian and $k_{ij} = 0$ if $i \nsim j$.*

Theorems (6.5) and (6.7) become

(6.5)′ Let $\chi_1(x) = \chi(Dx) = (\Sigma d_i^2|x_i|^2)^{\frac{1}{2}}$, for $x \in C_n$. Then, for all $A \in \mathscr{J}$, $\nu_1^0(A) = \chi_1^0(A) = (\rho(DAD^{-2}A^*D))^{\frac{1}{2}}$.

(6.7)′ If A is ν_1-normal, then DAD^{-1} is normal, and

$$\nu_1^0(A) = \chi_1^0(A) = v_1(A) = \rho(A),$$

where $v_1(A)$ is the numerical radius for ν_1.

Finally, $\mathscr{U}_1 = \exp(i\mathscr{H}_1) = D^{-1}\mathscr{U}D$, and

(7.8)′ The group of all isometries $\mathscr{V}_1$ is a semidirect product of $\mathscr{U}_1$ and $\mathscr{P}_1$, where $\mathscr{P}_1 = D^{-1}\mathscr{P}D$ is finite.

9

In [11], Tam presents several results which, restricted to C_n, are the special cases of some of our results when the norm is invariant under every permutation matrix.† We shall show that it is possible to obtain the conclusions of [11], Theorems 2 and 3 (restricted to C_n) under a somewhat weaker hypothesis.

Let G be a subgroup of S_n. Then G is called doubly-transitive if for all ordered pairs (i,j), $i \neq j$, and all ordered pairs (k,l), $k \neq l$, $\{i,j,k,l\} \subseteq \{1,\ldots,n\}$, there exists a permutation $\sigma \in G$ such that $\sigma(i) = k$ and $\sigma(j) = l$.

9.1

THEOREM *Let ν be an absolute norm on C_n with $\nu(e^1) = 1$, and let G be the subgroup of S_n defined by $\sigma \in G$, if P_σ is an isometry. If G is doubly transitive, then either*

a) $\nu = \chi$,

or

b) i) *$\mathcal{H}$ consists of all real diagonal matrices and, for $H \in \mathcal{H}$,*

$$\nu^0(H) = \max\{|h_{ii}| : i = 1,\ldots,n\}.$$

ii) *$\mathcal{U}$ consists of all diagonal matrices U with $|u_{ii}| = 1$, $i = 1,\ldots,n$.*

iii) *$\mathcal{P}$ consists of all P_σ, $\sigma \in G$.*

Proof Since $\nu(e^1) = 1$ and G is (doubly) transitive, it follows that $\nu(e^i) = (P_\sigma e^1)$, for suitable $\sigma \in G$, $i = 1,\ldots,n$. Hence ν is standardized.

Case (a) There exist distinct i, j in $\{1,\ldots,n\}$ such that $i \sim j$.

By (7.5) and the double-transitivity of G, $k \sim l$ for all k, l, with $k \neq l$, and $k, l \in \{1,\ldots,n\}$. Hence, by (2.3), $\nu = \chi$.

Case (*b*) Suppose all equivalence classes for $\sim$ are singletons. Then (i) follows from (6.2) and (6.5) and (ii) from (7.3). For (iii), observe that every permutation matrix which is an isometry is a block isometry.

9.2

Example Let $n \geq 3$ and $1 = a_1 > a_2 > \cdots > a_n \geq 0$. In C_n, set

$$\mu(x) = \sum_{i=1}^{n} a_i|x_i| \quad \text{and} \quad \nu(x) = \sup\{\mu(P_\sigma x) : \sigma \in A_n\},$$

where A_n is the alternating group on $\{1,\ldots,n\}$. (If $n \geq 4$, then A_n is doubly transitive.) Then ν is a standardized absolute norm on C_n, and if G is defined as in Theorem (9.1), then $A_n \subseteq G$. Let $x = (a_1,\ldots,a_n)$, $z = (a_2, a_1, a_3,\ldots,a_n)$. Then by a result found in Hardy, Littlewood and Polya ("Inequalities," (10.2)), it follows that $\nu(x) = \sum_{i=1}^{n} a_i^2 > \nu(z)$. Hence $G \neq S_n$, and so $G = A_n$.

† We are indebted to John Duncan for pointing this out to us. This section was written after the rest of this paper was completed.

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