COMPLETELY O-SIMPLE SEMIGROUPS

An Abstract Treatment of the Lattice of Congruences

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FOR THE READER

We propose in this monograph to introduce the reader to some of the basic tools of investigation in the algebraic theory of semigroups and to lead him through to some recent results in this theory. We will not presuppose more than the usual sophistication of a good first year graduate student in mathematics, and will attempt to make the monograph self contained. We will give basic definitions where necessary although the proofs of some of the easier propositions will be left to the reader. (Most of these can be found in the standard reference for this field by Clifford and Preston [2].)

The reader who is familiar with the theory of semigroups will be able to skim the preliminaries of §0 and pick up the investigation where it really begins in §2. We especially include for them the summary in §1.

CONTENTS

§0. PRELIMINARIES

A semigroup, S, is a collection of elements, $\{a, b, c \ldots\}$ closed with respect to a binary, associative operation, f. As usual, this operation will be written multiplicatively. Thus ab denotes the image of (a, b) under the binary operation $f : S \times S \to S$.

A relation $\mathcal{A}$ on the semigroup S is a subset of the cartesian product $S \times S$. We will alternatively write $x \mathcal{A} y$ whenever $(x, y) \in \mathcal{A}$. A relation $\mathcal{A}$ on S is said to be reflexive if $s \mathcal{A} s$ for each $s \in S$; it is symmetric if whenever $(x, y) \in \mathcal{A}$ we also have $(y, x) \in \mathcal{A}$; it is transitive if whenever $x \mathcal{A} y$ and $y \mathcal{A} z$ we have $x \mathcal{A} z$. An equivalence relation is a relation that is reflexive, symmetric and transitive.

A <u>congruence</u> $\mathcal{C}$ is an equivalence relation on S such that if $a \mathcal{C} b$ then $sa \mathcal{C} sb$ and $as \mathcal{C} bs$ for any $s \in S$.

Fundamental in the algebraic investigation of semigroups are the following relations defined on an arbitrary semigroup S called <u>Green's relations</u> (cf. [6]).

(0.1) <u>Definition</u>. Let S be a semigroup and $a, b \in S$. Let the relation $\mathcal{R}$ be defined by $\mathcal{R} = \{(a, b) \mid a = b$ or there exist $x, y \in S$ with $ax = b$ and $by = a\}$. Let $\mathcal{L} = \{(a, b) \mid a = b$ or there exist $u, v \in S$ with $ua = b$ and $vb = a\}$.

(0.2) <u>Proposition</u>. $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations. The intersection of any two equivalence relations is an equivalence relation.

The reader will note that these two equivalence relations are (left-right) dual to each other. Often in the following exposition we will make use of this duality and prove a theorem involving just one of these relations, leaving the obvious dualization for the reader. This left-right dualization will be more apparent after the following definitions and propositions.

(0.3) Definition 1. If S is a semigroup without an identity element then we can adjoin an identity element 1 to S by defining the product $s1 = 1s = s$ for any $s \in S$, $1 \cdot 1 = 1$ and leaving ab defined as in S whenever $a, b \in S$. The reader can readily check that $S \cup \{1\}$ is a semigroup. S^1 will denote the semigroup S when S already has an identity element or the semigroup $S \cup \{1\}$ just defined when S does not have an identity element.

2. For $a \in S$ we define the principal right ideal $R(a)$ generated by a by $R(a) = aS^1$, the principal left ideal $L(a)$ by $L(a) = S^1a$ and the principal (two-sided) ideal $J(a)$ by $J(a) = S^1aS^1$.

(0.4) Proposition. In any semigroup $a\mathcal{R}b$ if and only if $R(a) = R(b)$ (and dually $a\mathcal{L}b$ if and only if $L(a) = L(b)$).

(0.5) Definition. We can now define two more of Green's relations as follows. Define $a\mathcal{J}b$ whenever $J(a) = J(b)$ and $\mathcal{H}$ by $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$.

(0.6) <u>Proposition</u>. $\mathcal{H}$ and $\mathcal{J}$ are equivalence relations on a semigroup S.

We define the product of two (equivalence) relations $\mathcal{A}$, $\mathcal{B}$ on set S by $\mathcal{A} \circ \mathcal{B} = \{(a,c) \in S \times S \mid$ there is $b \in S$ such that $(a,b) \in \mathcal{A}$ and $(b,c) \in \mathcal{B}\}$. We will now show that the relations $\mathcal{R}$ and $\mathcal{L}$ defined above commute, i.e., $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.

(0.7) <u>Theorem</u>. Let S be a semigroup. Then $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.

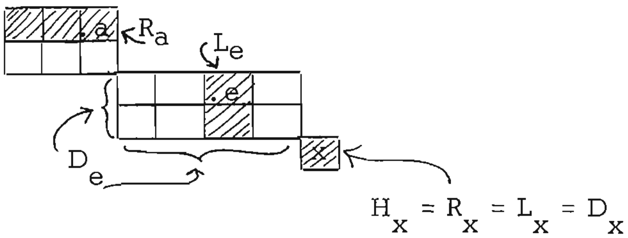
Proof. We will show that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{L} \circ \mathcal{R}$ leaving the other inclusion for the reader. Suppose then that $a \, \mathcal{R} \circ \mathcal{L} \, c$. Then by definition there is a $b \in S$ such that $a \mathcal{R} b$ and $b \mathcal{L} c$. If $b = a$ or $b = c$ we are done since $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations (use the symmetric and reflexive properties). If $b \neq a$ and $b \neq c$ then by definition there are $u, v, x, y \in S$ such that $ax = b$, $by = a$, $ub = c$ and $vc = b$. Let $d = cy$. Then $dx = (cy)x = ((ub)y)x = u(by)x = u(ax) = ub = c$ and $d = c(y)$ so that $cy = d \mathcal{R} c$. Similarly,

$ua = u(by) = (ub)y = cy = d$ and $vd = v(cy) = (vc)y = by = a$ so that $a \mathcal{L} d$. Thus $a \mathcal{L} d \mathcal{R} c$ and $(a, c) \in \mathcal{L} \circ \mathcal{R}$.

(0.8) Proposition. The product of two commuting equivalence relations is an equivalence relation.

(0.9) Definition. We define the last of Green's equivalence relations by $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.

Equivalence relations give rise to partitions of a set. Green's relations defined above give rise to the so-called egg-box structure of a semigroup; the partitioning sets will be the $\mathcal{R}$-classes, $\mathcal{H}$-classes, etc. One can picture $\mathcal{R}$-equivalent elements as lying in the same row ($\mathcal{R}$-class) and $\mathcal{L}$-equivalent elements as lying in the same column. The intersection of a row and a column when nonempty yields an $\mathcal{H}$-class, while the (intersecting) connected rows and columns build a $\mathcal{D}$-class. Thus, perhaps:



will represent the structure of a given semigroup S.

(0.10) Proposition. In any semigroup $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

Notation. Throughout this monograph we will denote the $\mathcal{R}$-equivalence class of an element $a \in S$ by R_a, the $\mathcal{L}$-class of $e \in S$ by L_e etc.

The reader will recall from previous algebra courses that one usually restricted the set of objects under consideration in order to obtain a more fruitful theory; thus, e.g., solvable groups when studying groups, or semi-simple rings in ring theory. The same is true in the algebraic theory of semigroups. We will now try to reach what will be for us such an interesting set throughout the remainder of the monograph—the set of completely 0-simple semigroups. In order to do so we will first have to give several definitions and results.

(0.11) Definition 1. A semigroup S is said to be regular if for each $a \in S$ there is a $x \in S$ such that

$a = axa$.

2. An element $e \in S$, a semigroup, is called an <u>idempotent</u> if $e^2 = e$.

(0.12) <u>Proposition</u>. Prove that S is regular if and only if $a \in aSa$ for each $a \in S$. If $a = axa$ prove that ax and xa are idempotents and that ax is a left identity on R_a while xa is a right identity on L_a.

Since ideals and especially minimal ideals figure heavily in puzzling out algebraic structure we record the following definitions:

(0.13) <u>Definition</u> 1. A nonempty subset $R[L]$ of a semigroup S is said to be a <u>right [left] ideal</u> of S if $RS \subseteq R[SL \subseteq L]$. A nonempty subset I of S is called an <u>ideal</u> if it is both a right and left ideal of S.

2. A right [left] ideal $R[L]$ of a semigroup S with zero, 0, is called <u>0-minimal right [left] ideal</u> if no nonzero right [left] ideal of S is properly contained in $R[L]$.

3. A similar definition is made for a <u>0-minimal ideal</u>.

We may now inquire as to the relationship between 0-minimal ideals and Green's equivalence classes. This relationship is given in the following proposition (and its dual).

(0.14) <u>Proposition</u>. If R is a 0-minimal right ideal in a semigroup S with 0 prove that $aS^1 = R$ for any $a \in R \setminus \{0\}$. Then prove that if R is a 0-minimal right ideal in a semigroup S with 0 that $R \setminus \{0\}$ is an $\mathcal{R}$-class.

(0.15) <u>Definition</u>. A semigroup S with 0 is said to be <u>0-bisimple</u> if it has just one nonzero $\mathcal{D}$-class. A semigroup S with 0 is said to be <u>0-simple</u> if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper two-sided ideal of S.

(0.16) <u>Proposition</u>. In any semigroup $G_0 = \{0\}$ where $\mathcal{G}$ is anyone of Green's relations.

The idempotents of a semigroup prove to be extremely useful wedges in separating out its structure. It is sometimes profitable to order them. Thus

(0.17) Definition. A partial order on a set S is a relation $\subseteq$ which is reflexive transitive and anti-symmetric, i.e., if $a \subseteq b$ and $b \subseteq a$ then $a = b$. If $a \subseteq b$ and $a \neq b$ we will write $a \subset b$.

Now let S be a semigroup and let $E = \mathcal{E}(S) = \{e \in S \mid e^2 = e\}$ be the set of idempotents of S. Define a relation $\leq$ on E by $e \leq f$ whenever $e = ef = fe$ for $e, f \in E$.

(0.18) Proposition. The relation $\leq$ defined above on $\mathcal{E}(S)$ is a partial ordering.

(0.19) Definition. A nonzero idempotent $f \in \mathcal{E}(S)$ is said to be primitive if whenever $e \leq f$ either $e = 0$ or $e = f$ (where $e \in \mathcal{E}(S)$).

We are now ready to give the usual definition of a completely 0-simple semigroup.

(0.20) Definition. A completely 0-simple semigroup is a semigroup S with 0 which is 0-simple and has at least one primitive idempotent.

We feel that it is far more preferable to depart now from what would be the usual approach—in which one proceeds to determine the element-wise behavior, i.e., where the product of two elements lie, etc., to another definition which takes, the ultimately determined behavior of a completely 0-simple semigroup as its starting point.

(0.21) Definition. A semigroup S with 0 is said to be absorbent if for any $a, b \in S$ we have $ab = 0$ or $ab \in R_a \cap L_b$.

Since an $\mathcal{R}$-class and an $\mathcal{L}$-class intersect precisely when they lie within the same $\mathcal{D}$-class it is easy to check the following:

(0.22) Proposition. In an absorbent semigroup each $\mathcal{D}$-class union $\{0\}$ is an ideal.

(0.23) Proposition. A semigroup is completely 0-simple if and only if it is a regular 0-simple absorbent semigroup.

The proof of this proposition is not hard. That a completely 0-simple semigroup (first definition) is absorbent is just one of the derived results in the usual development, cf. [2] Theorem 2.52. The converse follows immediately from [9] Proposition 3.3. However, we will show directly that each nonzero idempotent of an absorbent semigroup is primitive and leave it to the reader to put together the few remaining steps.

(0.24) Theorem. If S is an absorbent semigroup then every nonzero idempotent is primitive.

Proof. We must show that if $0 \neq e \leq f$ (for e, f idempotents) then $e = f$, i.e., f is primitive. By definition, since $e \leq f$ we have $e = ef = fe$. By absorbency, since $e \neq 0$ we have $ef \in R_e \cap L_f$ and thus $ef = e \in L_f$. Hence $L_e = L_f$. ($\mathscr{L}$ is an equivalence relation!) Now any idempotent is a right identity on its $\mathscr{L}$-class (show this). Whence it follows that $e = fe = f$ and f is primitive.

The absorbency condition also permits an easy proof of the following partial converse of (0.14) (cf. [2] Corollary 2.49):

(0.25) <u>Proposition</u>. Let S be an absorbent semigroup. Then every nonzero principal right ideal of S is 0-minimal.

Indeed, it is now not difficult to combine (0.14) and (0.25) to show:

(0.26) <u>Proposition</u>. Let S be a semigroup with 0. Then S is absorbent if and only if each nonzero principal right and left ideal is 0-minimal.

(0.27) <u>Proposition</u>. An absorbent 0-simple semigroup is 0-bisimple.

A few more observations about idempotents and subgroups of S, one more definition and we will then be ready to get into the monograph proper.

(0.28) <u>Proposition</u> 1. $\mathcal{R}$ is a left congruence in the sense that if $a\mathcal{R}b$ then $sa\mathcal{R}sb$. Dualize this.

2. If $e^2 = e$ then H_e is a maximal subgroup of S in the sense that no larger subgroup of S properly contains H_e. Conversely, if an $\mathcal{H}$-

class H is a group it contains an idempotent (cf. [2] Theorem 2.16).

The following theorem is directly related to the absorbent condition.

(0.29) Theorem. ([2] Theorem 2.17) Let S be a semigroup and $a, b \in S$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ is a group, in which case $H_a H_b = H_{ab}$.

Proof. The proof of this theorem uses techniques similar to those of Theorem (0.7) and Lemma (2.1). Since the proof of Lemma (2.1) is independent of this theorem we will use that result here. Suppose then that for a given $a, b \in S$ we have that $L_a \cap R_b$ is a group. By Proposition (0.28.2) $L_a \cap R_b$, a group $\mathcal{H}$-class contains an idempotent e. Thus $a \mathcal{L} e \mathcal{R} b$. As in Proposition (0.12) we can show that any idempotent is a left identity on its $\mathcal{R}$-class and a right identity on its $\mathcal{L}$-class. Hence using Proposition (0.28.1) and its dual we have $ab \mathcal{L} eb = b$ and $a = ae \mathcal{R} ab$.

It follows that $ab \in R_a \cap L_b$. But the above argument is the same for any $a' \in H_a$ and $b' \in H_b$. Whence $H_a H_b \subseteq H_{ab}$. Now since $\mathcal{H}$-classes within the same $\mathcal{D}$-class have the same number of elements and since by Lemmas (2.1) and (2.2) the translations (multiplications) are 1-1 and onto we can conclude that $H_a H_b = H_{ab}$.

Conversely, let us suppose that $ab \in R_a \cap L_b$. Then $a \mathcal{R} ab$ and as in Lemma (2.1) we can find a b' such that $(ab)b' = a$ and the mappings ρ_b $\rho_{b'}$ are mutually inverse, $\mathcal{R}$ -class preserving between L_a and $L_{ab} = L_b$. Now $\rho_{b'}$ maps b into $bb' \in L_a \cap R_b$. Now for any $x \in L_a$ we have $x\rho_b\rho_{b'} = xbb' = x$. Thus if we set $x = bb'$ it follows that bb' is an idempotent and that $H_{bb'} = L_a \cap R_b$ is a group by Proposition (0.28.2).

We shall eventually consider lattices:

(0.30) Definition. A lattice L is a partially ordered set which contains for every pair of elements $a, b \in L$ a greatest lower bound $(\inf(a, b) = a \wedge b)$ and a least upper bound (sup $(a, b) = a \vee b$).

A lattice L is <u>complete</u> if any subset (including the empty set $\emptyset$) has a greatest lower bound (glb) and least upper bound (lub).

(0.31) <u>Proposition</u>. The set of all subsets of a given set, partially ordered by inclusion, is a lattice, indeed it is a complete lattice. The set of all congruences on a [semi]group is a complete lattice under the inclusion ordering.

§1. SUMMARY AND NOTATION

In this monograph we will study the lattice $\underset{\sim}{C}$ of proper congruences on a completely 0-simple semigroup S. Let H be a nonzero group $\mathcal{H}$-class of S. We denote by $\underset{\sim}{N}$ the lattice of all normal subgroups of H, and by $\underset{\sim}{R}_*[\underset{\sim}{L}_*]$ the lattice of all equivalence relations on the set of $\mathcal{R}$-classes [$\mathcal{L}$-classes] of S. We identify an initial segment $\underset{\sim}{R}[\underset{\sim}{L}]$ of $\underset{\sim}{R}_*[\underset{\sim}{L}_*]$, such that $\underset{\sim}{C}$ is isomorphic to a complete sublattice $\underset{\sim}{T}$ of $\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$ (Theorems (7.8) and (8.4)). We investigate well-ordered chains in $\underset{\sim}{C}$ (e.g., Theorem (11.15)).

(1.1) <u>Outline and results</u>. In §2 we associate with every normal subgroup E of H a congruence $\mathfrak{C}(E)$ lying under $\mathcal{H}$ ((2.6)), and in §5 we show that $E \to \mathfrak{C}(E)$ is a complete lattice isomorphism of $\underset{\sim}{N}$ onto the lattice $\underset{\sim}{H}$ of congruences on S lying under $\mathcal{H}$ ((5.3)). In §3 we define an equivalence $\underset{\sim}{d}$ on the set of $\mathcal{R}$-classes of S, and with every equivalence $\underset{\sim}{r}$, $\underset{\sim}{r} \subseteq \underset{\sim}{d}$ we associate a congruence $\mathcal{r}(\underset{\sim}{r})$ on S ((3.5)) such that $\mathcal{r}(\underset{\sim}{r}) \cap \mathcal{R} = \Delta$, the diagonal. In §6 we show that $\underset{\sim}{r} \to \mathcal{r}(\underset{\sim}{r})$ is a complete lattice isomorphism between the lattice of equivalences lying under $\underset{\sim}{d}$ and the lattice of congruences $\mathcal{p}$ on S such that $\mathcal{p} \cap \mathcal{R} = \Delta$ ((6.5)). Dual results hold for an equivalence relation $\underset{\sim}{s}$ on the set of $\mathcal{L}$-classes of S, and congruences $\mathcal{p}$ such that $\mathcal{p} \cap \mathcal{L} = \Delta$.

