Bound Norms

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1. INTRODUCTION

Recently Daniel and Palmer [2] have given an interesting application of a little-known lemma due to A. E. Taylor** [3] concerning the existence of an "orthogonal" basis for a finite dimensional (real or complex) normed vector space. Here we shall apply Taylor's lemma to obtain some simple results which have apparently not been noticed before. Thus we shall show that, given any bound norm β, we can find a representation $\mathbf{M}$ so that, for each homomorphism T (linear transformation), the norm $\beta(T)$ is sandwiched between two easily computed subadditive functions of the matrix $\mathbf{M}(T)$ of T (Theorem 5.1), which do not depend on $\mathbf{M}$. Further we show that our results are the best possible: that is, there exists a bound norm for which our bounds for the norm are attained.

Our point of view owes much to concepts independently introduced by Wielandt [4] and Bauer [1]. Wielandt has defined norms on the set of all (real or complex) matrices, while Bauer observed that we may consider categories of vector spaces and corresponding norms.*** The two approaches are similar; we are in the happy position of wishing to make use of both. Thus we shall begin with a category of vector spaces and represent the homomorphisms as matrices. Our categories are sets; it would, however, be possible to state our results in terms of the category of all (real or complex) vector spaces, which is a proper class. Since we shall

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** See Note added in proof at the end of this paper.

*** Of course, norms on a single space or a pair of spaces have been considered previously.

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use the axiom of choice, this attractive formulation would require a more extensive reference to the axioms of set theory than seems appropriate in a paper on matrix theory.

Our terminology is close to that of Bauer [1]; however, it seems best to give some definitions in detail, at the expense of some paper.

2. CATEGORIES OF VECTOR SPACES AND MATRICES

2.1. A (small, preadditive) category is a couple $\mathscr{V} = (\mathscr{V}^0, \mathscr{V}^m)$, where $\mathscr{V}^0$ is a set of *objects* and $\mathscr{V}^m = \bigcup \{\mathrm{Hom}(C, D): \ (C, D) \in \mathscr{V}^0 \times \mathscr{V}^0\}$, where, for each C, D, $\mathrm{Hom}(C, D)$ is a vector space over a field F. Further, for $T \in \mathrm{Hom}(C, D)$ and $S \in \mathrm{Hom}(D', G)$, the product ST is defined if (and only if) $D' = D$.

Also,

$$(ST)U = S(TU), \tag{2.1.1}$$

$$(S_1 + S_2)T = S_1 T + S_2 T, \tag{2.1.2}$$

$$S(T_1 + T_2) = ST_1 + ST_2, \tag{2.1.3}$$

whenever the left-hand sides are defined.

2.2. From here on, F will stand consistently for *either* the complex field *or* the real field.

2.3. A *category of (finite dimensional) vector spaces (over F)* is a category $\mathscr{V} = (\mathscr{V}^0, \mathscr{V}^m)$, where $\mathscr{V}^0$ is a set of nonzero finite dimensional vector spaces such that $F \in \mathscr{V}^0$. If $(V, W) \in \mathscr{V}^0 \times \mathscr{V}^0$, then $\mathrm{Hom}(V, W)$ consists of all homomorphism (linear transformations) of V into W. Addition and multiplication and scalar multiplication are defined as usual.

2.4. Observe that $\mathrm{Hom}(F, V) \in \mathscr{V}^m$ is naturally isomorphic to $V \in \mathscr{V}^0$: $(x \to x_1)$ for $x \in \mathrm{Hom}(F, V)$. For this reason we shall denote elements of $\mathrm{Hom}(F, V)$ by x, y, etc., and, whenever convenient, identify $\mathrm{Hom}(F, V)$ and V. Observe also that, if $x \in V$ $[= \mathrm{Hom}(F, V)]$ and $y' \in V'$ $[= \mathrm{Hom}(V, F)]$, then $y'x \in F$ $[= \mathrm{Hom}(F, F)]$ and $xy' \in \mathrm{Hom}(V, V)$.

2.5. *The category of all matrices (over F)* is the category $\mathscr{M} = (\mathscr{M}^0, \mathscr{M}^m)$, where $\mathscr{M}^0$ is the set of positive integers $\{1, 2, 3, \ldots\}$. For $(n, m) \in \mathscr{M}^0 \times \mathscr{M}^0$, $\mathrm{Hom}(n, m)$ consists of all $(m \times n)$ matrices over F. We shall write

Mat(n, m) in place of Hom(n, m). Matrix addition and multiplication, as well as scalar multiplication, are defined as usual.

