Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 37, No. 2, February 1972
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Positive Eigenvectors of Order-preserving Maps

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1. The classical Perron–Frobenius theory for matrices has been extended by many authors, in the context of both finite- and infinite-dimensional spaces. The principal results of this paper concern the existence of "positive" eigenvectors of order preserving maps in a Banach space with a partial order induced by a cone. While the resolvent is a powerful tool for linear maps and fixed point theorems, for continuous ones, we assume neither linearity nor continuity and hence must rely heavily on order preservation. The type of proof we give can be traced to the work of H. Wielandt [1] on matrix problems. As an application we prove the existence of a positive eigenvector of a discontinuous Sturm–Liouville problem.

In addition to the existence theorems we present a new proof that the spectral radius is in the spectrum of a positive linear operator and extend comparison theorems for "spectral radii" to maps in a Banach space.

2. We let X denote a real Banach space with norm $\|\cdot\|$ and K, a cone in X, i.e., a closed convex set in X with the property that $x, y \in K$ implies $x + y \in K$ and $\alpha x \in K$ for $\alpha \geqslant 0$. We assume that K is *normal*; that is, there exists a $\delta > 0$ such that for $x, y \in K$, $\| x + y \| \geqslant \delta \| x \|$. We also assume K is *reproducing* which means that each $z \in X$ can be written in the form $z = x - y$ with $x, y \in K$.

LEMMA 2.1. *If K is a reproducing cone in X, then there is a constant $\eta > 0$ such that each $x \in X$ has a representation $x = x_1 - x_2$, $x_1, x_2 \in K$ with $\| x_i \| \leqslant \eta \| x \|$, $i = 1, 2$.*

Proof. This follows from the fact that the map $d : K \times K \to K$ taking x, y to $x - y$ is onto and hence open. A proof of the "open-mapping theorem" for d is contained in Ref. [2, Chap. II, Section 1].

One can use the cone K to introduce a partial order in X and we write

* Research partly supported by NSF Grants GP-9564 and GP-5574.

$x \leqslant y$ or $y \geqslant x$ if and only if $y - x \in K$. If A and B are linear maps, we write $A \leqslant B$ if and only if $B - A$ maps K into K.

DEFINITION. A cone $F \subset K$ is called a face of K if and only if, whenever $x \in F$, $y \in K$, and $x \geqslant y$, then $y \in F$. (This is equivalent to the definition of Vandergraff [3] in E^n, though he excludes K as a face.)

The set

$$F(x) = \{y \in K \mid \exists \alpha > 0 \ni \alpha y \leqslant x\}$$

is a face and is called the face generated by x. If in a cone K every chain $0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq \cdots$ of faces ends with K after a finite number of steps, we say K satisfies the finite chain condition. For example, if H is a Hilbert space, the cone $K = \{x \mid |(x, x_0)| \geqslant \alpha \| x \| \cdot \| x_0 \|\}$ for fixed x_0 and $0 < \alpha < 1$, satisfies the condition with the maximum length being three.

We denote the Banach space interior of K by K^o. It may be empty, e.g. $f \geqslant 0$ in $L^2[0, 1]$. We say that x is in the order interior of K, denoted K^{oo}, if and only if $F(x) = K$. If $x \in K^o$, then $x - K$ contains a neighborhood of 0 and for each $y \in K$, αy is in this neighborhood for some $\alpha \geqslant 0$. Thus $K^o \subset K^{oo}$. That K^{oo} may be strictly larger can easily be seen for an incomplete normed space, say $C[0, 1]$ with $L^2[0, 1]$ norm, but may also occur in a complete space. To see this let $X = C[0, 1]$ with the usual "sup" norm and let l be an unbounded linear functional defined on all of X with $l(1) = 1$. Let

$$K = \{f \in C[0, 1] \mid f \geqslant 0 \text{ and } l(f) \geqslant 0\}.$$

Then K is a normal, reproducing cone. Since l is unbounded the function 1 is not in K^o, but is in K^{oo}.