By considering factor semigroups in §4, we define for each normal subgroup E of H an equivalence relation $\underset{\sim}{d}(E)$ [$\underset{\sim}{s}(E)$] on the set of $\mathcal{R}$-classes [$\mathcal{L}$-classes] of S. A <u>permissible triple</u> $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ is then defined as an element of $\underset{\sim}{R}_* \times \underset{\sim}{N} \times \underset{\sim}{L}_*$ such that $\underset{\sim}{r} \subseteq \underset{\sim}{d}(E)$ and $\underset{\sim}{\ell} \subseteq \underset{\sim}{s}(E)$ ((4.4)). If $\underset{\sim}{T}$ is the lattice of all permissible triples, then we show in §7 that $\underset{\sim}{T}$ and the lattice $\underset{\sim}{C}$ of all proper congruences on S are isomorphic complete lattices ((7.8)).

Let $\mathbf{R}[\mathbf{L}]$ be the initial segment of $\mathbf{R}_*[\mathbf{L}_*]$ consisting of all equivalences $\mathbf{r}$ under $\mathbf{d}(H)$, [$\boldsymbol{\ell}$ under $\mathbf{s}(H)$], and put $\boldsymbol{\Lambda} = \mathbf{R} \times \mathbf{N} \times \mathbf{L}$. In §8 we show that $\mathbf{T}$ is a complete sublattice of $\boldsymbol{\Lambda}$ ((8.4)).

In §9, we determine necessary and sufficient conditions for the existence of a Brandt congruence on S, ((9.8)), and we investigate the sublattice of $\mathbf{C}$ consisting of all Brandt congruences. For example we show that the lattice of all Brandt congruences is a final segment of $\mathbf{C}$ ((9.9)).

In the last two sections of the monograph proper we will discuss chains of congruences on $\mathbf{C}$ (or $\mathbf{T}$). In §10 we show that τ_1 covers τ_2 in $\mathbf{T}$ if and only if τ_1 covers τ_2 in $\boldsymbol{\Lambda}$ ((10.3)). It follows quickly that $\mathbf{C}$ is an upper semimodular lattice ((10.5)) and hence satisfies the Jordan-Dedekind chain condition. In §11, we investigate ascending well-ordered (infinite) chains of congruences on S. If H has a well-ordered principal series, then for each proper congruence $\mathfrak{q}$ on S, there exists in $\underset{\sim}{C}$ a maximal ascending well-ordered chain from Δ to a $\mathfrak{q}$, and all such chains have the same length ((11.15)).

In §12 we break down and at last admit to the reader that we have indeed heard of the Rees matrix representation. We do this in order to construct an example showing that an inequality in a lattice that we have obtained may be strict.

(1.2) Related Papers. Congruences on a completely 0-simple semigroup have been considered before. (Gluskin [6,7] investigated congruences on a completely simple semigroup and showed that they satisfied a Jordan-Hölder theorem.) Preston [16] has obtained representations for congruences on a completely 0-simple semigroup though his representation of a congruence is not in general unique. Tamura [20] has obtained a unique representation in terms of a very special normalization of a sandwich matrix for S. Our representation is both unique and intrinsic. Preston [17] (cf. also [2], Vol. 2) has also considered finite chains of congruences on S. In the case of finite chains of congruences, the results of our §11 reduce to Preston's. A recent paper of Lallement [12] obtains similar results but again resorts to the Rees matrix representation. Howie [8] has also achieved these results starting with Tamura's normalized sandwich matrix.

(1.3) <u>Notation</u>. Our terminology and notation is essentially that of Clifford and Preston [2]. Relevant definitions and notation can be found in §0. When we consider a semigroup T, we shall use lower case letters for elements of T and capitals for subsets of T. Lower case letters, underlined, such as $\underset{\sim}{d}$, $\underset{\sim}{r}$ will denote equivalence relations on the set of $\mathcal{R}$-classes and $\mathcal{L}$-classes of T. We use lower case Gothic letters such as $\mathfrak{t}$, $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$ for congruences on T. Capitals, underlined, $\underset{\sim}{N}$, $\underset{\sim}{R}$, etc., will denote lattices.

(1.4) <u>Special Conventions. In what follows we shall assume that S stands for a completely 0-simple (c-0-s) semigroup, and H stands for a fixed non-zero group $\mathcal{H}$-class of S. By e we shall denote the identity of H. If $X \subseteq S$, then $\mathcal{E}(X)$ will be the set of idempotents in X.</u>

§2. THE CONGRUENCE $\mathcal{E} = \mathcal{E}(E)$ ASSOCIATED WITH A NORMAL SUBGROUP E OF A NONZERO GROUP $\mathcal{H}$-CLASS

In (0.9) we saw that $\mathcal{L}$ and $\mathcal{R}$ commute. The technique used in that proof can be used to prove the following modification of Green's Lemma ([2] Lemma 2.2). Indeed,

(2.1) Lemma. Let a and as be $\mathcal{R}$-equivalent elements of a semigroup T. Then the translation $\rho_s : x \to xs$ is a bijection of L_a onto L_{as} and further $x \mathcal{R} xs$ for all x in L_a. Moreover, there is an inverse mapping $\rho_{s'} : y \to ys'$ of L_{as} onto L_a where $(as)s' = a$. Dually, if $b \mathcal{L} tb$, then $\lambda_t : x \to tx$ is a bijection of R_b onto R_{tb} which is $\mathcal{L}$-class preserving: $x \mathcal{L} tx$ for x in R_b. Again there is an inverse mapping $\lambda_{t'}$ of R_{tb}

onto R_b where $t'(tb) = b$.

Proof: Since $a\mathcal{R}as$, there is an $s' \in T$ for which $(as)s' = a$. Suppose $x \in L_a$, say $x = ua$ and $a = vx$. Then clearly $xs = u(as)$ and $as = v(xs)$ whence $xs \in L_{as}$. Next note that $x = ua = uass' = xss'$. If $y \in L_{as}$, say $y = was$, then $ys's = wass's = was = y$. Hence $\rho_{s'}$ is the inverse map to ρ_s, and so ρ_s is a bijection of L_a onto L_{as}. Further, since $xss' = x$, it follows that $x\mathcal{R}xs$. The dual results are proved similarly.

From (0.25) we see that when S is a c-0-s semigroup, every nonzero principal right ideal is 0-minimal (cf. [2] Corollary 2.49). It follows that if $as \neq 0$, then $as\mathcal{R}a$. Similarly, if $ta \neq 0$ then $ta\mathcal{L}a$. Moreover we can then conclude from (0.29) that if $as \neq 0$, then $L_a \cap R_s$ is a group. These remarks will be used very often. In particular, the first two are used in combining the two parts of Lemma (2.1) into (recall (1.4)):

(2.2) Lemma. Let $a \in H$ and let $c = tas$. If $c \neq 0$, then the translation $\lambda_t\rho_s$ is a bijection of H

onto H_c with an inverse of form $\lambda_{t'}\rho_{s'}$. In every case, tHs is an $\mathcal{H}$-class.

Proof. Indeed, if $c \neq 0$, we have $a \mathcal{R} as \mathcal{L} t(as)$, and the first assertion follows by (2.1). If $tbs = 0$, for all $b \in H$, then $tHs = \{0\}$ which is an $\mathcal{H}$-class. If $tbs \neq 0$, for some $b \in H$, then tHs is an $\mathcal{H}$-class by the first part of the proof.

In order to fix the above result we should make the following:

(2.3) Definition. Let E be a nonempty subset of H. We shall call a subset E' of S a translate of E if and only if $E' = tEs$, for some $t, s \in S$.

Note that E, tE, and Es are translates of E since $E = eEe$, $tE = tEe$ and $Es = eEs$.

(2.4) Theorem. Let E be a normal subgroup of H. Then the set $\mathcal{C}$ of translates of E partitions S and $\mathcal{C}$ is itself a c-0-s semigroup under the induced multiplication.

Proof. Let e be the idempotent in E. Then since S is 0-simple $SeS = S$, and hence $SES = S$. Thus $\cup \mathcal{C} = S$.

Now suppose that two translates tEs and vEu have a nonempty intersection. Suppose that $0 \in tEs \cap vEu$. Thus $0 \in tEs \subseteq tHs$, whence $\{0\} = tEs = tHs$, by (2.2). Similarly $vEu = \{0\}$ and so $tEs = vEu$. If $0 \notin tEs \cap vEu$, then $tes = c$ where $c \neq 0$. Then by (2.2), $tHs = H_c$ and for suitable t', s' the mappings $\lambda_{t'}, \rho_{s'}$ are inverse mappings for the translations λ_t, ρ_s. Hence $E = t'(tEs)s'$ so that $t'(vEu)s' \cap E \neq \emptyset$. However $t'(vEu)s' = (t've)E(eus')$. But both $t've$ and eus' are elements of H. Hence $t'vEus'$ is a group coset of E meeting E, and so $t'vEus' = E$. Now applying λ_t and ρ_s we obtain $vEu = tEs$. We have shown that $\mathcal{C}$ partitions S.

Now let tEs and vEu be elements in $\mathcal{C}$. Then $(tEs)(vEu) = tE(esve)Eu = t(esve)Eu = (tesv)Eu \in \mathcal{C}$, since $esve \in H$ and E is normal in H. But $(tes)(veu) = (tesv)eu$, and since every element of S is of form tes, $tes \to tEs$ is a homomorphism of S onto $\mathcal{C}$. Thus $\mathcal{C}$ is a semigroup. It is now easy to show (cf. [2] Lemma 3.10) that a nontrivial homomorphic image of a c-0-s semigroup is also c-0-s. This will complete the proof.

An immediate consequence is:

(2.5) Corollary. Let $\mathcal{E}$ be the equivalence relation on S whose equivalence classes are the translates of a normal subgroup E of a nonzero $\mathcal{H}$-class H. Then $\mathcal{E}$ is a congruence on S and $\mathcal{E} \subseteq \mathcal{H}$.

Proof. Since $\mathcal{C}$ is a semigroup, $\mathcal{E}$ is a congruence. By (2.2) each translate tEs is contained in an $\mathcal{H}$-class, whence $\mathcal{E} \subseteq \mathcal{H}$.

(2.6) Definition. Let E be a normal subgroup of H, and let $\mathcal{E}$ be the congruence whose equivalence classes are the translates of E. We shall call $\mathcal{E}$ the congruence on S associated with the normal subgroup E of H and write $\mathcal{E} = \mathcal{E}(E)$ where convenient.

(2.7) Remark. Clearly $\mathcal{L}(H) = \mathcal{H}$ by (2.2). Thus $\mathcal{H}$ is a congruence on S.

§3. THE CONGRUENCES $\mathscr{r}$ AND $\mathscr{l}$ ASSOCIATED WITH THE EQUIVALENCE RELATIONS $\underset{\sim}{r}$ AND $\underset{\sim}{\ell}$

In this section we will define an equivalence relation, $\underset{\sim}{d}$, on the $\mathscr{R}$-classes of a c-0-s semigroup S so that for every equivalence relation $\underset{\sim}{r}$ defined on the $\mathscr{R}$-classes with $\underset{\sim}{r} \subseteq \underset{\sim}{d}$ there is an associated congruence, $\mathscr{r} = \mathscr{r}(\underset{\sim}{r})$ on S itself. We remark that an equivalence relation $\underset{\sim}{s}$ can be defined on the $\mathscr{L}$-classes of S which is exactly dual to $\underset{\sim}{d}$ and the definition of $\mathscr{l} = \mathscr{l}(\underset{\sim}{\ell})$ for an equivalence $\underset{\sim}{\ell} \subseteq \underset{\sim}{s}$ is also directly dual to that for $\mathscr{r} = \mathscr{r}(\underset{\sim}{r})$. Therefore, for each of the following lemmas and theorems there is a right-left dual.

We recall that $\mathscr{E}(X)$ is the set of idempotents in X.

(3.1) Theorem. Let R_1 and R_2 be two nonzero $\mathscr{R}$-

classes of S. Then the following conditions on R_1 and R_2 are equivalent:

(1) There exists an s in S such that $\mathcal{E}(R_1) = s\,\mathcal{E}(R_2)$.

(2) There exists an h in $\mathcal{E}(R_1)$ such that $\mathcal{E}(R_1) = h\,\mathcal{E}(R_2)$.

(3) $\mathcal{E}(R_1) = \mathcal{E}(R_1)\,\mathcal{E}(R_2)$.

(4) a. For any $\mathcal{L}$-class L we have $L \cap R_1$ is a group if and only if $L \cap R_2$ is a group, <u>and</u>

b. There exists an s in S such that if $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, and $e_1 \mathcal{L} e_2$ then $se_1 \neq 0$ and $se_1 = se_2$.

(5) Condition 4a <u>and</u>

5b. There exists an $h \in \mathcal{E}(S)$ such that if $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$ with $e_1 \mathcal{L} e_2$ then $he_1 \neq 0$ and $he_1 = he_2$.

(6) Condition 4a <u>and</u>

6b. For all $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, with $e_1 \mathcal{L} e_2$ and for all g in $\mathcal{E}(S)$ we have $ge_1 = ge_2$.

(7) Condition 4a <u>and</u>

7b. For all $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, with $e_1 \mathcal{L} e_2$ and for all t in S we have $te_1 = te_2$.

Proof. We shall prove $(2) \Leftrightarrow (3)$ and then $(1) \Rightarrow (7) \Rightarrow (6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$.

$(2) \Rightarrow (3)$. Assume (2). Since the idempotents in R_2 are left identities on R_2 we have $\mathcal{E}(R_2)\mathcal{E}(R_2) = \mathcal{E}(R_2)$. Hence $\mathcal{E}(R_1) = h\mathcal{E}(R_2) = h\,\mathcal{E}(R_2)\mathcal{E}(R_2)) = (h\mathcal{E}(R_2))\mathcal{E}(R_2) = \mathcal{E}(R_1)\mathcal{E}(R_2)$.

$(3) \Rightarrow (2)$. Assume (3) and let $h \in \mathcal{E}(R_1)$. Then $h\mathcal{E}(R_2) \subseteq \mathcal{E}(R_1)\mathcal{E}(R_2) = \mathcal{E}(R_1)$. Now if $e_1 \in \mathcal{E}(R_1)$ then by (3) $e_1 = e_1'e_2$ where $e_1' \in \mathcal{E}(R_1)$, $e_2 \in \mathcal{E}(R_2)$. Hence since $he_2 \neq 0$ we have $he_2 \mathcal{L} e_1'e_2 = e_1$. But also $he_2 \mathcal{R} e_1$ whence $he_2 \mathcal{H} e_1$ and since both elements are idempotent, $he_2 = e_1$.

$(1) \Rightarrow (7)$. (a) Assume (1) and let L be a given $\mathcal{L}$-class. An $\mathcal{H}$-class is a group if and only if it contains an idempotent ((0.28)). Hence if $L \cap R_2$ is a group then there is an idempotent $e_2 \in \mathcal{E}(R_2 \cap L)$ and then $se_2 \in \mathcal{E}(R_1)$. But $se_2 \mathcal{L} e_2$ and so $se_2 \in \mathcal{E}(R_1 \cap L)$ and hence $R_1 \cap L$ is a group. In almost the same manner one shows that if $L \cap R_1$ is a group then $L \cap R_2$ is a group.

(b) Now suppose $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$ with $e_1 \mathcal{L} e_2$. Let t be given in S. Since (7)a has already been demonstrated we can use the absorbency of S and (0.29) to conclude that $te_1 = 0$ implies $te_2 = 0$. If $te_1 \neq 0$, then there is an idempotent $e_1' \in L_t \cap R_1$, and for some $r \in S$, $t = re_1'$. By (1), $e_1' = se_2'$, for some $e_2' \in \mathcal{E}(R_2)$. Thus there is an r such that $r(se_2') = re_1' = t$. Hence $te_2 = (rse_2')e_2 = rse_2 = re_1 = (re_1')e_1 = te_1$, since $se_2 = e_1$, as se_2 is an idempotent in R_1 which is $\mathcal{L}$ equivalent to e_2. Thus in every case $te_1 = te_2$.

The implication (7) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (5). Assume (6)b. To prove (5)b we need only find an idempotent $h \in S$ for which $he_1 \neq 0$. But any idempotent in R_1 has this property.

The implication (5) $\Rightarrow$ (4) is also obvious.

(4) $\Rightarrow$ (2). Assume (4)a. Then for each $e_2 \in \mathcal{E}(R_2)$ there is an $\mathcal{L}$-equivalent idempotent $e_1 \in \mathcal{E}(R_1)$. Let s be as in (4)b. Since $se_1 \neq 0$ we have that $L_s \cap R_1$ is a group and hence there is an $h \in \mathcal{E}(L_s \cap R_1)$ and an r in S such that $rs = h$. Thus $se_1 = se_2$ implies $e_1 = he_1 = rse_1 = rse_2 = he_2$ and so we have proved that $h\mathcal{E}(R_2) \subseteq \mathcal{E}(R_1)$.

The proof of $\mathcal{E}(R_1) \subseteq h\mathcal{E}(R_2)$ is obtained by interchanging e_1 and e_2.

The obvious implication (2) $\Rightarrow$ (1) completes our proof.

(3.2) Definition. The $\mathcal{R}$-classes R_1 and R_2 are said to be d-equivalent (written $R_1 \underset{\sim}{d} R_2$) if and only if condition (3.1.7) holds.

It is obvious that $\underset{\sim}{d}$ is an equivalence relation on the set of $\mathcal{R}$-classes of S. Observe that the $\{0\}$ $\mathcal{R}$-class is $\underset{\sim}{d}$-equivalent only to itself. If R_1 and R_2 are $\underset{\sim}{d}$-equivalent nonzero $\mathcal{R}$-classes then obviously each condition in (3.1) holds.

(3.3) Lemma. Let T be an arbitrary semigroup. If $s = sf$ and $f\mathcal{L}s$ then $f^2 = f$.