2.6. Let F^n denote the space of n-tuples with elements in F (considered as column vectors). Let $\mathscr{F}$ denote the vector space category with $\mathscr{F}^0 = \{F, F^2, F^3, \ldots\}$. There is a natural mapping $\mathbf{h}$ of $\mathscr{M}^m$ onto $\mathscr{F}^m$ given by $\mathbf{h}(A)c = Ac$ for $A \in \mathrm{Mat}(n, m)$ and $c \in F^n$. Here $\mathbf{h}(A) \in \mathrm{Hom}(F^n, F^m)$.

2.7. An (isomorphic) *matrix representation* of $\mathscr{V}$ is a pair of mappings (dim, $\mathbf{M}$), such that $\mathbf{M}$ maps the vector space Hom(V, W) isomorphically onto Mat(n, m), where $n = \dim V$ and $m = \dim W$. Further,

$$\mathbf{M}(ST) = \mathbf{M}(S)\mathbf{M}(T) \tag{2.7.1}$$

and

$$\mathbf{M}(aT) = a\mathbf{M}(T), \qquad a \in F, \tag{2.7.2}$$

whenever the left-hand sides are defined. We shall refer to a representation $\mathbf{M}$ [in place of (dim, $\mathbf{M}$)].

2.8. To each family of (ordered) bases $\mathscr{B} = \{\mathscr{B}(V) \colon V \in \mathscr{V}^0\}$ there corresponds a representation $\mathbf{M}$, and conversely. For let $V \in \mathscr{V}^0$, $W \in \mathscr{V}^0$, with $\dim V = n$ and $\dim W = m$. If $\mathscr{B}(V) = (x_1, \ldots, x_n)$ and $\mathscr{B}(W) = (y_1, \ldots, y_m)$, define $A = \mathbf{M}(T)$, for $T \in \mathrm{Hom}(V, W)$, by $Tx_j = \sum_{i=1}^{m} y_i a_{ij}$, $j = 1, \ldots, n$. Further, $\mathbf{M}(x_i) = e_i{}^m$, the ith unit column m vector. Conversely, to any matrix representation $\mathbf{M}$ there corresponds a family of bases $\mathscr{B}$: For $V \in \mathscr{V}^0$, let x_i be given by $\mathbf{M}(x_i) = e_i^n$ and put $\mathscr{B}(V) = (x_1, \ldots, x_n)$. Thus

$$\mathscr{B}(V) = (x_1, \ldots, x_n); \quad \mathbf{M}(x_i) = e_i^n \tag{2.8.1}$$

defines a $1:1$ correspondence between families of bases for $\mathscr{V}^0$, and matrix representations for $\mathscr{V}$.

2.9. *Representations exist.* For, by the axiom of choice, we may choose a basis $\mathscr{B}(V)$ for each $V \in \mathscr{V}^0$.

3. NORMS

3.1. A *norm* ν *on* is a mapping of $\mathscr{V}^m$ into F^+ (the set of nonnegative elements in F) such that

$$(3.1.1) \qquad \nu(T) = 0 \quad \text{if and only if} \quad T = 0 \quad (T \text{ a zero homomorphism}),$$

$$(3.1.2) \qquad \nu(1) = 1, \qquad \text{for} \quad 1 \in F,$$

$$(3.1.3) \qquad \nu(aT) = |a|\nu(T), \qquad \text{for} \quad a \in F,$$

$$(3.1.4) \quad \nu(S + T) \leqslant \nu(S) + \nu(T),$$

$$(3.1.5) \qquad \nu(ST) \leqslant \nu(S)\nu(T),$$

whenever the left-hand sides are defined. Evidently, for all norms,

$$\nu(S) \geqslant \sup\{\nu(ST)/\nu(T) \colon ST \text{ exists and } T \neq 0\}.$$

3.2. If $V \in \mathscr{V}^0$, and $\nu|V$ (ν as a function on $V = \text{Hom}(F, V)$) satisfies (3.1.1), (3.1.3), and (3.1.5), we shall call ν a norm on V.