The set of bounded linear functionals x' in the dual of X satisfying $\langle x', x \rangle \geqslant 0$ for all $x \in K$ is called the *dual cone* to K and is denoted by K'. By the spectrum $\sigma(A)$ of a linear map A in X, we mean the spectrum of the extension of A to the complexification of X (cf., [4, p. 31]). The following theorem about the spectrum is central to the linear theory and we believe our proof is new:

THEOREM 2.2. *Let A be a bounded linear map of X into X taking a cone K into itself. Let A have spectral radius 1. Then $\lambda = 1$ is in $\sigma(A)$.*

Proof. If $1 \notin \sigma(A)$, then the spectral radius of $I + A$ is strictly less than 2. Thus for some $\eta > 0$,

$$\begin{aligned}(I - \eta - A)^{-1} &= (2 - \eta - (I + A))^{-1} \\ &= \sum_{k=0}^{\infty} \frac{(I + A)^k}{(2 - \eta)^{k+1}}\end{aligned}$$

exists and maps K into K. Since

$$(I - \eta - A)^{-1} = \sum_{k=0}^{N} \frac{A^k}{(1-\eta)^{k+1}} + \frac{A^{N+1}}{(1-\eta)^{N+1}} (I - \eta - A)^{-1},$$

$$0 \leqslant \frac{A^k}{(1-\eta)^{k+1}} \leqslant (1 - \eta - A)^{-1}$$

for all k. Since K is normal,

$$\left\| \frac{A^k x}{(1-\eta)^{k+1}} \right\| \leqslant \delta^{-1} \| (I - \eta - A)^{-1} x \|$$

for $x \geqslant 0$, from which it follows that $\| A^k \| \leqslant \delta^{-1} c(1-\eta)^k$, for some constant c, using Lemma 2.1. But then the spectral radius can be at most $1 - \eta$, a contradiction.

It has been shown by various authors, under conditions which insure that $\lambda = 1$ is a pole of $(\lambda - A)^{-1}$, that $Ax = x$ for a vector $x \in K$; that is, $Ax = \rho x$ where ρ is the spectral radius. It is this last result that we will extend.

We say that A is homogeneous if and only if $A(\alpha x) = \alpha Ax$ for α real and call A monotone if A maps K into itself and $x, y \in K$, $x \leqslant y$, implies $Ax \leqslant Ay$.

LEMMA 2.3. *Let A be a homogeneous, monotone map. If $x \in K$ and Ax is in the face $F(x)$, then A maps $F(x)$ into itself.*

Proof. If $y \in F(x)$, then $\alpha y \leqslant x$ for some $\alpha > 0$, so

$$\alpha Ay = A(\alpha y) \leqslant Ax \in F(x)$$

implying $Ay \in F(x)$.

Motivated by Vandergraff's definition of irreducibility in Ref. [3] we say that a homogeneous monotone map A is irreducible if and only if $x_1 \geqslant x_0 \geqslant 0$, $x_1 \neq x_0$, and $Ax_1 - Ax_0 \in F(x_1 - x_0)$ imply that $F(x_1 - x_0) = K$.

LEMMA 2.4. *Suppose A is irreducible and that $x_1 \geqslant x_2$, $x_1 \neq x_2$. Then $F_k = F((I + A)^k x_1 - (I + A)^k x_2) \subset F_{k+1}$ with strict inclusion unless $F_k = K$.*

Proof. We let $x_i^k = (I + A)^k x_i$, $i = 1, 2$, and note that the monotonicity of A implies that $x_1^k \geqslant x_2^k$ for all positive integers k. First,

$$x_1^{k+1} - x_2^{k+1} = x_1^k - x_2^k + Ax_1^k - Ax_2^k \geqslant x_1^k - x_2^k,$$

so $F_k \subset F_{k+1}$. If $F_k = F_{k+1}$, then for some $\alpha > 0$, $\alpha(x_1^{k+1} - x_2^{k+1}) \leqslant x_1^k - x_2^k$ so $\alpha(Ax_1^k - Ax_2^k) \leqslant (1 - \alpha)(x_1^k - x_2^k)$ which, since A is irreducible, implies $F_k = K$.

In Theorem 2.6 below we consider a homogeneous, monotone, upper semicontinuous map. As the next lemma points out such a map must be continuous in the interior of K. Of course, as noted, K^o may be empty.

LEMMA 2.5. *If A is monotone and homogeneous, then A is continuous in K^o.*

Proof. If $x \in K^o$ and $x_n \to x$ (the arrow denoting strong convergence), then for any $0 < \alpha < 1$, $\alpha x \leqslant x_n \leqslant \alpha^{-1}x$ for all large n. Thus $\alpha Ax \leqslant Ax_n \leqslant \alpha^{-1}Ax$. Now if N is any neighborhood of Ax, then using the normality of K one sees that for α sufficiently close to 1, the order interval $[\alpha Ax, \alpha^{-1}Ax]$ will be in N. Thus A is continuous at x.