Proof. Since $f\mathcal{L}s$ we can find an r in T^1 such that $rs = f$. Thus $s = sf$ implies $f = rs = rsf = f^2$.

(3.4) Theorem. Let R_1 and R_2 be $\underset{\sim}{d}$-equivalent $\mathcal{R}$-classes of S and let $x_i \in R_i$, $i = 1, 2$. If there is an s such that $sx_1 \neq 0$ and $sx_1 = sx_2$ then $tx_1 = tx_2$ for all t in S.

Proof. Let $x_i \in R_i$, $i = 1, 2$, and s be given as in the hypothesis. Since S is c-0-s, $sx_1 = sx_2 \neq 0$ implies that $L_s \cap R_i$, $i = 1, 2$, are groups and that $x_1 \mathcal{L} x_2$. Thus there are $\mathcal{L}$-equivalent idempotents $e_i \in \mathcal{E}(L_s \cap R_i)$, $i = 1, 2$. Since $e_1 \mathcal{R} x_1$, there exist u and u' such that $e_1 = x_1 u$ and $e_1 u' = x_1$ and then the translations ρ_u and $\rho_{u'}$ are inverse mappings of L_{x_1} and L_{e_1} upon each other by (2.1). These mappings are moreover $\mathcal{R}$-class preserving. Thus $x_2 u \in R_2 \cap L_{e_1}$. Now from $sx_1 = sx_2$ we have $s = se_1 = sx_1 u = sx_2 u$.

By (3.3) $x_2 u$ is the idempotent $e_2 \in \mathcal{E}(R_2 \cap L_s)$. Let t be given. Since $R_1 \underset{\sim}{d} R_2$ we have $te_1 = te_2$. Hence $t(x_1 u) = t(x_2 u)$ implies $t(x_1 uu') = t(x_2 uu')$, whence $tx_1 = tx_2$.

(3.5) Definition. Let $\underset{\sim}{r}$ be an equivalence relation on the set of $\mathcal{R}$-classes of S such that $\underset{\sim}{r} \subseteq \underset{\sim}{d}$. We define a relation $\mathcal{V} = \mathcal{V}(\underset{\sim}{r})$ on S by $x_1 \mathcal{V} x_2$ if and only if

1. $R_{x_1} \underset{\sim}{r} R_{x_2}$ and

2. $tx_1 = tx_2$ for all $t \in S$.

We say that $\mathcal{V}(\underset{\sim}{r})$ <u>is the equivalence associated with</u> $\underset{\sim}{r}$.

(3.6) Remarks.

1. It follows from (3.4) that in definition (3.5) we can replace (2) by the apparently weaker condition

(2') $sx_1 = sx_2 \neq 0$ for some s in S or $x_1 = x_2 = 0$.

2. It is obvious that $\mathcal{V} = \mathcal{V}(\underset{\sim}{r})$ is an equivalence relation with $\mathcal{V} \subseteq \mathcal{L}$.

3. Indeed $\mathcal{V} \cap \mathcal{R} = \Delta$. For suppose $x_1 (\mathcal{V} \cap \mathcal{R}) x_2$

and let $e_1 \in \mathcal{E}(R_{x_1})$. Then $x_1 = e_1 x_1 = e_1 x_2 = x_2$.

4. We further remark that if $R_1 \underset{\sim}{r} R_2$, $e_1 \in \mathcal{E}(R_1)$ and $x_2 \in R_2$ then $x_1 \mathscr{r} x_2$ where $x_1 = e_1 x_2$. In particular if $R_1 \underset{\sim}{r} R_2$ and $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, with $e_1 \mathcal{L} e_2$ then $te_1 = te_2$ by definition (3.1.7b). Whence $e_1 \mathscr{r} e_2$.

(3.7) Theorem. Let $\underset{\sim}{d}$ be as defined in (3.2) and let $\underset{\sim}{r}$, $\underset{\sim}{r} \subseteq \underset{\sim}{d}$, be an equivalence relation defined on the set of $\mathcal{R}$-classes of S. Let $\mathscr{r} = \mathscr{r}(\underset{\sim}{r})$ be the associated equivalence on S as defined in (3.5). Then $\mathscr{r}$ is a congruence.

Proof. In (3.6) we saw that $\mathscr{r}$ is an equivalence relation on S. Suppose now that $x_1 \mathscr{r} x_2$. Let $s \in S$. We have $sx_1 = sx_2$ and therefore, $sx_1 \mathscr{r} sx_2$. Since $tx_1 = tx_2$ for all t, it follows that $t(x_1 s) = t(x_2 s)$. Thus (3.5.2) holds for $x_1 s$ and $x_2 s$. It is clear that $x_1 \mathcal{L} x_2$ and hence $x_1 s = 0$ if and only if $x_2 s = 0$. In that case $x_1 s \mathscr{r} x_2 s$. Otherwise if $x_1 s \neq 0$, we have from (2.2), $R_{x_1 s} = R_{x_1} \underset{\sim}{r} R_{x_2} = R_{x_2 s}$ and again $x_1 s \mathscr{r} x_2 s$.

§4. THE CONGRUENCE $[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ ASSOCIATED WITH THE TRIPLE $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$

Let $\mathcal{P}$ be a congruence on S such that $\mathcal{P} \subseteq \mathcal{R}$. If $a \in S$, we put $\bar{a} = a^{\mathcal{P}}$, and then the relation $\mathcal{R}/\mathcal{P}$ on $S/\mathcal{P}$ is defined by $\bar{a}(\mathcal{R}/\mathcal{P})\bar{b}$ if and only if $a\mathcal{R}b$. We claim that in fact that $\mathcal{R}/\mathcal{P}$ is the $\mathcal{R}$-relation on $S/\mathcal{P}$ and the proof is easy. It follows that the $\mathcal{R}$-classes of S and those of $S/\mathcal{P}$ are in a one-one correspondence under a natural map $R \to R'$. Where necessary we will use primes to distinguish between S and $S/\mathcal{P}$.

To any equivalence relation $\underset{\sim}{r}$ defined on the set of $\mathcal{R}$-classes of S there obviously corresponds an equivalence relation $\underset{\sim}{r}'$ on the $\mathcal{R}$-classes of $S/\mathcal{P}$ defined by $R_1' \, \underset{\sim}{r}' \, R_2'$ if and only if $R_1 \, \underset{\sim}{r} \, R_2$, and in the same manner to any $\underset{\sim}{r}'$ defined on the $\mathcal{R}$-classes of $S/\mathcal{P}$ there corresponds an equivalence $\underset{\sim}{r}$ defined on the $\mathcal{R}$-classes of S. Because of this natural correspondence we will write $\underset{\sim}{r}$ for $\underset{\sim}{r}'$. In the sequel, we shall be concerned with semigroups $S/\mathfrak{C}$, where $\mathfrak{C} \subseteq \mathcal{H}$. Thus we shall write $\underset{\sim}{r}$ for a relation on the $\mathcal{R}$-classes of S or $S/\mathfrak{C}$, but we will find it necessary to distinguish the associated congruences (cf. (3.5)) on these semigroups. We shall write $\mathfrak{r}$ for the congruence on S and $\mathfrak{r}'$ for the congruence on $S/\mathfrak{C}$.

(4.1) Definition. Let E be a normal subgroup of the nonzero $\mathcal{H}$-class H, and let $\mathfrak{C} = \mathfrak{C}(E)$ be the associated congruence (cf. (2.6)). We denote by $\underset{\sim}{d}(E)$ the $\underset{\sim}{d}$-relation on $\mathcal{R}$-classes of $S/\mathfrak{C}$ defined by (3.2). The relation $\underset{\sim}{s}(E)$ is defined dually on the $\mathcal{L}$-classes of $S/\mathfrak{C}$.

(4.2) Remark. We note that $\underset{\sim}{d}(E)$ can be considered as a relation on the set of $\mathcal{R}$-classes of S since $\mathfrak{C} = \mathfrak{C}(E) \subseteq \mathcal{H} \subseteq \mathcal{R}$. Moreover, we then clearly have $\underset{\sim}{d}(E) \supseteq \underset{\sim}{d}(e) = \underset{\sim}{d}$. Indeed we can define $\underset{\sim}{d}(E)$ directly on S without going to $S/\mathfrak{C}$. Thus condition (7a) of (3.1) would remain the same while (7b) would read

(7b') For all $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, with $e_1 \mathcal{L} e_2$ and for all t in S we have $te_1 \mathfrak{C}\, te_2$.

Observe that $t^{\mathfrak{C}} e_i^{\mathfrak{C}} = (te_i)^{\mathfrak{C}} = t(e_i^{\mathfrak{C}})$ since $\mathfrak{C}$ is a congruence and hence $t(e_1^{\mathfrak{C}}) = t(e_2^{\mathfrak{C}})$ so that the above conditions are natural. Further if $E = H$, then by (2.7), $\mathfrak{C}(E) = \mathcal{H}$. But $e_1 \mathcal{L} e_2$ implies $te_1 \mathcal{H} te_2$, for all $t \in S$. Hence $\underset{\sim}{d}(H)$ is defined by (3.1.7a). More explicitly:

$R_1 \underset{\sim}{d}(H) R_2$ if and only if for any $\mathcal{L}$-class L, $R_1 \cap L$ is a group precisely when $R_2 \cap L$ is a group.

(4.3) <u>Lemma</u>. Let $\mathfrak{r}$, $\mathfrak{r} \subseteq \mathcal{L}$ be a right congruence and $\mathfrak{l}$, $\mathfrak{l} \subseteq \mathcal{R}$, be a left congruence on an arbitrary semigroup T. Then $\mathfrak{r} \circ \mathfrak{l} = \mathfrak{l} \circ \mathfrak{r}$. If $\mathfrak{r}$ and $\mathfrak{l}$ are congruences, then so is $\mathfrak{r} \circ \mathfrak{l}$ and $\mathfrak{r} \circ \mathfrak{l}$ is the smallest congruence containing both $\mathfrak{r}$ and $\mathfrak{l}$.

Proof. The first assertion of the lemma generalizes the result that $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ and can be proven as (0.7) making use of the first observation and dual of (0.28). The second assertion follows immediately since $\mathscr{r} \circ \mathscr{l}$ is an equivalence by (0.8). A special case of our lemma is found in Munn [14], Lemma 3.

(4.4) Definition of Permissible Triple and the Associated Congruence.

(a) Let H be the fixed nonzero group $\mathscr{H}$-class of S and let E be a normal subgroup of H. Let $\mathfrak{C} = \mathfrak{C}(E)$ be the associated congruence, cf. (2.6). Let $\underset{\sim}{d}(E)$ and $\underset{\sim}{s}(E)$ be the equivalence relations on the set of $\mathscr{R}$-classes and $\mathscr{L}$-classes, respectively, of S (or $S/\mathfrak{C}$), cf. (4.1). Let $\underset{\sim}{r}$ and $\underset{\sim}{\ell}$ be equivalence relations defined respectively on the same sets. Then $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ is said to be a permissible triple if and only if $\underset{\sim}{r} \subseteq \underset{\sim}{d}(E)$ and $\underset{\sim}{\ell} \subseteq \underset{\sim}{s}(E)$.

(b) Let $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ be a permissible triple. Let $\mathscr{r}' = \mathscr{r}(\underset{\sim}{r})$ and $\mathscr{l}' = \mathscr{l}(\underset{\sim}{\ell})$ be the congruences on $S/\mathfrak{C}$, $\mathfrak{C} = \mathfrak{C}(E)$, associated with the equivalences $\underset{\sim}{r}$ and $\underset{\sim}{\ell}$ respectively, cf. (3.5). Then $(\mathscr{r}' \circ \mathscr{l}')^{\mathfrak{C}}$ is said to be the congruence on S associated with $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$. This congruence will be denoted by $[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$.

(4.5) Remark. Note that for x and y in S, $x[\underset{\sim}{r}, E, \underset{\sim}{\ell}]y$ if and only if $x^{\mathcal{L}}(\mathfrak{r}' \circ \mathfrak{l}')y^{\mathcal{L}}$. It follows from Lemma (4.3) that $[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ is indeed a congruence. Thus $[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ is the kernel of the composite mapping

$$S \to S/\mathcal{L} \to S/\mathcal{L}/(\mathfrak{r}' \circ \mathfrak{l}').$$

(4.6) Observation. We observe from (3.6) and its dual that $\mathfrak{r}$ "collapses" exactly those $\mathcal{R}$-classes which are $\underset{\sim}{r}$-equivalent and is $\mathcal{L}$-class preserving, while $\mathfrak{l}$ "collapses" $\mathcal{L}$-classes which are $\underset{\sim}{\ell}$-equivalent and is $\mathcal{R}$-class preserving. Moreover, it is clear that null $\mathcal{H}$-classes can only go onto null $\mathcal{H}$-classes and group $\mathcal{H}$-classes onto group $\mathcal{H}$-classes.

§5. CONGRUENCES LYING UNDER $\mathcal{H}$

(5.1) Theorem. Let $\mathcal{G}$ be a congruence on S such that $\mathcal{G} \subseteq \mathcal{H}$. Let e be the nonzero idempotent in H and put $E = e^{\mathcal{G}}$ Then E is a normal subgroup of H and $\mathcal{G} = \mathcal{C}(E)$, the congruence associated with E.

Proof. Since $\mathcal{G}$ restricted to the group $\mathcal{H}$-class, H, is a congruence on H, it follows that $e^{\mathcal{G}} = E$ is a normal subgroup of H and by (2.6) there is an associated congruence $\mathcal{C} = \mathcal{C}(E)$. We must show that $\mathcal{G} = \mathcal{C}$. It is enough that every $\mathcal{G}$-congruence class is a translate of E. Let $a \in S$. If $a = 0$, then $a^{\mathcal{G}} = \{0\} = 0\,E\,0$. Suppose $a \neq 0$. Then, since $a \mathcal{D} e$, there exist

s, t, s', t' such that $tes = a$ and $t'as' = e$. Let $A = a^{\mathcal{F}}$. Then $A = (tes)^{\mathcal{F}} = t^{\mathcal{F}} Es^{\mathcal{F}} \supseteq tEs$, and similarly $E \supseteq t'As'$. Thus $A \supseteq tEs \supseteq tt'As's =$ A, by (2.2), whence $A = tEs$.

We remark that we have proved that if T is 0-bisimple, and $\mathcal{F}$ is a congruence such that $\mathcal{F} \subseteq \mathcal{H}$, then every congruence class is of the form $t\,B\,s$, where B is a fixed non-zero congruence class. In this more general case, we do not know if every $t\,B\,s$ is a congruence class.

(5.2) <u>Remark</u>. If the set $\underset{\sim}{N}$ of normal subgroups of H is ordered by set inclusion, then $\underset{\sim}{N}$ is a complete lattice (cf. (0.30) and (0.31)). The set of all congruences on S is also a complete lattice (cf. [3], p. 86). Since $\mathfrak{C}(H) = \mathcal{H}$ is a congruence on S, the set $\underset{\sim}{H}$ of all congruences lying under $\mathcal{H}$ is also a complete lattice.

(5.3) <u>Theorem</u>. Let $\underset{\sim}{H}$ be the set of congruences lying under $\mathcal{H}$, and let $\underset{\sim}{N}$ be the complete lattice of normal subgroups of H. Then $\underset{\sim}{N}$ and $\underset{\sim}{H}$ are isomorphic complete lattices under the mappings $E \to \mathfrak{C}(E)$ and $\mathfrak{C} \to e^{\mathfrak{C}}$.

Proof. Let e be the identity of H, and let $\alpha: \mathfrak{C} \to e^{\mathfrak{C}}$, and $\beta: E \to \mathfrak{C}(E)$ be mappings on the two sets mentioned above. By (5.1), $\alpha\beta$ is the identity on the set of congruences under $\mathcal{H}$. Let E be a normal subgroup of H. By (2.6), the image of E under $\beta\alpha$ is a translate of E and clearly contains e. Hence $E\beta\alpha = E$ and so α and β are mutually inverse. It is clear that α and β are order-preserving. Hence ([3], p. 22) α and β are complete lattice isomorphisms, and the corollary holds.

§6. CONGRUENCES LYING UNDER $\mathcal{L}$ AND $\mathcal{R}$

(6.1) Lemma. Let $\mathcal{P}$ be a congruence on S such that $\mathcal{P} \cap \mathcal{R} = \Delta$.

(1) If $e^2 = e$ and $x \mathcal{P} e$ then $x^2 = x$.

(2) If $x_1 \mathcal{P} x_2$ then $tx_1 = tx_2$ for all t in S.

(3) If $x_1 \mathcal{P} x_2$ then $x_1 \mathcal{L} x_2$.

Proof. (1) If $e = 0$ the result is trivial. Let $e \neq 0$. If $x \mathcal{P} e$ then $e^2 \mathcal{P} ex \mathcal{P} x^2$, hence $x \mathcal{P} x^2$ since $e^2 = e$. But then $x^2 \mathcal{R} x$; whence $x = x^2$, by $\mathcal{P} \cap \mathcal{R} = \Delta$.

(2) If $x_1 \mathcal{P} x_2$ then $tx_1 \mathcal{P} tx_2$. Since $\mathcal{P}$ is proper $tx_1 = 0$ implies $tx_2 = 0$. If $tx_1 \neq 0$, then $tx_1 \mathcal{R}\, t \mathcal{R}\, tx_2$ and whence $tx_1 = tx_2$ since $\mathcal{P} \cap \mathcal{R} = \Delta$.

(3) This is an immediate consequence of (2).

(6.2) Definition. Let $\mathcal{Q}$ be a proper congruence on S. Let $\underset{\sim}{r}(\mathcal{Q})$ be a relation on the $\mathcal{R}$-classes of S defined by $R_1 \underset{\sim}{r}(\mathcal{Q}) R_2$ if and only if there are $x_i \in R_i$, $i = 1, 2$ such that $x_1 \mathcal{Q} x_2$. We define $\underset{\sim}{\ell}(\mathcal{Q})$ dually.