3.3. A *bound norm* β on a category $\mathscr{V}$ of vector spaces is a norm β that, for $S \in \text{Hom}(V, W)$, satisfies

$$(3.3.1) \qquad \beta(S) = \sup\{\beta(Sx)/\beta(x) \colon x \in V, \ x \neq 0\}.$$

3.4. *Bound norms exist.* First define $\beta|F$ [β as a function on $F = \text{Hom}(F, F)$] by $\beta(a) = |a|$. Then, for $V \in \mathscr{V}^0$, $V \neq F$, choose $\beta|V$ to be any norm on V (3.2). Then, for $S \in \text{Hom}(V, W)$, $V \neq F$, define $\beta(S)$ by (3.3.1).

We mention without proof that a bound norm β also satisfies

$$(3.4.1) \ \beta(S) = \sup\{\beta(y'Sx)/\beta(y')\beta(x) \colon 0 \neq x \in V, \ 0 \neq y \in V' = \text{Hom}(V, F)\}$$

and

$$(3.4.2) \qquad \beta(S) = \sup\{\beta(ST)/\beta(T) \colon ST \text{ exists and rank } T = 1\}.$$

A norm β on $\mathscr{V}$ is a bound norm if and only if β is minimal in the set of all norms on $\mathscr{V}$. See Wielandt [4] for proofs of those results in the case of matrix norms.

3.5. Let $\mathscr{M}$ the category of matrices. A mapping μ of $\mathscr{M}^m$ into F^+ is called a *matrix norm*, if μ' is a norm on the vector space category $\mathscr{F}$, $\mathscr{F}^0 = \{F, F^2, F^3, \ldots\}$, where $\mu'(\mathbf{h}(A)) = \mu(A)$, $A \in \text{Mat}(n, m)$. We call μ a *matrix bound norm* if μ' is a bound norm on $\mathscr{F}$. (Intuitively: μ

is a norm on $\mathscr{M}$ considered as a vector space category.) We shall not distinguish further between A and $\mathbf{h}(A)$, μ' and μ.

4. TAYLOR'S LEMMA

4.1. Let $(x_1, \ldots, x_n)$ be a basis for V $[= \mathrm{Hom}(F, V)]$. Then there exist transformations $x_1', \ldots, x_n'$, V' $[= \mathrm{Hom}(V, F)]$ such that $x_i'x_i = 1$, and $x_i'x_j = 0$, $j \neq i$. It is well known that $(x_1', \ldots, x_n')$ is a basis for V'; it is called the basis *dual* to $(x_1, \ldots, x_n)$.

4.2. LEMMA (A. E. Taylor [3]). *Let β be a bound norm on the category $\{V, F\}$. Then there exists a basis $(x_1, \ldots, x_n)$ for V such that, if $(x_1', \ldots, x_n')$ is the dual basis, then*

$$1 = \beta(x_i) = \beta(x_i'), \qquad i = 1, \ldots, n.$$

Sketch of Proof. Let $\det: V^n \to F$ be the usual multilinear determinant function (normalized in an arbitrary fashion). Show that

$$s = \sup\{|\det(z_1, \ldots, z_n)|: \ \beta(z_i) \leqslant 1\}$$

is finite and nonzero and attained at, say, $(x_1, \ldots, x_n)$. Then show that $(x_1, \ldots, x_n)$ has the required property.

4.3. DEFINITIONS. (1) Let V be a vector space and let β be a bound norm on the category of vector spaces with two objects $\{V, F\}$. A basis for V satisfying the conditions of Lemma 4.2 will be called a *β-orthogonal basis* for V. (2) Let $\mathscr{V}$ be a category of vector spaces, let β be a bound norm on $\mathscr{V}$, and let $\mathscr{B}$ be a family of bases for $\mathscr{V}^0$. If $\mathscr{B}(V)$ is a β-orthogonal basis for all $V \in \mathscr{V}^0$, we shall call the matrix representation $\mathbf{M}$ corresponding to $\mathscr{B}$ a *β-orthogonal representation.*

Sometimes we shall use the phrase *norm-orthogonal* in place of β-orthogonal. If $\mathscr{B}$ is a family of β-orthogonal bases for $\mathscr{V}^0$, we refer to $\mathscr{B}$ as β-orthogonal.

4.4. If $V = F$, then (1) is the unique β-orthogonal basis for V.

4.5. PROPOSITION. *Let $\mathscr{V}$ be a category of vector spaces, and let β be a bound norm on $\mathscr{V}$. Then there exists a β-orthogonal matrix representation $\mathbf{M}$ of $\mathscr{V}$.*

Proof. By the axiom of choice and Taylor's Lemma 4.1, there exists a family of β-orthogonal bases for $\mathscr{V}^0$.