We say A is upper semicontinuous if and only if whenever $x_n \to x$ and $Ax_n \to z$, then $Ax \geqslant z$.

THEOREM 2.6. *Let X be a real Banach space and $X_1 \subset X$ a second Banach space having a compact injection in X. Let K be a cone in X and suppose that $K_1^{oo} \neq \varnothing$, where $K_1 = K \cap X_1$. Let $A \not\equiv 0$ be homogeneous, monotone, upper semicontinuous and bounded (i.e., map bounded sets to bounded sets) as a map from K to K. Suppose that:* (i) *there exists a homogeneous, monotone map B, bounded from K into K_1, such that $AB = BA$ and $Bx \geqslant \alpha Ax$ for some $\alpha > 0$ and all $x \in K$.* (ii) *There is a homogeneous, monotone map C with $AC = CA$ and such that $x, y \in K_1$, $y \geqslant x$, $y \neq x$ implies $Cy - Cx$ is in K_1^{oo}. Then there is an $x \in K_1^{oo}$ and $\rho > 0$ such that $Ax = \rho x$. Moreover, ρ is maximal; that is if $Aw = \mu w$ with $w \in K - \{0\}$, then $\mu \leqslant \rho$.*

Proof. Let

$$\rho = \sup\{\lambda \mid \exists x \in CB(K),\, x \neq 0,\, Ax \geqslant \lambda x\}. \tag{2.1}$$

Let x be a vector for which $Ax \neq 0$. Then $Bx \geqslant \alpha Ax \neq 0$, so $Bx \neq 0$ and thus $CBx \in K_1^{oo}$. For some $\beta > 0$ then, $CBx \geqslant \beta x$ and so

$$CBAx = ACBx \geqslant \beta Ax \neq 0.$$

Thus $CBAx$ is in K_1^{oo} and repeating the argument we see that $ACBAx = CBA^2x$ is in K_1^{oo}. That means there is an $\eta > 0$ for which $A(CBAx) \geqslant \eta CBAx$ from which it follows that $\rho > 0$. Let x_n be a sequence of vectors in $CB(K)$, normalized in X, i.e., with $\| x_n \| = 1$, and satisfying $Ax_n \geqslant (\rho - 1/n)\, x_n$. The sequences ABx_n and Bx_n are then bounded in K_1 and hence compact in K. Assuming we have taken subsequences if necessary, we can assert that $Bx_n \to x_0 \in K$ and $ABx_n \to z \in K$. We have

$$Bx_n \geqslant \alpha Ax_n \geqslant \alpha \left(\rho - \frac{1}{n}\right) x_n$$

which means $\| Bx_n \| \geqslant \delta\alpha(\rho - 1/n)$ or, that $x_0 \neq 0$. Similarly, $z \neq 0$. Now $ABx_n \geqslant (\rho - 1/n) Bx_n$ and A is upper semicontinuous so $Ax_0 \geqslant z \geqslant \rho x_0$. Then $ABx_0 = BAx_0 \geqslant \rho Bx_0 \geqslant \rho\alpha Ax_0 \neq 0$, so $Bx_0 \neq 0$ and is in K_1. If $ABx_0 = \rho Bx_0$, then $x = CBx_0 \in K_1^{oo}$ and $Ax = \rho x$, giving the desired eigenvector. Otherwise $ACBx_0 - \rho CBx_0 = CABx_0 - C\rho BX_0 \in K_1^{oo}$ and $ACBx_0 - (\rho + \epsilon) CBx_0 \geqslant 0$ for some $\epsilon > 0$, contradicting the characterization of ρ. The maximality of ρ follows easily from its characterization.

Corollary 2.7. *If in the statement of Theorem 2.6 the existence of C is replaced by the assumption that K_1 satisfies the finite chain condition and A is K_1-irreducible, then there is an $x \in K_1^{oo}$ and a maximal $\rho > 0$ such that $Ax = \rho x$.*

Proof. Let $C = (I + A)^m$, where m is the maximal chain length in K_1.

In the following theorem we replace homogeneity by a weaker condition ((i) below) and, in effect, replace B and C by a power of A.