(6.3) Remark. Let $\mathcal{Q}$ be any congruence on S, and suppose $y_1 \mathcal{Q} y_2$. If x_2 is any element in S with $x_2 \mathcal{R} y_2$ then one readily sees that there exists x_1 with $x_1 \mathcal{R} y_1$ such that $x_1 \mathcal{Q} x_2$ and $x_1 \mathcal{L} x_2$. For proof, observe that $x_2 = y_2 t$ for some t in S and put $x_1 = y_1 t$.

The relation $\underset{\sim}{r}(\mathcal{Q})$ defined in (6.2) is clearly reflexive and symmetric. That $\underset{\sim}{r}(\mathcal{Q})$ is also transitive is an easy consequence of the observation in the preceding paragraph. Thus $\underset{\sim}{r}(\mathcal{Q})$ is an equivalence on the set of $\mathcal{R}$-classes of S, which will be called the equivalence relation induced by $\mathcal{Q}$.

(6.4) Theorem. Let $\mathcal{P}$ be a proper congruence on S such that $\mathcal{P} \cap \mathcal{R} = \Delta$. Let $\underset{\sim}{r} = \underset{\sim}{r}(\mathcal{P})$ be the equivalence relation induced on the set of $\mathcal{R}$-classes of S as defined in (6.2). Then $\underset{\sim}{r} \subseteq \underset{\sim}{d}$ and $\mathcal{P} = \mathscr{r}(\underset{\sim}{r})$ (cf. (3.5)), the congruence associated with $\underset{\sim}{r}$.

Proof. Let $R_1 \underset{\sim}{r} R_2$. Suppose $e_1 \in \mathcal{E}(R_1)$. By (6.2), there is an $e_2 \in R_2$ such that $e_1 \mathcal{P} e_2$. By (6.1), $e_2 \in \mathcal{E}(R_2)$ and $e_1 \mathcal{L} e_2$. We can draw two conclusions from this.

First, suppose L is an $\mathcal{L}$-class for which $R_1 \cap L$ is a group. Then there is an idempotent $e_1 \in R_1 \cap L$ (0.28.2). But then $e_2 \in R_1 \cap L$ so that $R_2 \cap L$ is a group. Similarly, if $R_2 \cap L$ is a group, so is $R_1 \cap L$ and (3.1.4a) is verified.

Second, let $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, and $e_1 \mathcal{L} e_2$. By the first part of the proof, there is an idempotent $e_2' \in R_2$ such that $e_1 \mathcal{P} e_2'$ and $e_1 \mathcal{L} e_2'$. Hence $e_2 \mathcal{H} e_2'$, and as each $\mathcal{H}$-class has at most one idempotent, $e_2 = e_2'$. Thus $e_1 \mathcal{P} e_2$, whence by (6.1.2), $te_1 = te_2$, for all t in S.

This verifies (3.1.7b), whence $R_1 \underset{\sim}{d} R_2$.

We have shown $\underset{\sim}{r} \subseteq \underset{\sim}{d}$.

By the definition of $\mathscr{r}$ and (6.1.2) it is immediate that $\mathscr{p} \subseteq \mathscr{r} = \mathscr{r}(\underset{\sim}{r})$. Suppose now that $x_1 \mathscr{r} x_2$. Then by (6.3) there is an $x_2' \in R_2$ such that $x_1 \mathscr{p} x_2'$. Since $\mathscr{p} \subseteq \mathscr{r}$, we have $x_1 \mathscr{r} x_2'$ and hence $x_2 \mathscr{r} x_2'$. Thus $x_2 (\mathscr{r} \cap \mathscr{R}) x_2'$. But by (3.6), $\mathscr{r} \cap \mathscr{R} = \Delta$, whence $x_2 = x_2'$. Thus $x_1 \mathscr{p} x_2$ and it follows that $\mathscr{r} \subseteq \mathscr{p}$. This completes the proof.

(6.5) <u>Theorem</u>. Let $\underset{\sim}{R}$ be the set of equivalence relations on the $\mathscr{R}$-classes of S which are under $\underset{\sim}{d}$ and let $\underset{\sim}{P}$ be the set of all congruences $\mathscr{p}$ on S such that $\mathscr{p} \cap \mathscr{R} = \Delta$. Then $\underset{\sim}{R}$ and $\underset{\sim}{P}$ are complete lattices isomorphic under the mappings $\underset{\sim}{r} \to \mathscr{r}(\underset{\sim}{r})$ and $\mathscr{p} \to \underset{\sim}{r}(\mathscr{p})$.

Proof. We will show that the mappings $\alpha : \underset{\sim}{r} \to \mathscr{r}(\underset{\sim}{r})$ defined in (3.5) and $\beta : \mathscr{p} \to \underset{\sim}{r}(\mathscr{p})$ defined in (6.2) are mutually inverse mappings between $\underset{\sim}{R}$ and $\underset{\sim}{P}$. By (6.4), $\beta\alpha$ is the identity on $\underset{\sim}{P}$. That $\underset{\sim}{r}\alpha\beta \subseteq \underset{\sim}{r}$ is immediate from the definitions of $\mathscr{r}(\underset{\sim}{r})$, cf. (3.5.1), and $\underset{\sim}{r}(p)$, cf. (6.2). If $R_1 \underset{\sim}{r} R_2$, then by (3.6) there exist $x_1 \in R_1$ and $x_2 \in R_2$ such that $x_1 \mathscr{r}(\underset{\sim}{r}) x_2$. It follows that $\underset{\sim}{r} \subseteq \underset{\sim}{r}\alpha\beta$, whence $\alpha\beta$ is the identity on $\underset{\sim}{R}$.

When $\underset{\sim}{R}$ and $\underset{\sim}{P}$ are ordered in the usual fashion it is clear from the definitions that α and β are order-preserving, and that $\underset{\sim}{R}$ is a complete lattice. Since $\mathscr{r}(\underset{\sim}{d}) = \underset{\sim}{d}\alpha$ is the maximal element of $\underset{\sim}{P}$ it follows that $\underset{\sim}{P}$ is closed under arbitrary intersections whence $\underset{\sim}{P}$ is also a complete lattice. The conclusion is now immediate since α and β are order-preserving inverse mappings ([3], p. 22).

Since all of the above results can be dualized, we have that the set of equivalence relations, $\underset{\sim}{L}$ on the $\mathscr{L}$-classes of S which are under $\underset{\sim}{s}$ and the set $\underset{\sim}{Q}$ of all congruences, $\mathscr{q}$, on S, such that $\mathscr{q} \cap \mathscr{L} = \Delta$, are also isomorphic complete lattices.

§7. THE CORRESPONDENCE BETWEEN PROPER CONGRUENCES AND PERMISSIBLE TRIPLES

In this section we will finally show that the association of (4.4b) from the set of permissible triples to the set of congruences on S is 1-1 and onto. We will do this by factoring the given congruence, $\mathcal{P}$ through $\mathcal{H}$ —as $\mathcal{P} \cap \mathcal{H} = \mathcal{C}$ —and then factoring $\mathcal{P}/\mathcal{C}$ on $S/\mathcal{C}$ as the circle, $\circ$, product of two congruences $\mathcal{r}(\underset{\sim}{r})$ and $\mathcal{l}(\underset{\sim}{\ell})$.

We begin with two preliminary results.

(7.1) Theorems. Let $\mathcal{Q}$ be a proper congruence on S. Then $\mathcal{r} = \mathcal{Q} \cap \mathcal{L}$ and $\mathcal{l} = \mathcal{Q} \cap \mathcal{R}$ are proper congruences.

Proof. From the definitions, we have that $\mathcal{r}$ is a right congruence (cf. (0.28.1)) and $\mathcal{l}$ is a left congruence, and both are proper. Suppose $x \mathscr{r} y$ so that $x \mathscr{Q} y$ and $x \mathscr{L} y$. Then for any s, $sx \mathscr{Q} sy$. Thus $sx = 0$ if and only if $sy = 0$. In that case $sx \mathscr{r} sy$. Otherwise we have $sx \mathscr{L} x \mathscr{L} y \mathscr{L} sy$, whence $sx \mathscr{r} sy$. The proof for $\mathscr{I}$ is dual.

(7.2) Theorem. Let $\mathscr{Q}$ be a proper congruence on S. Then $\mathscr{Q} = (\mathscr{Q} \cap \mathscr{L}) \circ (\mathscr{Q} \cap \mathscr{R})$.

Proof. By (7.1), $\mathscr{Q} \cap \mathscr{L}$ and $\mathscr{Q} \cap \mathscr{R}$ are congruences which commute by (4.3). Hence $(\mathscr{Q} \cap \mathscr{L}) \circ (\mathscr{Q} \cap \mathscr{R}) \subseteq \mathscr{Q}$ by the last part of (4.3).

Conversely, suppose $a \mathscr{Q} b$, and $a \neq 0$. Then since S is regular, we can find an idempotent e, $\mathscr{R}$-equivalent to a. Hence $a \mathscr{Q} b$ implies $ea = a \mathscr{Q} eb$. Thus $eb \neq 0$ and both $eb \mathscr{R} e \mathscr{R} a$ and $eb \mathscr{L} b$. But $b \mathscr{Q} a \mathscr{Q} eb$ implies $b \mathscr{Q} eb$. Combining these, we see that for $a \neq 0$, $a \mathscr{Q} b$ implies $a(\mathscr{Q} \cap \mathscr{R})eb$ and $eb(\mathscr{Q} \cap \mathscr{L})b$, which is obviously also true for $a = 0$. We deduce that $\mathscr{Q} \subseteq (\mathscr{Q} \cap \mathscr{L}) \circ (\mathscr{Q} \cap \mathscr{R})$ and the equality follows.

For the sake of clarity, we shall now use primes to indicate relations on a factor semigroup $S/\mathfrak{C}$, where $\mathfrak{C}$ is a congruence (see beginning of §4). For example, $\mathcal{R}'$ is the $\mathcal{R}$-relation on $S/\mathfrak{C}$. If $\mathfrak{C} \subseteq \mathcal{H}$ then it is easy to check that $\mathcal{R}' = \mathcal{R}/\mathfrak{C}$, $\mathcal{L}' = \mathcal{L}/\mathfrak{C}$ and $\mathcal{H}' = \mathcal{H}/\mathfrak{C}$. Further, if $\mathfrak{p}$ is a congruence on S, then $(\mathfrak{p}/\mathfrak{C})^{\mathfrak{C}} = \mathfrak{p}$ and if $\mathfrak{p}'$ is a congruence on $S/\mathfrak{C}$ then $(\mathfrak{p}'^{\mathfrak{C}})/\mathfrak{C} = \mathfrak{p}'$. However, if R is an $\mathcal{R}$-class of S then the corresponding $\mathcal{R}$-class of $S/\mathfrak{C}$ will still be identified with R.

(7.3) Lemma. Let $\mathfrak{C}$ be a congruence lying under $\mathcal{H}$. Then for any proper congruence $\mathfrak{p}$ we have $\underset{\sim}{r}(\mathfrak{p}) = \underset{\sim}{r}((\mathfrak{p} \cap \mathcal{L})/\mathfrak{C})$ where $\underset{\sim}{r}(\;)$ is defined in (6.2). Dually $\underset{\sim}{\ell}(\mathfrak{p}) = \underset{\sim}{\ell}((\mathfrak{p} \cap \mathcal{R})/\mathfrak{C})$.

Proof. Let $\underset{\sim}{r} = \underset{\sim}{r}(\mathfrak{p})$ and $\underset{\sim}{r}' = \underset{\sim}{r}((\mathfrak{p} \cap \mathcal{L})/\mathfrak{C})$. If $R_1 \, \underset{\sim}{r}' \, R_2$ then there exists $x_i^{\mathfrak{C}} \in R_i$, $i = 1, 2$, such that $x_1^{\mathfrak{C}} ((\mathfrak{p} \cap \mathcal{L})/\mathfrak{C}) x_2^{\mathfrak{C}}$. Hence $x_1 (\mathfrak{p} \cap \mathcal{L}) x_2$ and clearly $R_1 \, \underset{\sim}{r} \, R_2$. Conversely, if $R_1 \, \underset{\sim}{r} \, R_2$ then there are $y_i \in R_i$, $i = 1, 2$ such that $y_1 \, \mathfrak{p} \, y_2$. But by (6.3) we can then find $x_i \in R_i$ with $x_1 \mathcal{L} x_2$ such that $x_1 \, \mathfrak{p} \, x_2$.

Hence $x_1(\mathscr{p} \cap \mathscr{L})x_2$ and $x_1^{\mathfrak{C}}((\mathscr{p} \cap \mathscr{L})/\mathfrak{C})x_2^{\mathfrak{C}}$ whence $R_1 \underset{\sim}{r}' R_2$. The equality follows.

(7.4) <u>Theorem</u>. Let $\mathscr{p}$ be a proper congruence on S and let

(1) $E = e^{(\mathscr{p} \cap \mathscr{H})}$

(2) $\underset{\sim}{r} = \underset{\sim}{r}(\mathscr{p})$

(3) $\underset{\sim}{\ell} = \underset{\sim}{\ell}(\mathscr{p})$.

Then $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ is a permissible triple and $\mathscr{p} = [\underset{\sim}{r}, E, \underset{\sim}{\ell}]$.

Proof. By (5.1), E is a normal subgroup of H, and $\mathfrak{C}(E) = \mathscr{p} \cap \mathscr{H} = \mathfrak{C}$, say. By (7.3), we have $\underset{\sim}{r} = \underset{\sim}{r}((\mathscr{p} \cap \mathscr{L})/\mathfrak{C})$. Now $(\mathscr{p} \cap \mathscr{L})/\mathfrak{C} \cap \mathscr{R}' = (\mathscr{p} \cap \mathscr{L} \cap \mathscr{R})/\mathfrak{C} = (\mathscr{p} \cap \mathscr{H})/\mathfrak{C} = \Delta'$ and similarly $(\mathscr{p} \cap \mathscr{R})/\mathfrak{C} \cap \mathscr{L}' = \Delta'$. By (6.4) $\underset{\sim}{r} \subseteq \underset{\sim}{d}' = \underset{\sim}{d}(E)$ and also $\underset{\sim}{\ell} \subseteq \underset{\sim}{s}' = \underset{\sim}{s}(E)$, so that $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ is a permissible triple.

Now put $\mathscr{r}' = \mathscr{r}(\underset{\sim}{r})$ and $\mathscr{l}' = \mathscr{l}(\underset{\sim}{\ell})$ on $S/\mathfrak{C}$. Then by (6.4), $\mathscr{r}' = (\mathscr{p} \cap \mathscr{L})/\mathfrak{C}$ and $\mathscr{l}' = (\mathscr{p} \cap \mathscr{R})/\mathfrak{C}$. Hence $(\mathscr{r}' \circ \mathscr{l}') = (\mathscr{p}/\mathfrak{C} \cap \mathscr{L}') \circ (\mathscr{p}/\mathfrak{C} \cap \mathscr{R}') = \mathscr{p}/\mathfrak{C}$ by (7.2). It follows that

$$[\underset{\sim}{r}, E, \underset{\sim}{\ell}] = (\mathfrak{r}' \circ \mathfrak{l}')^{\flat} = \mathfrak{p} .$$

(7.5) Lemma. If $\mathfrak{p}$ and $\mathfrak{q}$ are proper congruences on S such that $\mathfrak{p} \circ \mathfrak{q} = \mathfrak{q} \circ \mathfrak{p}$ then $(\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L} = (\mathfrak{p} \cap \mathcal{L}) \circ (\mathfrak{q} \cap \mathcal{L}) = (\mathfrak{p} \cap \mathcal{L}) \vee (\mathfrak{q} \cap \mathcal{L})$ and $(\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{R} = (\mathfrak{p} \cap \mathcal{R}) \circ (\mathfrak{q} \cap \mathcal{R}) = (\mathfrak{p} \cap \mathcal{R}) \vee (\mathfrak{q} \cap \mathcal{R})$ (cf. (0.30) and (0.31)).

Proof. By (7.1), $(\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L}$, $\mathfrak{p} \cap \mathcal{L}$, and $\mathfrak{q} \cap \mathcal{L}$ are all congruences. Since $\mathfrak{p} \cap \mathcal{L} \subseteq (\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L}$ and $\mathfrak{q} \cap \mathcal{L} \subseteq (\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L}$ we clearly have $(\mathfrak{p} \cap \mathcal{L}) \circ (\mathfrak{q} \cap \mathcal{L}) \subseteq (\mathfrak{p} \cap \mathcal{L}) \vee (\mathfrak{q} \cap \mathcal{L}) \subseteq (\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L}$. Conversely, suppose $x((\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L})y$. Then $x \mathcal{L} y$ and $x(\mathfrak{p} \circ \mathfrak{q})y$. Hence, there is a z such that $x \mathfrak{p} z \mathfrak{q} y$. But if $e \in \mathcal{E}(L_x)$, we have $x = xe \,\mathfrak{p}\, ze \,\mathfrak{q}\, ye = y$ and hence $ze \in L_x$. Thus $x(\mathfrak{p} \cap \mathcal{L})ze(\mathfrak{q} \cap \mathcal{L})y$ and therefore $x(\mathfrak{p} \cap \mathcal{L}) \circ (\mathfrak{q} \cap \mathcal{L})y$. Hence $(\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L} \subseteq (\mathfrak{p} \cap \mathcal{L}) \circ (\mathfrak{q} \cap \mathcal{L})$, and $(\mathfrak{p} \cap \mathcal{L}) \circ (\mathfrak{q} \cap \mathcal{L}) = (\mathfrak{p} \circ \mathfrak{q}) \cap \mathcal{L} = (\mathfrak{p} \cap \mathcal{L}) \vee (\mathfrak{q} \cap \mathcal{L})$ follows. Dually one obtains the other equality.

Combining the above results and that of (4.3), we have

(7.6) Corollary. If $\mathfrak{p}$ and $\mathfrak{q}$ are commuting congruences on S, then any pair of the following congruences commute: $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{p} \cap \mathcal{L}$, $\mathfrak{q} \cap \mathcal{L}$, $\mathfrak{p} \cap \mathcal{R}$, $\mathfrak{q} \cap \mathcal{R}$, $\mathfrak{p} \cap \mathcal{H}$ and $\mathfrak{q} \cap \mathcal{H}$.