5. BOUNDS FOR BOUND NORMS

5.1. THEOREM. *Let β be a bound norm on a category $\mathscr{V}$ of vector spaces. Let* **M** *be a β-orthogonal representation of $\mathscr{V}$. Then, for all $T \in \mathscr{V}^m$,*

$$\|M(T)\|_\infty \leqslant \beta(T) \leqslant \|M(T)\|_1, \tag{5.1.1}$$

where, for any $m \times n$ matrix A,

$$\|A\|_\infty = \max\{|a_{ij}|: i = 1, \ldots, m,\ j = 1, \ldots, n\}, \tag{5.1.2}$$

$$\|A\|_1 = \sum_{i=1,j=1}^{m,n} |a_{ij}|. \tag{5.1.3}$$

Proof. Let $T \in \operatorname{Hom}(V, W)$, $\dim V = n$, and $\dim W = m$. Let $(x_1, \ldots, x_n) = \mathscr{B}(V)$, $(y_1, \ldots, y_m) = \mathscr{B}(W)$, and let $\mathbf{M}(T) = A$. If $(y_1', \ldots, y_m')$ is the basis for $W' = \operatorname{Hom}(W, F)$ dual to $(y_1, \ldots, y_n)$, then $y_i'Tx_j = y_i'(\sum_{i=1}^m y_i a_{ij}) = a_{ij}$, whence, by (3.4.1), $\beta(T) \geqslant |a_{ij}|$. It follows that $\beta(T) \geqslant \|A\|_\infty$. Now let $z \in V$ with $\beta(z) = 1$, and let $w \in W'$ with $\beta(w') = 1$. Suppose $z = \sum_{i=1}^n x_i b_i$. If $(x_1', \ldots, x_n')$ is the dual basis to $(x_1, \ldots, x_n)$, it then follows from $x_j'z = b_j$ and $\beta(x_j') = 1$ that $|b_j| \leqslant 1$. Similarly, if $w' = \sum_{i=1}^m c_i y_i$, then $|c_i| \leqslant 1$. Hence $|y'Tx| = |\sum_{i=1,j=1}^{m,n} c_i a_{ij} b_j| \leqslant \sum_{i=1,j=1}^{m,n} |a_{ij}| = \|A\|_1$. The theorem is proved.

5.2. THEOREM. *Let β be a bound norm on a category $\mathscr{V}$ of vector spaces. Let* **M** *be a matrix representation of $\mathscr{V}$. Then the following are equivalent.*

(5.2.1) **M** *is a β-orthogonal representation.*

(5.2.2) $\|\mathbf{M}(T)\|_\infty \leqslant \beta(T) < \|\mathbf{M}(T)\|_1$, *for all $T \in \mathscr{V}^m$, where $\|\cdot\|_\infty$ and $\|\cdot\|_1$ are defined by* (5.1.2) *and* (5.1.3).

(5.2.3) $\|\mathbf{M}(x)\|_\infty \leqslant \beta(x) \leqslant \|\mathbf{M}(x)\|_1$, *for all $x \in V$ and $V \in \mathscr{V}^0$.*

(5.2.4) *If $\mathbf{M}(T)$ is a matrix with exactly one entry 1, and all other entries 0, then $\beta(T) = 1$.*

(5.2.5) *If $\mathbf{M}(y)$ is a unit row or column vector, then $\beta(y) = 1$.*

Proof. In the diagram below, the implication (1) $\Rightarrow$ (2) is given by Theorem 5.1 and all other implications are immediate [we write (1) for (5.2.1), etc.]:

$$\begin{array}{ccccc} (1) & \Rightarrow & (2) & \Rightarrow & (3) \\ & & \Downarrow & & \Downarrow \\ & & (4) & \Rightarrow & (5) \end{array}$$

Thus we need merely show that (5) implies (1). But, if $\mathscr{B}$ is the family of bases corresponding to $\mathbf{M}$ and $\mathscr{B}(V) = (x_1, \ldots, x_n)$, then $\mathbf{M}(x_i) = e_i^n$, whence, by (5), $\beta(x_i) = 1$. If $(x_1', \ldots, x_n')$ is the dual basis to $\mathscr{B}(V)$, then $\mathbf{M}(x_i) = (e_i^n)^T$ (the transpose of e_i^n), whence $\beta(x_i') = 1$. This proves that $\mathscr{B}$ is β-orthogonal, and (1) follows by Definition 4.3.2.