Theorem 2.8. *Suppose X, K, X_1, and K_1 are as in the previous theorem. Let A satisfy the following conditions:*

(i) *There exists an $a > 2$ such that if $0 < \| x \| \leqslant a^{-1}$, then $A(2x) \geqslant 2A(x)$ while, if $\| x \| \geqslant a$ and $Ax \geqslant \tau x$, then there is an x_1 with $a^{-1} < \| x_1 \| < a$ such that $Ax_1 \geqslant \tau x_1$. Further $A(0) = 0$.*

(ii) *A is monotone, upper semicontinuous and bounded from K into K.*

(iii) *There is an integer $\nu \geqslant 0$ such that for each ball B_α and each pair $0 < \rho_1 < \rho_2 < \infty$, the set*

$$S(\rho_1, \rho_2, \alpha, \nu) = \bigcup_{\rho_1 \leqslant r \leqslant \rho_2} \left(\frac{1}{r} A\right)^\nu (B_\alpha \cap K)$$

is bounded in K_1. Further, if $x \geqslant y \geqslant 0$ and $x \neq y$, then for each $r > 0$,

$$\left(\frac{1}{r} A\right)^\nu x - \left(\frac{1}{r} A\right)^\nu y \in K_1^{oo}.$$

Then there is an $x \in K_1^{oo}$ and a maximal $\rho > 0$ such that $Ax = \rho x$.

Proof. If $x \in K$ and $x \neq 0$, then $Ax \neq 0$, for $A^\nu x \in K_1^{oo}$ and $A(0) = 0$. Hence $A^{\nu+1}x \in K_1^{oo}$ and for some $\rho_1 > 0$, $A(A^\nu x) \geqslant \rho_1 A^\nu x$. From (ii) we know that for x in $K \cap B_a$, Ax is in $K \cap B_{\tilde{a}}$ for some $\tilde{a} > 0$. If we let ρ_2 be a real number greater than $\tilde{a}\delta^{-1}a$, where δ is the normality constant of K, then $Ax \geqslant \rho_2 x$ is not possible for any $x \neq 0$. For were it to hold for some x it would hold for an x with $a^{-1} < \| x \| < a$ together with the inequalities

$\tilde{a} \geqslant \| Ax \| \geqslant \delta\rho_2 \| x \| > \delta\rho_2 a^{-1}$, an impossibility. Using the ρ_1 and ρ_2 from above and the set $S(\rho_1, \rho_2, \alpha, \nu)$ from hypothesis (iii), we define

$$P_\alpha = \{x \in K_1 \mid x = \beta y, y \in S(\rho_1, \rho_2, \alpha, \nu), 1 \leqslant \beta \leqslant 2\delta^{-1}, x \notin B_{\alpha^{-1}}\}.$$

For α sufficiently large, P_α will be nonempty. Suppose we have chosen such an α. Now let

$$\rho = \sup\{\lambda \mid \exists\, x \in P_\alpha, Ax \geqslant \lambda x\},$$

noting that $\rho_1 \leqslant \rho \leqslant \rho_2$. The set P_α is bounded in K_1 and hence compact in K. Moreover, AP_α is compact. To show this it suffices to show that $AS(\rho_1, \rho_2, \alpha, \nu)$ is compact which will follow if $S(\rho_1, \rho_2, \alpha, \nu + 1)$ is compact. But since A is bounded $S(\rho_1, \rho_2, \alpha, \nu + 1)$ is contained in $S(\rho_1, \rho_2, \tilde{\alpha}, \nu)$ for some $\tilde{\alpha} > 0$ and this last set is compact. Hence, we can choose a sequence $x_n \in P_\alpha$ such that $x_n \to x_0$ in K with $\| x_0 \| \geqslant \alpha^{-1}$, $Ax_n \to z \in K$, and $Ax_n \geqslant (\rho - 1/n)\, x_n$. Upper semicontinuity then yields $Ax_0 \geqslant \rho x_0$. If $Ax_0 = \rho x_0$, then $A((1/\rho) A)^\nu x_0 = \rho((1/\rho) A)^\nu x_0$ so $((1/\rho) A)^\nu x_0$ is the desired eigenvector. Otherwise $((1/\rho) A)\, x_0 \geqslant x_0$ without equality and we may assume $a^{-1} < \| x_0 \| < a$. Then $((1/\rho) A)^{\nu+1} x_0 - ((1/\rho) A)^\nu x_0 \in K_1^{oo}$ or with $x = ((1/\rho) A)^\nu x_0$, $Ax - \rho x \in K_1^{oo}$ so $Ax \geqslant \tilde{\rho} x$ for some $\tilde{\rho} > \rho$. Since $x = ((1/\rho) A)^\nu x_0 \geqslant x_0$, $\| x \| \geqslant \delta a^{-1}$, and if $\| x \| < a^{-1}$, then for some power $2^p \leqslant 2\delta^{-1}$, $x_1 = 2^p x$ will be in P_α. Using hypothesis (i) p times, we find $Ax_1 \geqslant \tilde{\rho} x_1$ which is impossible for $\tilde{\rho} > \rho$. If $\| x \| > a$, hypothesis (i) again provides such an $x_1 \in P_\alpha$ and a contradiction.