In order to state our main theorem, we order the set of permissible triples in an obvious fashion.

(7.7) Definition. If $\underset{\sim}{T}$ is the set of permissible triples on S, then we partially order $\underset{\sim}{T}$ by $(\underset{\sim}{r}, E, \underset{\sim}{\ell}) \subseteq (\underset{\sim}{r}', E', \underset{\sim}{\ell}')$ if and only if $\underset{\sim}{r} \subseteq \underset{\sim}{r}'$, $E \subseteq E'$ and $\underset{\sim}{\ell} \subseteq \underset{\sim}{\ell}'$.

(7.8) Main Theorem. Let $\underset{\sim}{C}$ be the set of proper congruences on S, and let $\underset{\sim}{T}$ be the set of permissible tripes. Then $\underset{\sim}{C}$ and $\underset{\sim}{T}$ are isomorphic complete lattices.

Proof. Let α map T into $\underset{\sim}{C}$ by $(\underset{\sim}{r}, E, \underset{\sim}{\ell})\alpha = [\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ and β map $\underset{\sim}{C}$ into $\underset{\sim}{T}$ by $\mathfrak{p}\beta = (\underset{\sim}{r}(\mathfrak{p}), e^{\mathfrak{p} \cap \mathcal{H}}, \underset{\sim}{\ell}(\mathfrak{p}))$, where $[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ is defined by (4.4) and $\underset{\sim}{r}(\mathfrak{p})$ and $\underset{\sim}{\ell}(\mathfrak{p})$ by (6.2).

Theorem (7.4) asserts that $\beta\alpha$ is the identity on $\underset{\sim}{C}$. We shall now prove that $\alpha\beta$ is the identity on $\underset{\sim}{T}$. So let $(\underset{\sim}{r}, E, \underset{\sim}{\ell}) \in \underset{\sim}{T}$, and let $\mathfrak{p} = (\underset{\sim}{r}, E, \underset{\sim}{\ell})\alpha = (\mathfrak{r}' \circ \mathfrak{l}')^{\mathfrak{C}}$, where $\mathfrak{C} = \mathfrak{C}(E)$, $\mathfrak{r}' = \mathfrak{r}(\underset{\sim}{r})$ and $\mathfrak{l}' = \mathfrak{l}(\underset{\sim}{\ell})$.

By (2.5), $\mathfrak{C} \subseteq \mathcal{H}$; and so we observe that $(\mathfrak{p} \cap \mathcal{L})/\mathfrak{C} = \mathfrak{p}/\mathfrak{C} \cap \mathcal{L}' = (\mathfrak{r}' \circ \mathfrak{l}') \cap \mathcal{L}' = ((\mathfrak{r}' \cap \mathcal{L}') \circ (\mathfrak{l}' \cap \mathcal{L}'))$ by (7.5). Since by (3.6) and its dual, $\mathfrak{r}' \subseteq \mathcal{L}'$ and $\mathfrak{l}' \cap \mathcal{L}' = \Delta'$, we obtain that $\mathfrak{p} \cap \mathcal{L}/\mathfrak{C} = \mathfrak{r}'$. Now $\mathfrak{p} \cap \mathcal{H}/\mathfrak{C} = \mathfrak{p}/\mathfrak{C} \cap \mathcal{H} = (\mathfrak{p}/\mathfrak{C} \cap \mathcal{L}') \cap \mathcal{R}' = \mathfrak{r}' \cap \mathcal{R}' = \Delta'$, and so $\mathfrak{C} = \mathfrak{p} \cap \mathcal{H}$. By (5.2), $E = e^{\mathfrak{C}}$, and $E = e^{\mathfrak{p} \cap \mathcal{H}}$ follows.

Again, since $\mathfrak{C} \subseteq \mathcal{H}$ we have by (7.3) that $\underset{\sim}{r}(\mathfrak{p}) = \underset{\sim}{r}(\mathfrak{p} \cap \mathcal{H}/\mathfrak{C}) = \underset{\sim}{r}(\mathfrak{r}')$, where $\mathfrak{r}' = \mathfrak{r}(\underset{\sim}{r})$ on $S/\mathfrak{C}$. Hence by (6.5), $\underset{\sim}{r}(\mathfrak{p}) = \underset{\sim}{r}$. Similarly $\underset{\sim}{\ell}(\mathfrak{p}) = \underset{\sim}{\ell}$, and hence $\alpha\beta$ is the identity on $\underset{\sim}{T}$.

We have proved that α and β are bijections between $\underset{\sim}{T}$ and $\underset{\sim}{C}$, and from the definitions it is clear that they preserve order. Thus α and β are order isomorphisms. Clearly $(\underset{\sim}{d}(H), H, \underset{\sim}{\ell}(H))$

is the maximum element of T, whence $(\underset{\sim}{d}(H), H, \underset{\sim}{\ell}(H))\alpha = [\underset{\sim}{d}(H), H, \underset{\sim}{\ell}(H)] = \mathfrak{m}$, say, is the maximum element of $\underset{\sim}{C}$. But the collection of all congruences on S is a complete lattice, and $\underset{\sim}{C}$ consists of all congruences under $\mathfrak{m}$, whence $\underset{\sim}{C}$ is itself a complete lattice. Since β is an order isomorphism, it follows that $\underset{\sim}{T}$ is an isomorphic complete lattice.

§8. THE LATTICE STRUCTURE OF $\underset{\sim}{T}$

In the previous section we have shown that $\underset{\sim}{T}$ is a complete lattice under the natural ordering. It is of interest to describe explicitly the lattice operations on $\underset{\sim}{T}$.

(8.1) Notation. Let $\underset{\sim}{K}$ be a lattice and let A be an index set. If $\{k_\alpha\}_{\alpha \in A}$ is a family of elements in $\underset{\sim}{K}$, then the infinum and supremum of $\{k_\alpha\}_{\alpha \in A}$ in $\underset{\sim}{K}$ will be denoted by $\bigwedge_{\underset{\sim}{K}} k_\alpha$ and $\bigvee_{\underset{\sim}{K}} k_\alpha$ respectively. The index set A will be implicit in this notation. Since we shall have to refer to many lattices, we give a list here:

$\underset{\sim}{R}$ — the lattice of equivalences on the $\mathcal{R}$-classes of S, lying under $\underset{\sim}{d}(H)$;

$\underset{\sim}{L}$—the latticeof equivalences on the $\mathscr{L}$-classes of S lying under $\underset{\sim}{s}(H)$;

$\underset{\sim}{N} = \underset{\sim}{N}(H)$—the lattice of normal subgroups of H;

$\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$;

$\underset{\sim}{T}$—the lattice of permissible triples;

$\underset{\sim}{C}$—the lattice of proper congruences on S;

$\underset{\sim}{H}$—the lattice of congruences on S which lie under $\mathscr{H}$.

(8.2) <u>Remark</u>. By (5.2), $E \to \mathfrak{C}(E)$ is a complete lattice isomorphism of $\underset{\sim}{N}(H)$ onto $\underset{\sim}{H}$. Thus for any family $\{E_\alpha\}_{\alpha \in A}$ of normal subgroups of H, $\mathfrak{C}(\wedge_{\underset{\sim}{N}} E_\alpha) = \wedge_{\underset{\sim}{H}} \mathfrak{C}(E_\alpha)$ and $\mathfrak{C}(\vee_{\underset{\sim}{N}} E_\alpha) = \vee_{\underset{\sim}{H}} \mathfrak{C}(E_\alpha)$. Observe that unless A is empty $\wedge_{\underset{\sim}{H}}$ can be replaced by $\wedge_{\underset{\sim}{C}}$, but if A is empty $\wedge_{\underset{\sim}{H}} \mathfrak{C}(E_\alpha) = \mathscr{H}$ while $\wedge_{\underset{\sim}{C}} \mathfrak{C}(E_\alpha)$ is the maximal proper congruence on S.

(8.3) <u>Lemma</u>. Let $\{E_\alpha\}_{\alpha \in A}$ be a collection of normal subgroups of H. Then $\wedge_{\underset{\sim}{R}} \underset{\sim}{d}(E_\alpha) = \underset{\sim}{d}(\wedge_{\underset{\sim}{N}} E_\alpha)$, $\wedge_{\underset{\sim}{L}} \underset{\sim}{s}(E_\alpha) = \underset{\sim}{s}(\wedge_{\underset{\sim}{N}} E_\alpha)$, $\vee_{\underset{\sim}{R}} \underset{\sim}{d}(E_\alpha) \subseteq \underset{\sim}{d}(\vee_{\underset{\sim}{N}} E_\alpha)$ and

$\bigvee_{\underset{\sim}{L}} \underset{\sim}{s}(E_\alpha) \subseteq \underset{\sim}{s}(\bigvee_{\underset{\sim}{N}} E_\alpha)$, where $\underset{\sim}{d}$ and $\underset{\sim}{s}$ are defined as in (3.2) and its dual.

Proof. Let $\underset{\sim}{d}^* = \bigwedge_{\underset{\sim}{R}} \underset{\sim}{d}(E_\alpha)$ and $\underset{\sim}{d}_* = \underset{\sim}{d}(\bigwedge_{\underset{\sim}{N}} E_\alpha)$. Then $R_1 \underset{\sim}{d}^* R_2$ if and only if R_1 and R_2 satisfy (3.1.7a) and for $e_i \in \mathcal{E}(R_i)$, with $e_1 \mathcal{L} e_2$ we have $te_1 \mathfrak{C}(E_\alpha) te_2$ for all $t \in S$ and each $\alpha \in A$. Hence $R_1 \underset{\sim}{d}^* R_2$ is equivalent to $te_1 (\bigwedge_{\underset{\sim}{H}} \mathfrak{C}(E_\alpha)) te_2$ for all t in S. But $R_1 \underset{\sim}{d}_* R_2$ if and only if R_1 and R_2 satisfy (3.1.7a) and for e_1 and e_2 as above we have $te_1 \mathfrak{C}(\bigwedge_{\underset{\sim}{N}} E_\alpha) te_2$ for all t. But $\mathfrak{C}(\bigwedge_{\underset{\sim}{N}} E_\alpha) = \bigwedge_{\underset{\sim}{H}} \mathfrak{C}(E_\alpha)$ by (8.2) and the first assertion follows. The second is dual.

The final two inequalities are immediate, since for normal subgroups E and F of H with $E \subseteq F$ it follows that $\underset{\sim}{d}(E) \subseteq \underset{\sim}{d}(F)$ and $\underset{\sim}{s}(E) \subseteq \underset{\sim}{s}(F)$.

It will be shown by an example in the appendix, §12, that the last two inequalities of the lemma are sometimes strict.

(8.4) <u>Theorem</u>. Let $\underset{\sim}{T}, \underset{\sim}{R}, \underset{\sim}{N}$, and $\underset{\sim}{L}$ be defined as in (8.1). Then $\underset{\sim}{T}$ is a complete sublattice of $\underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L} = \underset{\sim}{\Lambda}$.

Proof. We will show that if $\{(\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha)\}_{\alpha \in A}$ is a collection of permissible type triples, then $(\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$ and $(\bigvee_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigvee_{\underset{\sim}{N}} E_\alpha, \bigvee_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$, are in $\underset{\sim}{T}$ and $\bigwedge_{\underset{\sim}{T}} (\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha) = (\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha) = \bigwedge_{\underset{\sim}{\Lambda}} (\underset{\sim}{r}_\alpha, E_\alpha \underset{\sim}{\ell}_\alpha)$ and $\bigvee_{\underset{\sim}{T}} (\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha) = (\bigvee_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigvee_{\underset{\sim}{N}} E_\alpha, \bigvee_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha) = \bigvee_{\underset{\sim}{\Lambda}} (\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha)$.

Since each triple is permissible, $\underset{\sim}{r}_\alpha \subseteq \underset{\sim}{d}(E_\alpha)$ and $\underset{\sim}{\ell}_\alpha \subseteq \underset{\sim}{s}(E_\alpha)$ for each $\alpha \in A$. Thus $\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha \subseteq \bigwedge_{\underset{\sim}{R}} \underset{\sim}{d}(E_\alpha) \subseteq \underset{\sim}{d}(\bigwedge_{\underset{\sim}{N}} E_\alpha)$ by (8.3) and dually $\bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha \subseteq \underset{\sim}{s}(\bigwedge_{\underset{\sim}{N}} E_\alpha)$. It follows that $(\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$ is in $\underset{\sim}{T}$. Since $\underset{\sim}{T}$ is a complete lattice by (7.8), we may put $\bigwedge_{\underset{\sim}{T}} (\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha) = (\underset{\sim}{r}, E, \underset{\sim}{\ell})$. Thus $\underset{\sim}{r} \subseteq \underset{\sim}{r}_\alpha$ for all $\alpha \in A$, whence $\underset{\sim}{r} \subseteq \bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha$. Similarly $E \subseteq \bigwedge_{\underset{\sim}{N}} E_\alpha$ and $\underset{\sim}{\ell} \subseteq \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha$, it follows that $(\underset{\sim}{r}, E, \underset{\sim}{\ell}) \subseteq (\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$. But

$(\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$ is a lower bound for $\{(\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha)\}_{\alpha \in A}$ and $\bigwedge_{\underset{\sim}{T}} (\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha) = (\bigwedge_{\underset{\sim}{R}} \underset{\sim}{r}_\alpha, \bigwedge_{\underset{\sim}{N}} E_\alpha, \bigwedge_{\underset{\sim}{L}} \underset{\sim}{\ell}_\alpha)$ follows. The proof of the second equality is similar.

(8.5) Remark. Observe that $\underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L} = \underset{\sim}{\Lambda}$ is an initial segment of $\underset{\sim}{R}_* \times \underset{\sim}{N} \times \underset{\sim}{L}_* = \underset{\sim}{\Lambda}_*$ where $\underset{\sim}{R}_*[\underset{\sim}{L}_*]$ is the lattice of all equivalences on the set of $\mathscr{R}$- $[\mathscr{L}]$-classes of S. Thus, except for empty intersections, lattice operations in $\underset{\sim}{\Lambda}$ and $\underset{\sim}{\Lambda}_*$ coincide and we see that $\underset{\sim}{T}$ is essentially a complete sublattice of the Cartesian product $\underset{\sim}{R}_* \times \underset{\sim}{N} \times \underset{\sim}{L}_*$.

§9. THE LATTICE OF BRANDT CONGRUENCES ON S

We will now determine necessary and sufficient conditions for the existence of a Brandt congruence on S and show that the set of Brandt congruences $\underset{\sim}{B}$ (if nonempty) on S forms a complete lattice contained in the lattice of all congruences.

(9.1) <u>Definition</u>. 1. A semigroup, T, with zero, 0, is a <u>Brandt semigroup</u> if (a) for each $a \neq 0$, $a \in T$, there are unique elements e and f such that $ea = a$, $af = a$ and a unique element a' such that $a'a = f$, and if (b) for any nonzero idempotents e, f of T we have $eTf \neq 0$.

2. A congruence, $\mathscr{B}$, on a semigroup, G,

is a Brandt congruence if and only if $G/\mathscr{V}$ is a Brandt semigroup.

One readily checks that the elements e and f above are idempotents. This follows from their uniqueness ($e^2a = ea = a$, etc.). Indeed $aa' = e$ since $(aa')a = a(a'a) = af = a$. It is easily seen that $a\mathscr{R}e$ and $a\mathscr{L}f$. Now let $b \neq 0$ be any other element in T; let f be the right identity for b and let e be the left identity for a. Then by (9.1.1b) we can find a $c \neq 0$ in eTf. Since e and f are idempotent $ec = c = cf$ and we have, as above, $c\mathscr{R}e$ and $c\mathscr{L}f$. Thus $a\mathscr{R}e\mathscr{R}c\mathscr{L}f\mathscr{L}b$ and it follows that $a\mathscr{D}b$. Now if e and f are nonzero idempotents of T, then just from $e = ee = fe$ we can conclude $e = f$ (uniqueness) so that each nonzero idempotent of a Brandt semigroup is primitive (cf. p. 9). Thus it can be seen that a Brandt semigroup is completely 0-simple.

The reader should now be able to complete the proof of the following lemma which provides a characterization for Brandt semigroups (cf. [2] Theorems 3.9 and 1.17). In what follows the reader may wish to think of Brandt semigroups in terms of this characterization.

(9.2) Lemma. A semigroup T is a Brandt semigroup if and only if it is completely 0-simple and the idempotents of T commute; or equivalently, if and only if it is completely 0-simple and each $\mathcal{R}$- and $\mathcal{L}$-class of T contains exactly one idempotent.

(9.3) Definition of Condition Br. The semigroup S will be said to satisfy condition Br if and only if $\{R_i \cap L_j\}$, $i, j = 1, 2$, never contains exactly three distinct groups for any two $\mathcal{R}$-classes R_1, R_2 and any two $\mathcal{L}$-classes L_1, L_2.

(9.4) Lemma. Let S(c-0-s) be given, and suppose the $\mathcal{R}$-classes R_1 and R_2 of S contain two $\mathcal{L}$-equivalent idempotents. If S satisfies Br then $R_1 \underset{\sim}{d}(H)R_2$.

Proof. Let L_1 be the $\mathcal{L}$-class of the $e_i \in \mathcal{E}(R_i)$, $i = 1, 2$, and let L_2 be any $\mathcal{L}$-class. Since $R_i \cap L_1$, $i = 1, 2$ contains an idempotent, both $R_1 \cap L_1$ and $R_2 \cap L_1$ are groups. Hence, by Br, if $L_2 \neq L_1$, then

$R_1 \cap L_2$ is a group if and only if $R_2 \cap L_2$ is a group, and this is trivial if $L_2 = L_1$. It now follows by (4.2) that $R_1 \underset{\sim}{d}(H)R_2$.

(9.5) Lemma. If $\mathscr{b}$ is a Brandt congruence on S and R_1 and R_2 contain $\mathscr{L}$-equivalent idempotents then $R_1 \underset{\sim}{r}(\mathscr{b}))R_2$.