6. STANDARD MATRIX NORMS

6.1. THEOREM. *Let σ be a matrix bound norm. Then the following are equivalent.*

(6.1.1) $||C||_\infty \leqslant \sigma(C) \leqslant ||C||_1$, *for all matrices, C.*

(6.1.2) $||x||_\infty \leqslant \sigma(x) \leqslant ||x||_1$, *for any column vector x.*

(6.1.3) *If E is any matrix with exactly one entry equal to* 1, *and all others* 0, *then* $\sigma(E) = 1$.

(6.1.4) *If e is any unit row or column vector, then* $\sigma(e) = 1$.

Proof. By Theorem 5.2 all of the above are equivalent to the σ-orthogonality of the identity representation $\mathbf{M}$: $\mathbf{M}(A) = A$.

6.2. DEFINITION. A matrix bound norm σ satisfying any of the equivalent conditions (6.1.1)–(6.1.4) will be called a *standard* (matrix) norm.

Observe that the matrix bound norm associated with a Hölder vector norm $||x||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ (or $||x||_\infty = \max_i |x_i|$), for $x \in F^n$, $1 \leqslant p < \infty$, is a standard norm.

6.3. THEOREM. *Let μ be a matrix bound norm. There exist nonsingular $n \times n$ matrices P_n, $n = 1, 2, \ldots,$ with $P_1 = 1$, such that*

(6.3.1) *for each column* x_i *of* P_n, $\mu(x_i) = 1$;

(6.3.2) *for each row* x_i' *of* P_n^{-1}, $\mu(x_i') = 1$;

(6.3.3) *the mapping* σ *of* $\mathscr{M}^m$ *into* F^+ *defined by* $\sigma(C) = \mu(P_m C P_n^{-1})$, *for an* $m \times n$ *matrix* C, *is a standard norm.*

Proof. Consider $C \in \mathrm{Mat}(n, m)$ as an element of $\mathrm{Hom}(F^n, F^m)$. Choose a family of μ-orthogonal bases for $\mathscr{F}^0 = \{F, F^2, \ldots\}$. If $(x_1, \ldots, x_n)$ is the basis for F^n, let P_n be the matrix $[x_1, \ldots, x_2]$. By (4.4), $P_1 = 1$. If x_i' is the ith row of P_n^{-1}, then $(x_1', \ldots, x_n')$ is dual to $(x_1, \ldots, x_n)$, and (6.3.1) and (6.3.2) follow. Now observe that, if $\mathbf{M}$ is the corresponding matrix representation, and $\mathbf{M}(C) = A$, then $Cx_j = \sum_{i=1}^m y_i a_{ij}$, $j = 1, \ldots, n$, where $(y_1, \ldots, y_n)$ is the chosen basis for F^m. Hence $CP_n = P_m A$, and so $\mathbf{M}(C) = P_m^{-1} C P_n$. If σ is defined as in (6.3.3), then it is easy to verify that σ is a matrix bound norm. Further, as $\mathbf{M}$ is a μ-orthogonal representation, it follows by Theorem 5.2 that

$$\sigma(C) \leqslant \mu(P_m C P_n^{-1}) \leqslant \|\mathbf{M}(P_m C P_n^{-1})\|_1 = \|C\|_1,$$

and similarly $\sigma(C) \geqslant \|C\|_\infty$. Hence, by (6.1.1) and Definition 6.2, σ is a standard norm.

7. THE BOUNDS ARE ATTAINED

7.1. Let V be any vector space and let $(x_1, \ldots, x_s)$ be any finite family of vectors that spans V. For each $x \in V$, define

$$\nu(x) = \inf \left\{ \sum_{i=1}^{s} |a_i| : \sum_{i=1}^{s} a_i x_i = x \right\}. \tag{7.1.1}$$

It is not hard to see that $\nu|V$ norms V [i.e., satisfies (3.1.1), (3.1.3), and (3.1.5)], and a compactness argument in F^n shows that the infimum is achieved, viz., for suitable a_i, $i = 1, \ldots, s$. $x = \sum_{i=1}^s a_i x$ and $\nu(x) = \sum_{i=1}^s |a_i|$. We shall call $\nu|V$ defined by (7.1.1) the *convex hull norm* belonging to $(x_1, \ldots, x_s)$. [The name is natural since $\nu(x) \leqslant 1$ if and only if x lies in the convex hull of $(x_1, \ldots, x_s)$].