As an application of the previous theorem consider the following situation. Let X be $C[0, 1]$ with the "sup" norm and let $H^k[0, 1]$ be the space of $L^2[0, 1]$ functions f having k L^2 derivatives with the norm

$$\| f \|_k^2 = \| f \|_0^2 + \| Df \|_0^2 + \cdots + \| D^k f \|_0^2,$$

$\| \ \|_0$ being the $L^2[0, 1]$ norm and D being differentiation. We let K be the cone of nonnegative functions.

We define a differential operator L with domain

$$\mathscr{D}(L) = \{f \in H^2[0, 1] \mid f(0) = f(1) = 0\}$$

and defined by $Lf = -D^2 f$ for $f \in \mathscr{D}(L)$. For each $h \in L^2$, $g = L^{-1}h = Gh$ is defined by

$$g(x) = \int_0^x s(1 - x)\, h(s)\, ds + \int_x^1 x(1 - s)\, h(s)\, ds.$$

It follows that

$$Dg(0) = \int_0^1 h(s)\,ds + \int_0^1 sh(s)\,ds \qquad \text{and} \qquad Dg(1) = -\int_0^1 sh(s)\,ds.$$

Suppose we let $X_1 = H^2[0, 1]$. Then by a theorem of Rellich, H^2 has a compact injection in H^1. Since H^1 has a continuous injection in $C[0, 1]$, X_1 has a compact injection in X. Now any $g \in K_1 = K \cap X_1$ which is positive on $(0, 1)$ and satisfies $Dg(0) > 0$, $Dg(1) < 0$ is in K_1^o with respect to the H^2 topology and hence in K_1^{oo}. Thus, if $h \in K$, $h \not\equiv 0$, then $g = Gh \in K_1^{oo}$.

Let $a(s, t)$ be a real-valued function defined on $[0, 1] \times [0, \infty)$ satisfying the following conditions:

(i) for each s, $a(s, t)$ is upper semicontinuous in t (uniformly in s) and strictly increasing in t;

(ii) there is an $\tilde{\alpha} > 2$ such that for $0 < t < \tilde{\alpha}^{-1}$, $a(s, 2t) \geqslant 2a(s, t) > 0$ while if $t \to \infty$ then $t^{-1}a(s, t) \to 0$ uniformly in s;

(iii) for each $f(s) \in K$, $a(s, f(s))$ is measurable and $a(s, f(s)) \equiv 0$ if and only if $f \equiv 0$.

Using the notation of the preceding paragraphs we can state a result as follows:

COROLLARY 2.9. *There is a* $\mu > 0$ *and an* $f \in \mathscr{D}(L)$ *with* $f > 0$ *on* $(0, 1)$, $Df(0) > 0$, *and* $Df(1) < 0$ *such that*

$$Lf(s) = \mu a(s, f(s)).$$

Moreover, μ *is the smallest eigenvalue corresponding to such an* f.

Proof. The problem is equivalent to finding an f in K_1^o ($= K_1^{oo}$ in this case), such that

$$Af \equiv Ga(s, f(s)) = \mu^{-1} f = \lambda f,$$

so it will suffice to show that A satisfies the conditions of Theorem 2.8. Since for $\|f\| \leqslant \tilde{\alpha}^{-1}$, $a(s, 2f(s)) \geqslant 2a(s, f(s))$ and G is monotone, the first part of hypothesis (i) will be satisfied for any number $a > \tilde{\alpha}$. To satisfy the second part it suffices to show that if $\tau > 0$ is fixed and we assume $Af_n \geqslant \tau f_n \geqslant 0$ for a sequence f_n with $\|f_n\| \to +\infty$, we are led to a contradiction. Under such an assumption, for each n, $\| Ga(s, f_n(s))\| \geqslant \tau \|f_n\|$. For a given $\epsilon > 0$, we can choose t_0 so that for $t > t_0$, $a(s, t) < \epsilon t$ yielding

$$a(s, f_n(s)) \leqslant a(s, t_0) + \epsilon f_n(s) \qquad \text{and} \qquad \| a(s, f_n(s)\|_0 \leqslant \| a(s, t_0)\|_0 + \epsilon \|f_n\|_0 .$$