Proof. Let R_i, $i = 1,2$ and $\mathscr{b}$ be as above and suppose that the idempotents e_i belong to $R_i \cap L$, $i = 1,2$, for some $\mathscr{L}$-class L. Then $e_1^{\mathscr{b}} \mathscr{L} e_2^{\mathscr{b}}$ in $S/\mathscr{b}$ since a congruence respects Green's relations and whence by (9.2) $e_1 \mathscr{b} e_2$. It follows that $R_1 \underset{\sim}{r}(\mathscr{b})R_2$ by (6.2).

(9.6) Theorem. Let $(\underset{\sim}{r}, E, \underset{\sim}{\ell})$ be a permissible triple on S and let $\mathscr{b} = [\underset{\sim}{r}, E, \underset{\sim}{\ell}]$. Then $\mathscr{b}$ is a Brandt congruence if and only if

(1) S satisfies Br and

(2) $\underset{\sim}{r} = \underset{\sim}{d}(H)$ and $\underset{\sim}{\ell} = \underset{\sim}{s}(H)$.

Proof. Suppose $\mathscr{b}$ is a Brandt congruence on S. Let R_i, $i = 1,2$, be $\mathscr{R}$-classes and suppose that $R_i \cap L$, $i = 1,2$, is a group, for some nonzero $\mathcal{L}$-class L. Observe that $R_i \cap L$, $i = 1,2$ each contains an idempotent e_i, and $e_1 \mathcal{L} e_2$. Whence $R_1 \underset{\sim}{r}(\mathcal{b})R_2$ by (9.5). But by (7.4) and (7.8), $\underset{\sim}{r} = \underset{\sim}{r}(\mathcal{b})$, hence $R_1 \underset{\sim}{r} R_2$.

Now, (i) let R_i, L_i, $i = 1,2$ be $\mathcal{R}$ and $\mathcal{L}$-classes and suppose $R_1 \cap L_1$, $R_2 \cap L_1$ and $R_1 \cap L_2$ are groups. Then $R_1 \underset{\sim}{r} R_2$, whence $R_1 \underset{\sim}{d}(H)R_2$, since $\underset{\sim}{r} \subseteq \underset{\sim}{d}(E) \subseteq \underset{\sim}{d}(H)$. Hence by (4.2) $R_2 \cap L_2$ is a group. Dually, (ii) if $R_1 \cap L_1$, $R_1 \cap L_2$ and $R_2 \cap L_1$ are groups, so is $R_2 \cap L_2$. Hence S satisfies Br, and (1) is proved. Next let $R_1 \underset{\sim}{d}(H)R_2$. For some $\mathcal{L}$-class L, $R_1 \cap L$ is a group. By (4.2), $R_2 \cap L$ is a group. Hence by the first part of the proof, $R_1 \underset{\sim}{r} R_2$. Thus $\underset{\sim}{d}(H) \subseteq \underset{\sim}{r}$, whence $\underset{\sim}{d}(H) = \underset{\sim}{r}$. Thus (2) is proved.

Conversely, suppose (1) and (2) hold. Let e_1' and e_2' be $\mathcal{L}$-equivalent nonzero idempotents in $S/\mathcal{b}$, say $e_i' \in L'$, $i = 1,2$, where L' is an $\mathcal{L}$-class of $S/\mathcal{b}$. It is easy to prove that there is an $\mathcal{L}$-class L_1 in S whose image under $\mathfrak{h}$ is L'. Then there exist $x_i \in L_i$, $i = 1, 2$ with $x_i^{\mathfrak{h}} = e_i'$. Further the x_i belong to group $\mathcal{H}$-classes H_i of S, since they cannot be nilpotent. But $\mathfrak{h}$ restricted to H_i is a group congruence, hence it follows that $e_i^{\mathfrak{h}} = e_i'$, where e_i is the identity of H_i. Let $H_i = R_i \cap L_1$, $i = 1, 2$ and let L_2 be any $\mathcal{L}$-class. Since S satisfies Br, and $R_i \cap L_1$, $i = 1, 2$ are both groups, $R_1 \cap L_2$ is a group precisely when $R_2 \cap L_2$ is a group. Hence by (4.2), $R_1 \underset{\sim}{d}(H) R_2$, and so by assumption $R_1 \underset{\sim}{r} R_2$, and this relation also holds in $S/\mathfrak{C}$, where $\mathfrak{C} = \mathfrak{C}(E)$. Hence, applying (3.6) to $S/\mathfrak{C}$, we obtain $e_1^{\mathfrak{C}} \ \underset{\sim}{r}' \ e_2^{\mathfrak{C}}$, where $\mathfrak{r}' = \mathfrak{r}(\underset{\sim}{r})$ in $S/\mathfrak{C}$. Since $\mathfrak{h} = (\mathfrak{r}' \circ \mathfrak{l}')^{\mathfrak{C}}$, we obtain $e_1 \mathfrak{h} e_2$. It follows that $e_1' = e_2'$, and so each $\mathcal{L}$-class of $S/\mathfrak{h}$ contains exactly one idempotent. Dually, the same is true for $\mathcal{R}$-classes of $S/\mathfrak{h}$, and it follows by (9.2) that $\mathfrak{h}$ is a Brandt congruence.

The following corollaries are now immediate:

(9.7) *Corollary*. If S satisfies Br, then the maximal proper congruence $[\underset{\sim}{d}(H), H, \underset{\sim}{s}(H)]$ is a Brandt congruence.

(9.8) *Corollary*. There exists a Brandt congruence on S if and only if S satisfies Br.

(9.9) *Corollary*. Let $\underset{\sim}{B}$ be the collection of Brandt congruences on S, and suppose $\underset{\sim}{B}$ is nonempty. Then there is a minimal normal subgroup F of H such that $\underset{\sim}{d}(F) = \underset{\sim}{d}(H)$ and $\underset{\sim}{s}(F) = \underset{\sim}{s}(H)$. The collection $\underset{\sim}{B}$ consists of all congruences $[\underset{\sim}{d}(H), E, \underset{\sim}{s}(H)]$, where $E \supseteq F$. Further, $\underset{\sim}{B}$ is a complete lattice which is a final segment of $\underset{\sim}{C}$, and is isomorphic to a final segment of $\underset{\sim}{N}(H)$. If $\mathfrak{n} = [\underset{\sim}{d}(H), F, \underset{\sim}{s}(H)]$ and $\mathfrak{m} = [\underset{\sim}{d}(H), H, \underset{\sim}{s}(H)]$ and if $\mathfrak{b} \in \underset{\sim}{B}$ then $S/\mathfrak{b}$ is a homomorphic image of $S/\mathfrak{n}$ and can be mapped homomorphically onto $S/\mathfrak{m}$.

(9.10) *Corollary*. Let S be a completely simple semi-group with 0 adjoined. Then $\underset{\sim}{B}$ is nonempty and for every $\mathfrak{b} \in \underset{\sim}{B}$, $S/\mathfrak{b}$ is a group with 0.

(9.11) <u>Corollary</u> ([18] Theorem 6). If S is a completely simple semigroup with adjoined 0 and if F is the minimal normal subgroup of H such that $\underset{\sim}{d}(F) = \underset{\sim}{d}(H)$ and $\underset{\sim}{s}(F) = \underset{\sim}{s}(H)$ then for $\mathfrak{b} = [\underset{\sim}{d}(F), F, \underset{\sim}{s}(F)]$ $S/\mathfrak{b}$ is the maximal group image of S with adjoined zero.

§10. FINITE CHAINS OF CONGRUENCES ON S

For the sake of completeness, we give two lattice theoretic definitions, cf. [1], [19].

(10.1) <u>Definition</u>. In a partially ordered set, $\underset{\sim}{K}$, a is said to <u>cover</u> b (written $a \succ_{\underset{\sim}{K}} b$ or $b \prec_{\underset{\sim}{K}} a$) if $a \geq b$ but there is no c such that $a > c > b$.

(10.2) <u>Definition</u>. A lattice, $\underset{\sim}{K}$ is (upper) <u>semimodular</u> if whenever $a \succ_{\underset{\sim}{K}} c$ and $b \succ_{\underset{\sim}{K}} c$ where $a \neq b$, then $a \vee b \succ_{\underset{\sim}{K}} a$ and $a \vee b \succ_{\underset{\sim}{K}} b$.

We shall show that the lattice $\underset{\sim}{C}$ of proper congruences on S is semimodular. This will be done by considering $\underset{\sim}{T}$, the lattice of permissible triples of S (cf. (7.8)).

(10.3) <u>Lemma</u>. Let $\underset{\sim}{T}$ be the lattice of permissible triples of S, and let $\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$, where $\underset{\sim}{R}, \underset{\sim}{N}, \underset{\sim}{L}$ are defined in (8.1). If $\tau_i = (\underset{\sim}{r}_i, E_i, \underset{\sim}{\ell}_i) \in \underset{\sim}{T}$, (i = 1, 2) then $\tau_1 \succ_{\underset{\sim}{T}} \tau_2$ if and only if $\tau_1 \succ_{\underset{\sim}{\Lambda}} \tau_2$.

Proof. Since $\underset{\sim}{T} \subseteq \underset{\sim}{\Lambda}$, $\tau_1 \succ_{\underset{\sim}{\Lambda}} \tau_2$ implies $t_1 \succ_{\underset{\sim}{T}} \tau_2$.

Suppose conversely that τ_1 does not cover τ_2 in $\underset{\sim}{\Lambda}$, say $(\underset{\sim}{r}_1, E_1, \underset{\sim}{\ell}_1) > (\underset{\sim}{r}, E, \underset{\sim}{\ell}) > (\underset{\sim}{r}_2, E_2, \underset{\sim}{\ell}_2)$. There are three possible cases: (1) $E_1 \supset E \supset E_2$ (we use $\supset$ for proper containment), or (2) $E_1 = E$, or (3) $E_1 \supset E = E_2$. In case (1), $\sigma_1 = (\underset{\sim}{r}_2, E, \underset{\sim}{\ell}_2) \in \underset{\sim}{T}$ and in case (2), $\sigma_2 = (\underset{\sim}{r}, E, \underset{\sim}{\ell}) \in \underset{\sim}{T}$. In case (3), $\sigma_3 = (\underset{\sim}{r}_2, E_1, \underset{\sim}{\ell}_2)$ $\in \underset{\sim}{T}$ and either $\underset{\sim}{r} > \underset{\sim}{r}_2$ or $\underset{\sim}{\ell} > \underset{\sim}{\ell}_2$. Thus for one of i = 1, 2, 3, $\sigma_i \in \underset{\sim}{T}$ and $\tau_1 > \sigma_i > \tau_2$ so that τ_1 does not cover τ_2 in $\underset{\sim}{T}$. This contradiction completes the proof of the lemma.

(10.4) <u>Lemma</u>. The lattice $\underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L} = \underset{\sim}{\Lambda}$ is semimodular.

Proof. The lattice $\underset{\sim}{N}$ of normal subgroups of H, is modular (cf. (10.7) and (10.8)), and it is easily shown that a modular lattice is also semimodular. Now $\underset{\sim}{R}[\underset{\sim}{L}]$ consists of all equivalences on the set of $\mathcal{R}[\mathcal{L}]$-classes which lie under $\underset{\sim}{d}(H)$ $[\underset{\sim}{s}(H)]$. It is not hard to see that the lattice of all equivalences on a set is upper semimodular (cf. (10.9)) so are $\underset{\sim}{R}$ and $\underset{\sim}{L}$. But one easily verifies that semimodularity is preserved under direct products, and this completes the proof.

(10.5) Theorem. The lattice $\underset{\sim}{C}$ of all proper congruences on S is semimodular.

Proof. By the isomorphism theorem (7.8) it is enough to prove that the lattice $\underset{\sim}{T}$ of all permissible triples is semimodular. By (8.4), $\underset{\sim}{T}$ is a sublattice of $\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$ and for $\tau_i \in \underset{\sim}{T}$, $i = 1, 2$, $\tau_1 \succ_{\underset{\sim}{T}} \tau_2$ if and only if $\tau_1 \succ_{\underset{\sim}{\Lambda}} \tau_2$ by (10.3). But by (10.4) $\underset{\sim}{\Lambda}$ is semimodular and the theorem then follows immediately.

See also Lallement [12].

(10.6) Corollary. The Jordan-Dedekind Chain Condition holds on $\underset{\sim}{C}$; viz, all finite maximal chains between two elements of $\underset{\sim}{C}$ have the same length.

Proof. This follows directly from the semimodularity of $\underset{\sim}{C}$ (cf. [7], Theorem 8.3.4).

(10.7) Definition. A lattice $\underset{\sim}{L}$ is said to be modular if whenever $a \geq c$ in $\underset{\sim}{L}$ then $a \wedge (b \vee c) = (a \wedge b) \vee c$ for any $b \in \underset{\sim}{L}$.

(10.8) Lemma. The set, $\underset{\sim}{N}$, of normal subgroups of a (fixed) group G is modular lattice under the inclusion relation.

Proof. It is easily shown that $\underset{\sim}{N}$ is a lattice. Now suppose $A \supseteq C$ for $A, C \in \underset{\sim}{N}$ and let $B \in \underset{\sim}{N}$. We must show $A \wedge (B \vee C) = (A \wedge B) \vee C$. Since $A \supseteq A \wedge B$, $A \supseteq C$ and $B \vee C \supseteq A \wedge B$, $B \vee C \supseteq C$ we have $A \wedge (B \vee C) \supseteq (A \wedge B) \vee C$ so that we need now only check the reverse containment.

In order to show $A \wedge (B \vee C) \subseteq (A \wedge B) \vee C$ we make use of the lattice operations in $\underset{\sim}{N}$ checking that $N_1 \wedge N_2 = N_1 \cap N_2$ and $N_1 \vee N_2 = N_1 N_2 = \{n_1 n_2 \mid n_1 \in N, n_2 \in N_2\}$ for $N_1, N_2 \in \underset{\sim}{N}$. Thus we must show $A \cap (BC) \subseteq (A \cap B)C$. Let $x \in A \cap BC$. Then $x = a \in A$ and $x = bc$ for $b \in B$, $c \in C$. From $x = a = bc$, we have $b = ac^{-1} \in A$ since $A \supseteq C$. Thus $b \in A \cap B$ and $x = bc \in (A \cap B)C$ and the result follows.

(10.9) Lemma. The set, $\underset{\sim}{E}$, of all equivalences on a set is a semimodular lattice.

Proof. Let $\underset{\sim}{E}$ be the set of equivalence relations on a fixed set X. One readily verifies that $\underset{\sim}{E}$ is a lattice under the inclusion ordering. Let $a, b, c \in \underset{\sim}{E}$ where $a \succ c$ and $b \succ c$ and $a \neq b$. It is obvious that $a \succ c$ (and $b \succ c$) precisely when a (and b) identifies exactly two equivalence classes, say X_1, X_2, (say X_3, X_4), induced by c on X, so that the equivalence classes of a are those of c excluding X_1 and

X_2 but including $(X_1 \cup X_2)$. Since $a \neq b$ either three or four equivalence classes of c are identified by a and b. Suppose then there are just three equivalence classes identified and $X_2 = X_3$. Then the equivalence classes of $a \vee b$ are those of c excluding X_1, X_2 and X_3 but including $(X_1 \cup X_2 \cup X_3)$ and it is then clear that $a \vee b \succ a$ and $a \vee b \succ b$. The proof of the other case is similar.

§11. WELL-ORDERED CHAINS OF CONGRUENCES ON S

In this section we will examine well-ordered chains of congruences on S. We will show that under a certain condition all maximal well-ordered chains in $\underset{\sim}{C}$ are of the same length. Some general lattice theoretical definitions must be given and then we must first consider the lattices $\underset{\sim}{N}$, of normal subgroups of H, and $\underset{\sim}{Eq}(X)$—the lattice of equivalences on a set X.

Recall that a partial ordered set, P, is <u>well-ordered</u> if every nonempty subset, Q, has a first element q_1, i.e., there is a $q_1 \in Q$ such that $q_1 \subseteq q$ for all $q \in Q$. A <u>chain</u> is a partially ordered set P in which any two elements are comparable, i.e., for $p, q \in P$ either $p \leq q$ or $q \leq p$.

(11.1) <u>Definition</u>. Let $\underset{\sim}{K}$ be an arbitrary lattice with minimum element Δ and maximum element ν.

(1) Let A be a well-ordered index set. A collection $\{k_\alpha\}_{\alpha \in A}$, of elements of $\underset{\sim}{K}$ will be called an (strictly) <u>ascending well-ordered chain</u>, indexed by A if and only if $k_\alpha \subset k_\beta$ in $\underset{\sim}{K}$ whenever $\alpha < \beta$ in A. For short, $\{k_\alpha\}_{\alpha \in A}$ will be called a <u>chain</u> (indexed by A).

(2) If in $\underset{\sim}{K}$ $\{k_\alpha\}_{\alpha \in A}$ is an ascending well-ordered chain indexed by A, then the cardinal $|A| - 1$, is said to be its <u>length</u>.

(3) An element $k \in \underset{\sim}{K}$ will be called <u>accessible</u> if and only if there exists an ascending well-ordered chain $\{k_\alpha\}_{\alpha \in A}$ from Δ to k which is maximal in $\underset{\sim}{K}$. Here $k_\lambda = k$ precisely when λ is the maximal element in A. The lattice $\underset{\sim}{K}$ will be called <u>accessible</u> if and only if every element in $\underset{\sim}{K}$ is accessible.

(4) If $k \in \underset{\sim}{K}$ is accessible and if all ascending well-ordered maximal chains from Δ to k are of the same length, then we will say that the <u>height</u>, $\|k\|$, of k is the common length of these chains. If ν has height then we say that the height $\|\underset{\sim}{K}\|$ of $\underset{\sim}{K}$ is $\|\nu\|$.

(5) The lattice $\underset{\sim}{K}$ will be called an <u>accessible lattice with height</u> if and only if every $k \in \underset{\sim}{K}$ is accessible and has height.