7.2. Lemma. *Let* $(x_1, \ldots, x_n)$ *be a basis for the vector space* V, *and let* $u = \sum_{i=1}^n x_i$. *Let* γ *be the bound norm on* $\{V, F\}$ *determined by the* *convex hull norm on V belonging to $(x_1, \ldots, x_n, u)$. Then $(x_1, \ldots, x_n)$ is a γ-orthogonal basis for V.*

Proof. Let $z \in V$ such that $\gamma(z) = 1$. Then there exist $a_i \in F$, $i = 1, \ldots, n$ and $c \in F$ such that $z = \sum_{i=1}^n a_i x_i + cu$, and $\sum_{i=1}^n |a_i| + |c| = 1$. Let $(x_1', \ldots, x_n')$ be the dual basis to $(x_1, \ldots, x_n)$. Then $x'u = 1$, and so $|x'z| = |a_i + c| \leqslant 1$. Since $x_i'x_i = 1$, it follows that $\gamma(x_i') = 1$, $i = 1, \ldots, n$.

It is clear that, for any $m \times n$ matrix A, $||A||_0 \leqslant mn||A||_1$. The following theorem (really an example in disguise) shows that the bounds of Theorem 5.1 may be achieved.

7.3. THEOREM. *Let $\mathscr{V}$ be a category of vector spaces and let $\mathbf{M}$ be any representation of $\mathscr{V}$. Then there exists a bound norm γ such that $\mathbf{M}$ is γ-orthogonal and, for every V, $W \in \mathscr{V}^0$, we can find S, $T \in \mathrm{Hom}(V, W)$ for which*

$$||\mathbf{M}(S)||_\infty = \gamma(S) = (mn)^{-1}||\mathbf{M}(S)||_1, \tag{7.3.1}$$

$$mn||\mathbf{M}(T)||_\infty = \gamma(T) = ||\mathbf{M}(T)||_1 \tag{7.3.2}$$

where $n = \dim V$, and $m = \dim W$.

Proof. Let $\mathscr{B}$ be a family of bases for $\mathscr{V}^0$. For $V \in \mathscr{V}^0$, suppose $\mathscr{B}(V) = (x_1, \ldots, x_n)$, and set $u = \sum_{i=1}^n x_i$. Let $\gamma|V$ be the convex hull norm on V belonging to $(x_1, \ldots, x_n, u)$ and let γ be the corresponding bound norm on $\mathscr{V}$. By Lemma 7.2, $\mathscr{B}$ is a family of γ-orthogonal bases for $\mathscr{V}$. Let $V, W \in \mathscr{V}^0$, and let $(x', \ldots, x_n')$, $x_i' \in V'$ be the dual to $(x_1, \ldots, x_n) = \mathscr{B}(V)$. Set $u' = \sum_{i=1}^n x_i$ and $v' = \sum_{i=1}^n (-1)^{i-1} x_i' = x_1' - x_2' + \cdots + x_n'$. Let $\mathscr{B}(W) = (y_1, \ldots, y_m)$, $w = \sum_{i=1}^m y_i$ and $z = \sum_{i=1}^m (-1)^{i-1} y_i$. We claim that

$$\gamma(w') = 1, \quad \gamma(z) = m, \quad \gamma(u') = n, \quad \gamma(v') = 1. \tag{7.3.3}$$

To avoid tedious computations, we shall prove only two of these: $\gamma(v') = 1$, and $\gamma(z) = m$. By definition, $\gamma(v') = \sup\{|v'x| : x \in V,\ \gamma(x) = 1\}$ and $\gamma(x) = \inf\{\sum_{i=1}^n |a_i| + c_i x : x = \sum_{i=1}^n a_i x_i + cu\}$. So suppose $x = \sum_{i=1}^n \cdot a_i x_i + cu$, where $\sum_{i=1}^n |a_i| + |c| = 1$. Then $v'x = \sum_{i=1}^n (-1)^{i-1} a_i + \delta c$, where $\delta = 0$ or 1 according as n is even or odd. In either case, $|v'x| \leqslant \sum_{i=1}^n |a_i| + |c| = 1$. Since $v'x_1 = 1$, it follows that $\gamma(v') = 1$. To show that $\gamma(z) = m$, let $z = \sum_{i=1}^m a_i y_i + cw$. Then, for odd i,