Since G is a bounded map from L^2 to H^2 and $C^1[0, 1]$ has a bounded injection in H^2, there are constants c_1 and c_2 such that

$$\begin{aligned}\tau \| f_n \| \leqslant c_1 \| Ga(s, f_n(s)) \|_2 &\leqslant c_1 c_2 (\| a(s, t_0) \|_0 + \epsilon \| f_n \|_0) \\ &\leqslant c_1 c_2 (\| a(s, t_0) \|_0 + \epsilon \| f_n \|).\end{aligned}$$

But we may choose ϵ so that $c_1 c_2 \epsilon < \tau$ which, for large $\| f_n \|$, gives a contradiction. Recalling that $a(s, 0) = 0$ we see that $A(0) = 0$ completing the demonstration that hypothesis (i) of Theorem 2.8 is satisfied.

A will clearly be monotone and bounded. To show upper semicontinuity suppose $f_n \to f$ and that $Af_n \to g$. The sequence $a(s, f_n(s))$ will be bounded in $L^2[0, 1]$ and hence will have a subsequence $a(s, f_{n_k}(s))$ converging weakly to an element $h \in L^2$. If $a(s, f(s)) - h(s)$ is not in K, there will be a vector $\psi \geqslant 0$ in L^2 such that $(\psi, a(s, f(s)) - h) = \eta < 0$, the inner product being that in L^2, and there is an integer N_0 such that for $k > N_0$,

$$d_k = (\psi, a(s, f(s)) - a(s, f_{n_k}(s))) < \frac{\eta}{2}.$$

From the upper semicontinuity of a, however, we conclude that $\liminf d_k \geqslant 0$, so we must have $a(s, f(s)) \geqslant h$. Since G maps weakly convergent sequences to strongly convergent ones (in L^2), $Ga(s, f_{n_k}(s))$ converges to Gh. But we assumed the sequence Af_n converged to g so we have

$$Af = Ga(s, f(s)) \geqslant Gh = g,$$

and the satisfaction of hypothesis (ii).

For (iii) we may take $\nu = 1$. We have already remarked that G and hence A is bounded from L^2 to H^2 and hence from X to H^2. If $f_1 \geqslant f_2 \geqslant 0$ and $f_1 \neq f_2$, then $q(s) = a(s, f_1(s)) - a(s, f_2(s))$ will be positive on a set of positive measure since $a(s, t)$ is strictly increasing in t. But then $Af_1 - Af_2 = Gq$ will be in $K_1{}^o$ as will any positive multiple of Gq, completing the demonstration.

We wish to give some results on the size of $\rho = \rho(A)$ and for simplicity will limit ourselves to the situation of Theorem 2.6 and Corollary 2.7. The next two results are immediate consequences of Theorem 2.6.

COROLLARY 2.10. *Under the hypotheses of Theorem 2.6,*

$$\rho(A) = \sup_{x \in CB(K)} \inf_{x' \in K'} \frac{\langle x', Ax \rangle}{\langle x', x \rangle}.$$

Proof. For each $x \in CB(K)$, $Ax - \rho(A)\, x$ is either not in K or is 0. In the first case we can separate $Ax - \rho(A)\,x$ and K by a hyperplane using the Hahn–Banach theorem which means there is an $x' \in K'$ such that

$$\langle x', Ax - \rho(A)\,x\rangle < 0,$$

or

$$\inf_{x' \in K} \frac{\langle x', Ax\rangle}{\langle x', x\rangle} < \rho(A).$$

In the second case the quotient is $\rho(A)$ for all x'.