(6) If $k \leq k'$ in $\underset{\sim}{K}$, then k' is said to be accessible from k if there is a maximal ascending well-ordered chain from k to k'. If every such maximal chain is of the same length that common length will be called the height of k' over k and will be written $\|k'/k\|$.

(7) If H is a group then a maximal ascending well-ordered chain to H in the lattice of all normal subgroups, $(\underset{\sim}{N}(H)$—cf. (8.1)), will be called a <u>principal series</u> for H.

(11.2) Remark. (1) When convenient, our index set A will be totally ordered, but possibly not well-ordered.

(2) If A is a well-ordered set we will write $W = W(A)$ to be the collection of all elements in A which have a predecessor in A, i.e., $W(A) = \{b \in A \mid \text{there is an } a \in A \text{ with } a \prec b\}$. It is easy to verify that $|W| = |A| - 1$. In particular, when A is infinite $|W| = |A|$.

(3) It will be seen that our definition of principal series coincides with that of Kurosh ([10], p. 173). Indeed, H has a principal series if and only if it is accessible in $\underset{\sim}{N} = \underset{\sim}{N}(H)$. Moreover, we shall show that the accessibility of H in $\underset{\sim}{N}$ is sufficient to guarantee the accessibility of $\underset{\sim}{N}$.

(11.3) Lemma. Let $\{H_\alpha\}_{\alpha \in A}$ be a chain in $\underset{\sim}{N}$ indexed by A and $E \in \underset{\sim}{N}$. Then $(\bigvee_{\alpha < \beta} H_\alpha) \wedge E = \bigvee_{\alpha < \beta} (H_\alpha \wedge E)$ for $\beta \in A$.

Proof. Clearly $(\bigvee_{\alpha < \beta} H_\alpha) \wedge E \supseteq \bigvee_{\alpha < \beta} (H_\alpha \wedge E)$.

Conversely, we note that $\bigvee_{\alpha<\beta} H_\alpha = \bigcup_{\alpha<\beta} H_\alpha$.

Thus if $h \in (\bigvee_{\alpha<\beta} H_\alpha) \wedge E$ then $h \in H_\gamma \wedge E$ for some $\gamma < \beta$. Hence $h \in \bigvee_{\alpha<\beta} (H_\alpha \wedge E)$. Thus $(\bigvee_{\alpha<\beta} H_\alpha) \wedge E \subseteq \bigvee_{\alpha<\beta} (H_\alpha \wedge E)$ and the equality follows immediately.

(11.4) <u>Theorem</u>. Let $\{H_\alpha\}_{\alpha \in A}$ be a principal series for H. Then for each $E \in \underset{\sim}{N}$ there is a suitable subset $B \subseteq A$ such that $\{H'_\beta\}_{\beta \in B}$ is a maximal well-ordered ascending chain to E in $\underset{\sim}{N}(H)$, where $H'_\alpha = H_\alpha \wedge E$.

Proof. Let $\beta \in B$ if and only if β is the smallest α of A such that $H'_\alpha = H'_\gamma$ with $\gamma \in A$. Clearly, B is well-ordered. If $\{H'_\beta\}_{\beta \in B}$ is not maximal we can find an $F \in \underset{\sim}{N}$ such that for some $\gamma \in B$, $F \subset H'_\gamma$ and $H'_\alpha \subset F$ for all $\alpha < \gamma$.

Case 1) Suppose γ has a predecessor δ in A. By the construction of B, it follows that $H'_\delta = H_\delta \wedge E \subset F$. (We do not claim that $\delta \in B$

but of course there is a $\delta' \in B$ with $H_\delta' = H'_{\delta'}$.)
We will now produce a contradiction. We
clearly have $H_\delta \wedge H'_\gamma = H_\delta \wedge H_\gamma \wedge E = H_\delta \wedge E =$
H'_δ and hence we can also readily deduce
$H_\delta \wedge F = H'_\delta$. From $F \subset H'_\gamma \subset H_\gamma$ we have
$H_\delta \subseteq F \vee H_\delta \subseteq H_\gamma$ and since $\{H_\alpha\}_{\alpha \in A}$ is maximal $F \vee H_\delta = H_\gamma$ or $F \vee H_\delta = H_\delta$. In the former
case, it would follow that $H'_\gamma \vee H_\delta = H_\gamma$. Thus
we have a five point sublattice as in Fig. 1.
By [3], p. 66, 67, $\underset{\sim}{N}$ is nonmodular and this
is a contradiction. Hence $F \vee H_\delta = H_\delta$, whence
$F \subseteq H_\delta \wedge E \subseteq H'_\gamma$, again a contradiction.

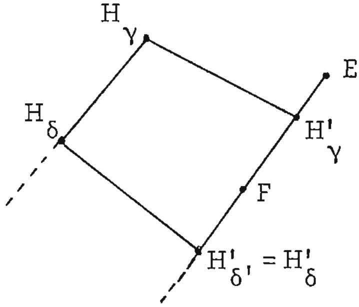


Figure 1

Case 2) Suppose γ has no predecessor in A. Then since $\{H_\alpha\}_{\alpha\in A}$ is maximal we must have $\bigvee_{\alpha<\gamma} H_\alpha = H_\gamma$. But then by (11.3), $H'_\gamma = \bigvee_{\alpha<\gamma} (H_\alpha \wedge E) = \bigvee_{\alpha<\gamma} H'_\alpha$. But for each $\alpha < \gamma$,

$H'_\alpha \subset F \subset H'_\gamma$, whence $H'_\gamma = \bigvee_{\alpha<\gamma} H'_\alpha \subseteq F \subset H'_\gamma$, a contradiction. This completes the proof that $\{H'_\beta\}_{\beta\in B}$ is maximal.

It is easy to see that $\{H'_\beta\}_{\beta\in B}$ is a strictly ascending chain. Whence we can conclude that $\{H'_\beta\}_{\beta\in B}$ is a maximal, well-ordered ascending chain to E and so E is accessible in $\underset{\sim}{N}$.

(11.5) Corollary. A group H is accessible in $\underset{\sim}{N}(H)$ if and only if $\underset{\sim}{N}(H)$ is an accessible lattice.

(11.6) Theorem. Let E be a normal subgroup of H which is accessible in $\underset{\sim}{N}(H)$. Then E has height.

Proof. Consider E as a group with operators, where the set of operators consists of all inner automorphisms of H restricted to E. It is proved in Kurosh [10], p. 175, that any two (well-ordered) principal series of an arbitrary group with operators are isomorphic and hence have the same length. But such principal series for E are precisely the maximal well-ordered ascending chains in $\underset{\sim}{N}(H)$ to E, and the result follows.

Since the triples in $\underset{\sim}{T}$ involve equivalences on sets we will first develop the necessary theory of $\underset{\sim}{Eq}(X)$, the lattice of equivalences on a set X before proceeding to $\underset{\sim}{T}$.

(11.7) Definition. Let $\mathcal{Q}$ be an equivalence on a set X. A *transversal* Q *for* $\mathcal{Q}$ will be a subset of X consisting of precisely one element from each equivalence class determined by $\mathcal{Q}$.

(11.8) Proposition. Let A be a well-ordered set and $\{\mathcal{Q}_\alpha\}_{\alpha \in A}$ be a (strictly) ascending chain of equivalences on X from $\mathcal{Q}_0 = \Delta$ to $\mathcal{Q}_\lambda = \mathcal{Q}$. If Q is a transversal for $\mathcal{Q}$ then $|A| - 1 \leq |X \setminus Q| \leq |X| - 1$.

Proof. We well-order X and put

$Q_\alpha = \{x \mid x$ is the first element of a $\mathcal{Q}_\alpha$-class of $X\}$.

Without loss of generality we suppose $Q = Q_\lambda$. Let $T_\alpha = X \setminus Q_\alpha$. We observe that $T_0 = \emptyset$ and that $\{T_\alpha\}_{\alpha \in A}$ is a strictly ascending chain.

Let $W = W(A)$ (cf. (11.2)) and if $\beta \in W$ let β^* be the predecessor of β. Since $\mathcal{Q}_{\beta^*} < \mathcal{Q}_\beta$, $T_\beta \setminus T_{\beta^*} \neq \emptyset$. Let $x(\beta)$ be the first element of $T_\beta \setminus T_{\beta^*}$. Since λ is the greatest element in A, we have $X \setminus Q_\lambda = T_\lambda \supseteq T_\beta$ for all β in A, whence $\{x(\beta)\}_{\beta \in W} \subseteq T_\lambda$.

If $\beta, \gamma \in W$ with $\gamma < \beta$ then $x(\gamma) \in T_\gamma \subseteq T_{\beta^*}$ but $x(\beta) \notin T_{\beta^*}$. Thus $x(\gamma) \neq x(\beta)$ and so the map $\beta \to x(\beta)$ of W into T_λ is 1-1. Since $|A| - 1 = |W|$, the first inequality follows.

The second inequality is trivial.

(11.9) Theorem. Let X be a set and let $\underset{\sim}{\mathrm{Eq}}(X)$ be the lattice of equivalences on X. Then $\underset{\sim}{\mathrm{Eq}}(X)$ is an accessible lattice with height and, for

$\mathcal{Q} \in \underset{\sim}{\mathrm{Eq}}(X)$, $\|\mathcal{Q}\| = |X \setminus Q|$ where Q is a transversal for $\mathcal{Q}$.

Proof. We again well-order X and let Q consist of all first members of equivalence classes. Let $T = X \setminus Q$. Let $\mathcal{J}$ be the set of initial segments of T, ordered by inclusion. Since X is well-ordered, so is $\mathcal{J}$. For $I \in \mathcal{J}$ define an equivalence $\mathcal{Q}_I$ on X by $x \mathcal{Q}_I y$ if and only if either (1) $x, y \in I \cup Q$ and $x \mathcal{Q} y$ or (2) $x = y$. Thus $\{\mathcal{Q}_I\}_{I \in \mathcal{J}}$ is an ascending chain in $\underset{\sim}{\mathrm{Eq}}(X)$ from $\Delta = \mathcal{Q}_\emptyset$ to $\mathcal{Q} = \mathcal{Q}_T$. Now $\{\mathcal{Q}_I\}_{I \in \mathcal{J}}$ is clearly a well-ordered chain which is maximal and $|\mathcal{J}| - 1 = |T| = |X \setminus Q|$.

Now let $\{\mathcal{Q}_\alpha\}_{\alpha \in A}$ be a maximal ascending well-ordered chain of equivalences from Δ to $\mathcal{Q} = \mathcal{Q}_\lambda$. We define Q_α, T_α, W and $x(\beta)$ as in (11.8). As in the proof of that proposition, the mapping $\beta \to x(\beta)$ from W to T_λ is 1 - 1. We shall now show that it is in this case also onto.

Let $x \in T_\lambda$. Let β be the first element of of A for which $x \in T_\beta$. Since $T_0 = \emptyset$, $\beta \neq 0$. Suppose, contrary to our hopes, β has no predecessor. Then $\{\mathcal{Q}_\alpha\}_{\alpha<\beta} = \mathcal{Q}_\beta$ by the maximality of the given chain. Hence $\bigcap_{\alpha<\beta} \{Q_\alpha\} = Q_\beta$ and so $T_\beta = \bigcup_{\alpha<\beta} \{T_\alpha\}$. But then $x \in T_\alpha$ for some $\alpha < \beta$, a contradiction. Thus β has a predecessor β and $x \in T_\beta \setminus T_{\beta^*}$. Indeed, since the chain is maximal, $\mathcal{Q}_\beta$ identifies exactly two $\mathcal{Q}_{\beta^*}$ classes, from which it follows that $T_\beta \setminus T_{\beta^*} = \{x\}$ and therefore $x(\beta) = x$. The map $\beta \to x(\beta)$ is therefore a bijection of W onto T_λ. Whence $|W| = |T_\lambda| = |X \setminus Q|$. But $|A| - 1 = |W|$, whence $|A| - 1 = |X \setminus Q|$ and this is true for all maximal well-ordered chains. Hence by definition (11.1.4) $\|\mathcal{Q}\|$ exists and $\|\mathcal{Q}\| = |X \setminus Q|$.

(11.10) <u>Corollary</u>. The lattices $\underset{\sim}{R}$ and $\underset{\sim}{L}$ are accessible with height.

Proof. The lattice $\underset{\sim}{R}$ is an initial segment of the lattice $\underset{\sim}{R}_*$ of all equivalences on the set of $\mathcal{R}$-classes of S, and $\underset{\sim}{R}_*$ is accessible with height by (11.9). A similar argument proves the result for $\underset{\sim}{L}$.

(11.11) Lemma. Let $\underset{\sim}{K}$ and $\underset{\sim}{K}'$ be two lattices. If $k \in \underset{\sim}{K}$, $k \in \underset{\sim}{K}'$ are accessible with height, then (k, k') is accessible in $\underset{\sim}{K} \times \underset{\sim}{K}'$ with height and $\|(k, k')\| = \|k\| + \|k'\|$.

Proof. Let $\{k_\beta\}_{\beta \in B}$ and $\{k'_\gamma\}_{\gamma \in C}$ be maximal well-ordered ascending chains for k and k' in $\underset{\sim}{K}$ and $\underset{\sim}{K}'$ respectively. We suppose that B and C are disjoint except that the last element of B is the first element of C. Let $A = B \cup C$. Define

$$p_\alpha = \begin{cases} (k_\alpha, \Delta) & \text{if } \alpha \in B \\ (k, k'_\alpha) & \text{if } \alpha \in C. \end{cases}$$

It is clear that $\{p_\alpha\}_{\alpha \in A}$ is a maximal ascending well-ordered chain for (k, k') in $\underset{\sim}{K} \times \underset{\sim}{K}'$ and that $|A| - 1 = (|B| - 1) + (|C| - 1) = \|k\| + \|k'\|$.

Let $\{(k_\alpha, k'_\alpha)\}_{\alpha \in A}$ be any maximal well-ordered chain for (k, k'). For $\beta \in A$ put β in B if and only if $k_\alpha = k_\beta$ implies $\alpha \geq \beta$, when $\alpha \in A$. Similarly put γ in C if and only if

$k'_\delta = k'_\gamma$ implies $\delta \geq \gamma$. It is then easily seen $\{k_\beta\}_{\beta \in B}$ is a maximal well-ordered chain for k, and that $\{k'_\gamma\}$ is a maximal well-ordered chain for k'. Let $\delta \in W(A)$, and suppose $(k_\gamma, k'_\gamma) \prec (k_\delta, k'_\delta)$. Then either $k_\gamma = k_\delta$ and $k'_\gamma \prec k'_\delta$ or $k_\gamma \prec k_\delta$ and $k_\gamma = k'_\delta$. Since $\gamma < \delta$, either $\delta \in B$ or $\delta \in C$ but not both. Now, if A is finite, then $W = A \setminus \{0\}$, where 0 is the first element in A, and $B \cap C = \{0\}$. Thus $|A| - 1 = |W| = |(B \cup C) \setminus \{0\}| = |B \cup C| - 1 = (|B| - 1) + (|C| - 1) = \|k\| + \|k'\|$.

Otherwise, if $|A|$ is infinite, then $W \subseteq B \cup C \subseteq A$ and $|W| = |A|$. Hence $|A| = |B| + |C|$. Thus $|A| - 1 = |A| = |B| + |C| = |B| - 1 + |C| - 1 = \|k\| + \|k'\|$. The result follows.

Clearly, if $\underset{\sim}{K}$ and $\underset{\sim}{K}'$ are accessible lattices with height, then (11.11) implies $\underset{\sim}{K} \times \underset{\sim}{K}'$ is also an accessible lattice with height.

(11.12) <u>Lemma</u>. If $\{\tau_\alpha\}_{\alpha\in A}$ is a maximal ascending well-ordered chain to τ in $\underset{\sim}{T}$, then it is also a maximal ascending well-ordered chain to τ in $\underset{\sim}{\Lambda}$.

Proof. This is immediate from the (8.4) and the "covering lemma" (10.3).

(11.13) <u>Theorem</u>. Let $\underset{\sim}{C}$ be the lattice of proper congruences on S. Let $\mathfrak{p} = [\underset{\sim}{r}, E, \underset{\sim}{\ell}] \in \underset{\sim}{C}$. Then $\mathfrak{p}$ is accessible in $\underset{\sim}{C}$ if and only if E is accessible in $\underset{\sim}{N}(H)$ where H is the fixed subgroup of S. Moreover, in this case

$$\|\mathfrak{p}\| = \|\underset{\sim}{r}\| + \|E\| + \|\underset{\sim}{\ell}\|.$$

Proof. By (7.8) there is a complete lattice isomorphism between $\underset{\sim}{C}$ and $\underset{\sim}{T}$. We will thus consider the accessibility of $\tau = (\underset{\sim}{r}, E, \underset{\sim}{\ell})$ in $\underset{\sim}{T}$.

Suppose τ is accessible, and let $\{\tau_\alpha\}_{\alpha\in A} = \{(\underset{\sim}{r}_\alpha, E_\alpha, \underset{\sim}{\ell}_\alpha)\}_{\alpha\in A}$ be a maximal well-ordered chain to τ. Obviously, for a suitable B contained in A, $\{E_\alpha\}_{\alpha\in B}$ is a maximal well-ordered chain for E in $\underset{\sim}{N}(H)$. Hence E is accessible in $\underset{\sim}{N}(H)$.