$$a_i = 1 + c$$

and, for even i,

$$a_i = -1 + c.$$

Hence $|a_{2i} - a_{2i-1}| \geqslant 2$, $i = 1, 2, \ldots$, and so, for $m = 2p$,

$$\sum_{i=1}^{m} |a_i| + |c| \geqslant \sum_{i=1}^{p} |a_{2i} - a_{2i-1}| \geqslant 2p = m$$

and, for $m = 2p - 1$,

$$\sum_{i=1}^{m} |a_i| + c \geqslant \sum_{i=1}^{p} |a_i - a_{2i-1}| + |a_{2p+1} + c|$$

$$\geqslant 2m + 1 = n.$$

Hence $\gamma(z) \geqslant n$. But $z = \sum_{i=1}^{n} y_i$, whence $\gamma(z) = n$.

Now let $S = wv'$. Then $Sx = wv'x$, for $x \in V$. Hence $\gamma(Sx) = \gamma(w)|v'x|$, and it follows that $\gamma(S) = \gamma(w)\gamma(v') = 1$. Further, if $A = \mathbf{M}(S)$, then $A = \mathbf{M}(w)\mathbf{M}(v)$ and hence $a_{ij} = (-1)^{j-1}$, $j = 1, \ldots, n$. Thus $||\mathbf{M}(S)||_\infty = 1$, while $||\mathbf{M}(S)||_1 = mn$. This proves (7.3.1). Similarly, for $T = zu'$, we have $\gamma(T) = \gamma(z)\gamma(v') = mn$, while $B = \mathbf{M}(T)$, where $b_{ij} = (-1)^{i-1}$. Thus $||\mathbf{M}(T)||_\infty = 1$ and again $||\mathbf{M}(T)||_1 = mn$, yielding (7.3.2). The theorem is proved.

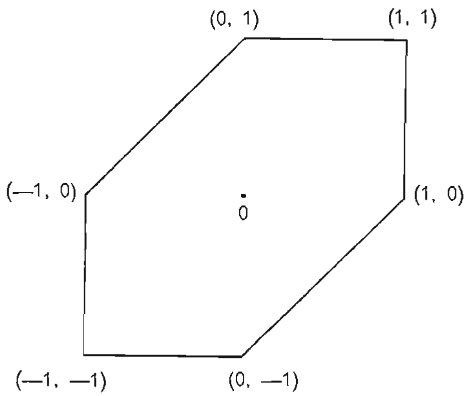

FIG. 1.

To help intuition, we sketch in Fig. 1 the norm body $\{x\colon \gamma(x) \leqslant 1\}$, where F is the real field, $n = 2$, and the basis (x_1, x_2) is $(e_1{}^2, e_2{}^2)$.

7.4. COROLLARY. *Let σ be a bound norm on the category of matrices such that for $n = 1, 2, \ldots$, $\sigma | F^n$ is the convex hull norm belonging to* $(e_1^n, e_2^n, \ldots, e_n^n, f^n)$, where e_i^n is the ith unit vector and $f_{(i)} = 1$, $i = 1, \ldots, n$. Let A, B be the $m \times n$ matrices given by $a_{ij} = (-1)^{j-1}$, $b_{ij} = (-1)^{i-1}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then σ is a standard matrix norm and

(7.4.1) $$\|A\|_\infty = \sigma(A) = (mn)^{-1}\|A\|_1,$$

(7.4.2) $$mn\|B\|_\infty = \sigma(B) = \|B\|_1.$$

8. AN OPEN QUESTION

Let σ be a standard matrix norm, and let C be a matrix for which $\sigma(C) = 1$. Then, by (6.1.1),

(8.0.1) $$\|C\|_\infty \leqslant 1 \leqslant \|C\|_1.$$

However, it is easy to find a matrix C satisfying (8.0.1) such that $\sigma(C) < 1$ [or $\sigma(C) > 1$] for all standard norms σ. Thus it seems interesting to characterize the set of matrices C such that $\sigma(C) = 1$ for some standard norm σ.

Note added in proof: B. Grunbaum [*Math. Rev.* **9**(1964), 4429] has given an interesting history of this lemma in a review of a paper by A. Sobczyk, and I thank W. W. Kahan for drawing my attention to this review. An equivalent geometric result was published by M. M. Day [*Trans. American Math. Soc.* **62**(1947), 315–139] almost simultaneously with Taylor's paper. The form of the result needed in this paper is Taylor's.

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