COROLLARY 2.11. *Suppose A, B, and C are given as in Theorem 2.6 and that there is a similar triple $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ with $\tilde{A}x \geqslant Ax$ for $x \in K$. Then $\rho(\tilde{A}) \geqslant \rho(A)$.*

Proof. Let $Ax_0 = \rho(A)x_0$, where $x_0 \in K_1^{oo}$ and suppose

$$\tilde{A}x_0 \geqslant (\rho(\tilde{A}) + \epsilon)\,x_0$$

for some $\epsilon > 0$. Then

$$\tilde{A}\tilde{B}x_0 - \rho(\tilde{A})\tilde{B}x_0 \in K_1 - \{0\}$$

and

$$\tilde{A}\tilde{C}\tilde{B}x_0 - \rho(\tilde{A})\tilde{C}\tilde{B}x_0 \in K_1^{oo}$$

yielding $\rho(\tilde{A}) > \rho(\tilde{A})$, an impossibility. It follows that adding x_0 to the set $\tilde{C}\tilde{B}(K)$ in the definition of $\rho(\tilde{A})$ (cf., (2.1)) will not increase $\rho(\tilde{A})$. Then since $\tilde{A}x_0 \geqslant Ax_0 \geqslant \rho(A)\,x_0$, we see that $\rho(\tilde{A}) \geqslant \rho(A)$.

THEOREM 2.12. *Suppose that the conditions of Corollary 2.7 are satisfied for a linear pair A, B. Likewise, assume a second linear pair $\tilde{A}$, $\tilde{B}$ satisfies the hypotheses of that corollary with the exception of irreducibility for $\tilde{A}$. If $\tilde{A}x \geqslant Ax$ for $x \in K$ and $\tilde{A} \not\equiv A$ on K_1, then $\tilde{A}$ is irreducible, $\rho(\tilde{A})$ and $\rho(A)$ are the spectral radii of $\tilde{A}$ and A, respectively, and $\rho(\tilde{A}) > \rho(A)$.*

Proof. The irreducibility of $\tilde{A}$ follows easily. Let r be the spectral radius of A. Clearly $\rho(A) \leqslant r$. Since r is a boundary point of $\sigma(A)$, $\|(A - r + 1/n))^{-1}\|$ must be unbounded as $n \to \infty$. Otherwise, choosing any vector y and setting $z_n = (A - (r + 1/n))^{-1}\,y$ we have

$$(A - r)\,z_n = y + (1/n)\,z_n\,,$$

showing that $A - r$ has dense range. Then r is in the point or continuous spectrum and $(A - (r + 1/n))^{-1}$ must be unbounded in n. In particular, there must be unit vectors $x_n \in K$ for which $(A - r)\,x_n \to 0$ as $n \to \infty$.

Then $ABx_n - rBx_n \to 0$ and since B is compact, we can assume, without loss of generality, that Bx_n converges to a vector w in K, satisfying $Aw - rw = 0$. Having $Bx_n \geqslant \alpha Ax_n \geqslant \alpha r x_n$ we see that $\| Bx_n \| \geqslant \delta \alpha r$ and that $w \neq 0$. Then $ACw - rCw = 0$ and we see that $\rho(A) \geqslant r$. Now let y be a vector in K_1 for which $\tilde{A}y \neq Ay$. Then

$$E_{k+1} = F((I + \tilde{A})^{k+1} y - (I + A)^{k+1} y)$$

properly contains E_k or $E_k = K$. For if $E_{k+1} = E_k \neq K$, we must have

$$\begin{aligned}(I + A)\,[(I + \tilde{A})^k y - (I + A)^k y] &\leqslant (I + \tilde{A})^{k+1} y - (I + A)^{k+1} y \\ &\leqslant \beta((I + \tilde{A})^k y - (I + A)^k y)\end{aligned}$$

for large β, contradicting the irreducibility of A. From the finite chain condition it follows that $E_p = K$, for some integer p. Now if $Ax_0 = \rho(A)\, x_0$ for an $x_0 \in K_1^{oo}$, $\alpha y \leqslant x_0$ for some $\alpha > 0$ and

$$\begin{aligned}(I + \tilde{A})^p x_0 - (1 + \rho(A))^p x_0 &\geqslant ((I + \tilde{A})^p - (I + A)^p)\, \alpha y \\ &\quad + (I + A)^p x_0 - (I + \rho(A))^p x_0 \\ &= h,\end{aligned}$$

where $h \in K_1^{oo}$. As before we can conclude that $\rho((I + \tilde{A})^p) > (1 + \rho(A))^p$ which, combined with the spectral mapping theorem, yields $\rho(\tilde{A}) > \rho(A)$.

As a corollary we obtain Theorem 4.6 of Vandergraff [3].

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Printed in Belgium by the St. Catherine Press Ltd., Tempelhof 37, Bruges