Suppose now E is accessible in $\underset{\sim}{N}(H)$. To show that τ is accessible we begin as in (11.11). Since by (11.10), $\underset{\sim}{r}$ and $\underset{\sim}{\ell}$ are also accessible, we can find maximal well-ordered ascending chains $\{\underset{\sim}{r}_\gamma\}_{\gamma \in C}$, $\{E_\beta\}_{\beta \in B}$, and $\{\underset{\sim}{\ell}_\delta\}_{\delta \in D}$ to $\underset{\sim}{r}$, E, and $\underset{\sim}{\ell}$, respectively. We assume moreover that the last element of B is the first element of C, that the last element of C is the first element of D and that B, C and D are otherwise disjoint. Let $A = B \cup C \cup D$ and order A in the obvious way. Define

$$\tau_\alpha = \begin{cases} (\Delta, E_\alpha, \Delta) & \text{if } \alpha \in B \\ (\underset{\sim}{r}_\alpha, E, \Delta) & \text{if } \alpha \in C \\ (\underset{\sim}{r}, E, \underset{\sim}{\ell}_\alpha) & \text{if } \alpha \in D. \end{cases}$$

It is easy to see that $\{\tau_\alpha\}_{\alpha \in A}$ is a maximal well-ordered ascending chain in $\underset{\sim}{T}$. Whence τ is accessible in $\underset{\sim}{T}$. By (11.12) $\{\tau_\alpha\}_{\alpha \in A}$ is also a maximal well-ordered chain to τ in $\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$. By (11.6) E has height, and since $\underset{\sim}{r}$ and $\underset{\sim}{\ell}$ also have height it follows by double use of (11.11) that $|A| - 1 = \|\underset{\sim}{r}\| + \|E\| + \|\underset{\sim}{\ell}\|$. This proves the theorem.

(11.14) Corollary. Let $\mathfrak{p}$ be a proper congruence on S such that there exists a finite chain of congruences (of length n) to $\mathfrak{p}$ which is maximal. Then all chains to $\mathfrak{p}$ have at most length n and all maximal chains to $\mathfrak{p}$ have precisely length n.

Proof. By (11.13) all well-ordered chains to $\mathfrak{p}$ have length at most n, and all maximal well-ordered chains have length n. Suppose there is a totally ordered chain to $\mathfrak{p}$ of length greater than n. Then we can select a subchain of length $(n+1)$ which is a contradiction. The result follows.

(11.15) Theorem. Let $\underset{\sim}{C}$ be the lattice of proper congruences on S and let H be the fixed subgroup of S. Then $\underset{\sim}{C}$ is an accessible lattice with height if and only if H is a group with principal series. Moreover, if $\mathfrak{p} = [\underset{\sim}{r}, E, \underset{\sim}{\ell}] \in C$, then

$\|\mathfrak{p}\| = \|\underset{\sim}{r}\| + \|E\| + \|\underset{\sim}{\ell}\|.$

Proof. Let E be a normal subgroup of H. By (11.4), E is accessible in $\underset{\sim}{N}(H)$. The theorem now follows by (11.13).

(11.16) Remark. By slight generalization of our arguments we can obtain results for well-ordered chains from $\mathfrak{p}$ to $\mathfrak{p}'$ in $\underset{\sim}{C}$. Thus (11.13) would become

(11.13)' Theorem. Let $\mathfrak{p} = [\underset{\sim}{r}, E, \underset{\sim}{\ell}]$ and $\mathfrak{p}' = [\underset{\sim}{r}', E', \underset{\sim}{\ell}]$ be in $\underset{\sim}{C}$, and let $\mathfrak{p} \subseteq \mathfrak{p}'$. Then $\mathfrak{p}'$ is accessible from $\mathfrak{p}$ if and only if E' is accessible from E in $\underset{\sim}{N}(H)$. Moreover in this case,

$\|\mathfrak{p}'/\mathfrak{p}\| = \|\underset{\sim}{r}'/\underset{\sim}{r}\| + \|E'/E\| + \|\underset{\sim}{\ell}'/\underset{\sim}{\ell}\|.$

Then (11.15) would become a theorem due to Preston [17].

(11.15)' Corollary. Let $\mathfrak{p}$ and $\mathfrak{p}'$ be proper congruences on S such that there exists a finite chain of congruences (of length n) from $\mathfrak{p}$ to $\mathfrak{p}'$ which is maximal. Then all chains from $\mathfrak{p}$ to $\mathfrak{p}'$ have at most length n, and all maximal chains from $\mathfrak{p}$ to $\mathfrak{p}'$ have precisely length n.

§12. Appendix. The Regular Rees Matrix Semigroups

Of great importance in the study of c-0-s semigroups are the regular Rees matrix semigroups. It can be shown that these semigroups are c-0-s and every c-0-s semigroup is isomorphic to a semigroup of this type (cf. [2], Theorem 3.5). These semigroups were first introduced by D. Rees (On Semi-groups, Proc. Cambridge Philos. Soc. 36(1940), 387-400). We will briefly develop some of this theory in order to give an example of the lattice operations involving permissible triples in $\underset{\sim}{T}$ and to give an example in which the strict inequality is obtained in Lemma (8.3).

(12.1) Definition. 1. Let I, Λ be index sets and let G be a group (G is written multiplicatively and $0 \notin G$). A $\Lambda \times I$ matrix P with entries in $G \cup \{0\}$ is called regular if P has at least one nonzero entry in each row and column (P is called a matrix over $G \cup \{0\}$).

2. Let S be a collection of $I \times \Lambda$ matrices over $G \cup \{0\}$ such that each $A \in S$ has at most one nonzero entry in G. Let P be a regular $\Lambda \times I$ matrix. For $A, B \in S$, define $A \circ B = APB$, where the latter is the regular matrix product. Then S is called a Rees $I \times \Lambda$ matrix semigroup, and is denoted by $\mathcal{M}^0(I, G, \Lambda; P)$. Note that we can write $A = (i, g, \lambda)$ for $A \in \mathcal{M}^0(I, G, \Lambda; P)$, if g, the unique nonzero entry, occurs in position (i, λ). Also, $\mathcal{M}^0(I, G, \Lambda; P)$ is clearly a semigroup since $(A \circ B) \circ C = (APB)PC = AP(BPC) = A \circ (B \circ C)$.

The structure of $S = \mathcal{M}^0(I, G, \Lambda; P)$ is easily determined. If we let $[\lambda i]$ be the entry in P in position (λ, i), then direct calculation shows that $A \circ B = APB = (i, g, \lambda)P(j, h, \mu) = (i, g[\lambda j]h, \mu)$ where $A = (i, g, \lambda)$ and $B = (j, h, \mu)$. Thus the product $A \circ B$ is 0 if and only if $[\lambda j] = 0$. Using the regularity of P one can easily check the following theorem.

(12.2) Theorem. Let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be a regular Rees matrix $I \times \Lambda$ semigroup. Then

(1) $R_i = \{(i, g, \lambda) \mid g \in G, \lambda \in \Lambda\}$ is an $\mathcal{R}$-class for each $i \in I$.

(2) $L_\lambda = \{(i, g, \lambda) \mid g \in G, i \in I\}$ is an $\mathcal{L}$-class for each $\lambda \in \Lambda$.

(3) $H_{i\lambda} = \{(i, g, \lambda) \mid g \in G\}$ is an $\mathcal{H}$-class for each $i \in I$, $\lambda \in \Lambda$.

(4) $\mathcal{E}(\mathcal{M}^0) = \{(i, [\lambda i]^{-1}, \lambda) \mid$ the λi^{th} entry in P, $[\lambda i] \neq 0\}$.

Proof. For example, we show $(i, g, \lambda)\,\mathcal{R}\,(i, h, \mu)$. Since P is regular, there is a nonzero entry in the λ^{th} row, say $[\lambda j] \neq 0$, and a nonzero entry in the μ^{th} row, say $[\mu k] \neq 0$ (remember P is a $\Lambda \times I$ matrix). Then direct calculation shows $(i, g, \lambda)(j, [\lambda j]^{-1} g^{-1} h, \mu) = (i, h, \mu)$ and $(i, h, \mu)(k, [\mu k]^{-1} h^{-1} g, \lambda) = (i, g, \lambda)$.

(12.3) Theorem. The regular Rees matrix semigroup $S = \mathcal{M}^0(I, G, \Lambda; P)$ is completely 0-simple.

Proof. We will verify the conditions of Proposition (0.23). The regularity of S is a direct consequence of the regularity of P. If $A = (i, g, \lambda)$ and $[\lambda j] \neq 0$, $[\mu i] \neq 0$, then $(i, g, \lambda)(j, [\lambda j]^{-1} g^{-1} [\mu i]^{-1}, \mu)(i, g, \lambda) = (i, g, \lambda)$. If $B = (k, h, \nu)$ then we have $(k, h, g^{-1}[\mu i]^{-1}, \mu) \cdot (i, g, \lambda)(j, [\lambda j]^{-1}, \nu) = (k, h, \nu)$ and in a similar fashion we can find $X, Y \in S$ such that $XBY = A$. It readily follows that S is 0-simple. Now the $\mathcal{R}$ and $\mathcal{L}$-classes of S are precisely those determined in Theorem (12.2) so that the absorbency condition will follow directly from the definition of $A \circ B$ which, if not 0, is $(i, g[\lambda k]h, \nu) \in R_i \cap L_\nu = R_A \cap L_B$ where A, B are as above. Thus $S = \mathcal{M}^0(I, G, \Lambda; P)$ is a completely 0-simple semigroup.

Rees' theorem is (mainly) the converse of (12.3): Let S be a completely 0-simple semigroup. Then there exist index sets I and Λ, a group G, and a regular $\Lambda \times I$ matrix P such that $S \cong \mathcal{M}^0(I, G, \Lambda; P)$ ([2], Theorem 3.5). Further ([2], Corollary (3.12), suppose $\mathcal{M}^0(I, G, \Lambda, P) \cong \mathcal{M}^0(I', G', \Lambda'; P')$ then there exists a bijection $i \to i'$ of I onto I', a bijection $\lambda \to \lambda'$ of Λ onto Λ', and an isomorphism $g \to g'$ of G onto G' such that the element of P' in position (λ', i') is $u_\lambda [\lambda i]' v_i$ where $\{u_\lambda \mid \lambda \in \Lambda\}$ and $\{v_i \mid i \in I\}$ are families of elements in G. Conversely, if the above conditions on the various mappings denoted by $'$ are all satisfied, then $\mathcal{M}^0(I, G, \Lambda; P) \cong \mathcal{M}^0(I', G', \Lambda'; P')$.

We shall not prove the results, as, after all, in all our theory we have proceeded intrinsically, i.e., without reference to representations.

We will now proceed to construct the example promised after Lemma (8.3). Let $S = \mathcal{M}^0(I, G, \Lambda; P)$ be a regular Rees matrix semigroup where I and Λ contain a common element 1, $H_{11} = R_1 \cap L_1$ is a group, and where in P we have $[11] = e$, the identity of the group G. The H fixed in (1.4) will be identified with H_{11}.

It is easy to check that $g \to (1, g, 1)$ is an isomorphism of G onto H_{11}. Indeed, we can also check that $g \to (i, g[\lambda i]^{-1}, \lambda)$ is again an isomorphism of G onto the group $\mathcal{H}$-class $H_{i\lambda}$, since $H_{i\lambda}$ is a group $\mathcal{H}$-class precisely when $[\lambda i] \neq 0$ in P. Moreover, we have that E is normal in G precisely when $(1, E, 1)$ is normal in $(1, G, 1) = H_{11}$. Thus the nonzero translates, tEs of (2.3), are of the form (i, gE, λ) for some $g \in G$, $i \in I$, $\lambda \in \Lambda$. The partitioning of S by the translates of $E = (1, E, 1)$ is now obvious.

Furthermore, we can now check directly on P the conditions of (3.1.7) and (4.2). For E normal in $H = H_{11} = G$, we have:

(12.4) $\underset{\sim}{R}_i \, \underset{\sim}{d}(E) R_j$ $(i, j \in I)$ if and only if

(a) $[\nu i] = 0$ precisely when $[\nu j] = 0$; and

(b) for some $g \in G$, $[\mu i]^{-1}[\mu j] \in gE$ whenever $[\mu i] \neq 0$.

A dual formulation can be given for $L_\lambda \, \underset{\sim}{s}(E) L_\mu$.

It is easy to derive (12.4.b) from (4.2.7b'). Suppose that $L_\lambda \cap R_i$ and $L_\lambda \cap R_j$ are groups for some fixed λ. Then they contain the idempotents $e_i = (i, [\lambda i]^{-1}, \lambda)$ and $e_j = (j, [\lambda j]^{-1}, \lambda)$, respectively. Thus under the congruence $\mathfrak{C} = \mathfrak{C}(E)$, we have

$e_i^{\mathcal{L}} = (i, E[\lambda i]^{-1}, \lambda)$ and $e_j^{\mathcal{L}} = (j, E([\lambda j]^{-1}, \lambda)$. Now if $[\mu i] \neq 0$ (then also $[\mu j] \neq 0$ by (12.4.a)), let $t = (k, e, \mu)$ and compute $te_i^{\mathcal{L}} = te_j^{\mathcal{L}}$ to obtain $(k, [\mu i]E[\lambda i]^{-1}, \lambda) = (k, [\mu j]E[\lambda j]^{-1}, \lambda)$. Equating middle coordinates we have $[\mu i]E[\lambda i]^{-1} = [\mu j]E[\lambda j]^{-1}$. Since E is normal in $H = G = H_{11}$, we have $[\lambda i]^{-1}[\lambda j]E = E[\lambda i]^{-1}[\lambda j] = [\mu i]^{-1}[\mu j]E$ and it follows that $[\mu i]^{-1}[\mu j] \in gE$ for all such $[\mu i] \neq 0$ where $g = [\lambda i]^{-1}[\lambda j]$.

A dual formulation can be made for $L_\lambda \underset{\sim}{s}(E) L_\mu$.

We remark that P can be normalized so that certain "leading" entries are e, the identity of G (Tamura [20]). In that case condition (12.4.b) may be replaced by (b'): $[\mu i]^{-1}[\mu j] \in E$, whenever $[\mu i] \neq 0$ (cf. Howie [8]).

The reader should now recall the lattices of (8.1).

(12.5) <u>Example</u>. Let G be the cyclic group Z^6. Let $I = \{1, 2, 3\}$, $\Lambda = \{1, 2\}$,

$$P = \begin{bmatrix} e & e & e \\ e & a^2 & a \end{bmatrix},$$

and let $S = \mathcal{M}^0(I, G, \Lambda; P)$.

Let $E_1 = \{e, a^2, a^4\}$ and $E_2 = \{e, a^3\}$. First observe that (12.4.a) does not apply since P has no zero entries. Next note that

$[11]^{-1}[12] = e \in E_1$ and $[21]^{-1}[22] \in E_1$ so that $R_1 \underset{\sim}{d}(E_1)R_2$. On the other hand $[11]^{-1}[13] = e \in E_1$ determining the coset eE_1 of (12.4.b) while $[21]^{-1}[23] = a \notin eE_1$ and it follows that R_1 and R_3 are not $\underset{\sim}{d}(E_1)$ equivalent. Thus the equivalent classes for $\underset{\sim}{d}(R_1)$ are $\{R_1, R_2\}$ and $\{R_3\}$.

In a similar fashion one sees that the "leading" entries of P in the first row (and in position (2, 1)) will always fix E_2 as the coset of E_2 determined in (12.4.b). But in this case $[21]^{-1}[22] \notin E_2$, $[21]^{-1}[23] \notin E_2$ and $[22]^{-1}[23] \in E_2$ so that the equivalence classes of $\underset{\sim}{d}(E_2)$ are just $\{R_1\}$, $\{R_2\}$ and $\{R_3\}$. Hence $\bigvee_{\underset{\sim}{R}} \underset{\sim}{d}(E_i) = \underset{\sim}{d}(E_1)$, (the supremum being taken over $\{1, 2\}$). But $\bigvee_{\underset{\sim}{N}}(E_i) = G$, and by (12.4.b) $R_i \underset{\sim}{d}(G) R_j$ for all i and j. Thus $\underset{\sim}{d}(G)$ is the universal congruence on $\{R_1, R_2, R_3\}$ with the one equivalence class $\{R_1, R_2, R_3\}$. It follows that $\bigvee_{\underset{\sim}{R}} \underset{\sim}{d}(E_i) \subset \underset{\sim}{d}(\bigvee_{\underset{\sim}{N}} E_i)$ and thus the containments of (8.3) are sometimes proper.

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LIST OF SYMBOLS

$\underset{\sim}{R}_*[\underset{\sim}{L}_*]$, 16, 18	Lattice of all equivalences on the set of $\mathcal{R}[\mathcal{L}]$-classes of S
$\underset{\sim}{R}[\underset{\sim}{L}]$, 16, 18, 47, 48	Initial segment of $\underset{\sim}{R}_*[\underset{\sim}{L}_*]$
$\underset{\sim}{\Lambda} = \underset{\sim}{R} \times \underset{\sim}{N} \times \underset{\sim}{L}$, 16	Cartesian product lattice
$\underset{\sim}{T}$, 16, 54, 57ff	Sublattice of $\underset{\sim}{\Lambda}$ of permissible triples
$\underset{\sim}{H}$, 17, 42	Lattice of congruences on S lying under $\mathcal{H}$
$\mathcal{C}$, 24	Set of translates
$\underset{\sim}{d}$, $\underset{\sim}{r}$, 17, 32; $\underset{\sim}{d}(E)$, $\underset{\sim}{r}(E)$, 17, 34; [$\underset{\sim}{s}$, $\underset{\sim}{\ell}$, $\underset{\sim}{s}(E)$, $\underset{\sim}{\ell}(E)$]	Equivalence relations on the set of $\mathcal{R}[\mathcal{L}]$-classes of S
$\mathfrak{C}(E)$, 16, 26	Congruence associated with E
ρ, $\rho(\underset{\sim}{r})[\lambda$, $\lambda(\underset{\sim}{\ell})]$, 17, 34	Congruence associated with $\underset{\sim}{r}[\underset{\sim}{\ell}]$
$(\underset{\sim}{r}, E, \underset{\sim}{\ell})$, 17, 39	Permissible triple
$[\underset{\sim}{r}, E, \underset{\sim}{\ell}]$, 40	Congruence associated with permissible triple
$\underset{\sim}{r}(\mathcal{T})$, $\underset{\sim}{\ell}(\mathcal{T})$, 45	Induced equivalence relations
$\underset{\sim}{B}$, 62ff	Lattice of Brandt congruences
Br, 64	Condition Br
$\succ$, 70	Covers
$\underset{\sim}{Eq}(X)$, 76, 84	Lattice of equivalences on the set X
W(A), 79	Set of predecessors in the set A

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