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The Numerical Range of a Continuous Mapping of a Normed Space

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(Dedicated to A. Ostrowski on the occasion of his 75th birthday)

1. Introduction

Let X denote a normed linear space over the real or complex field, X' the dual space of X, $G(X)$ the set of all continuous mappings of $S(X)$ into X, where $S(X)$ is the unit sphere of X, i.e. the set of all $x \in X$ such that $\|x\| = 1$. Given $x \in S(X)$, let $D(x) = \{f \in X' : f(x) = \|f\| = 1\}$; and, given $T \in G(X)$, let $V(T, x) = \{f[T(x)] : f \in D(x)\}$. The *numerical range* $V(T)$ of a continuous mapping $T \in G(X)$ is defined by

$$V(T) = \bigcup \{V(T, x) : x \in S(X)\}.$$

In the special case when X is a Hilbert space and $D(x)$ can be identified with $\{x\}$, the numerical range of a linear operator has a long history [7]. Under the name 'field of values', the concept has been extended by F. L. BAUER [1] to linear operators on all finite dimensional normed linear spaces. The numerical range of a linear operator on a semi-inner-product space has been studied by G. LUMER [4]. A normed linear space X has, in general, many semi-inner-products that correspond to the norm of X. The choice of one of these semi-inner-products corresponds to the choice of a mapping $x \to f_x$ of $S(X)$ into X' such that $f_x \in D(x)$ for each x. Then the numerical range $W(T)$ for this semi-inner-product is given by

$$W(T) = \{f_x(T x) : x \in S(X)\}.$$

Thus if T is a continuous linear operator, $V(T)$ is the union of all the numerical ranges $W(T)$ in the sense of LUMER.

When T is a continuous linear operator it is classical that $V(T)$ is a convex set if X is a Hilbert space [7], but an example is given in [5] of a linear operator on a two dimensional normed linear space for which $V(T)$ is not convex. Our main result is that $V(T)$ is connected for every normed linear space X and every $T \in G(X)$ (unless both X is the real numbers and $T(-\alpha) \neq -T(\alpha)$ where $S(X) = \{\alpha, -\alpha\}$). We give two proofs of this result and include an example of a (continuous) linear operator on a real or complex two dimensional semi-inner-product space for which $W(T)$ is not connected.

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Our first proof of the connectedness of $V(T)$ depends on the upper semi-continuity of the set-valued mapping $x \to D(x)$ with respect to the weak* topology on X'. We show that this mapping is also upper semi-continuous for the norm topology on X' when $X = (c_0)$, the sup norm space of sequences converging to 0, (and also of course when X has finite dimension) but not when $X = (c)$, the sup norm space of convergent sequences, or for certain other spaces. As an alternative to upper semi-continuity we also work with a topology on the set of subsets of X', which, although it is probably less familiar than upper semi-continuity, permits us to use the class of continuous functions and their widely studied properties in place of the class of upper semi-continuous mappings. Our second proof shows that when X is not the real numbers $P = \{(x, f) \in X \times X' : x \in S(X), f \in D(x)\}$ is connected in certain topologies, and this may be of interest in itself.

2. Connectedness of the numerical range

Let $\mathscr{P}(U)$ denote the set of all subsets of the set U. If E is a topological space, let $\{\mathscr{P}(U) : U \subseteq E;\ U \text{ is open}\}$ be a basis for the *τ-topology* on $\mathscr{P}(E)$. Adjectives used with reference to the τ-topology will bear the prefix 'τ-', e.g. τ-open. A mapping $x \to A(x)$ of a topological space F into the set of subsets of a topological linear space E is *upper semi-continuous (usc)* on F if and only if for every $x \in F$ and every neighbourhood U of 0 in E there exists a neighbourhood V of x such that for all $y \in V$, $A(y) \subseteq A(x) + U$ (cf. [6], pages 35–36). There are other definitions of upper semi-continuity currently in use (cf. [2]). In fact, what we are calling τ-continuous is sometimes called upper semi-continuous.

LEMMA 1. *Let F be a topological space and let E be a topological linear space. If the mapping $x \to A(x)$ is τ-continuous, then it is usc. If for every x in F, $A(x)$ is a compact subset of E, then the function A is τ-continuous if and only if it is usc.*

Proof: Assume A is τ-continuous. If $x \in F$ and U is an open neighbourhood of 0 in E, then, since $\mathscr{P}(A(x) + U)$ is a τ-open subset of $\mathscr{P}(E)$, it follows that $A^{-1}[\mathscr{P}(A(x) + U)] = \{y \in F : A(y) \subseteq A(x) + U\}$ is an open subset of F. Hence $A^{-1}[\mathscr{P}(A(x) + U)]$ is a V whose existence is required in the definition of usc, and A is usc.

Assume that A is usc. Let $\mathscr{P}(U)$, with U an open subset of E, be a basic τ-open set. If $x \in A^{-1}(\mathscr{P}(U))$ then $A(x) \subseteq U$. The compactness hypothesis now provides a neighbourhood of 0 in E, denoted G, such that $A(x) + G \subseteq U$ (see [3] pages 35 and 36 for the details). Since A is usc there is a neighbourhood V of x such that for each $y \in V$, $A(y) \subseteq A(x) + G \subseteq U$. That is, $A(y) \in \mathscr{P}(U)$. Thus $x \in V \subseteq A^{-1}[\mathscr{P}(U)]$. So $A^{-1}[\mathscr{P}(U)]$ is open and A is τ-continuous.

An application of the Hahn-Banach Theorem shows that for each $x \in S(X)$ we have $D(x) \neq \emptyset$. In the weak* topology, $D(x)$ is a closed subset of the (solid) unit ball in X' and hence is compact (cf. [3] page 155). Since $D(x)$ is convex, $D(x)$ is connected in any topology which makes X' a topological linear space, because in any such topology $\alpha \to \alpha f + (1-\alpha) g$, $0 \leqslant \alpha \leqslant 1$ is a continuous function.

LEMMA 2. *Let $S(X)$ have the norm topology and let X' have the weak* topology. Then the mapping $x \to D(x)$ is τ-continuous and usc.*

Proof: Since $D(x)$ is compact in the weak* topology, Lemma 1 shows that it is sufficient to prove that $x \to D(x)$ is usc. Suppose that the mapping is not usc. Then there exist $x \in S(X)$ and a weak* neighbourhood U of 0 in X' such that for every positive integer n there exists $y_n \in S(X)$ and $f_n \in D(y_n)$ satisfying $\|y_n - x\| < 1/n$ and $f_n \notin D(x) + U$. Since $\|f_n\| = 1$, there exists a weak* cluster point g of $\{f_n\}$ with $\|g\| \leqslant 1$. Then

$$\begin{aligned} |g(x) - 1| &\leqslant |g(x) - f_n(x)| + |f_n(x) - f_n(y_n)| \\ &\leqslant |g(x) - f_n(x)| + \|x - y_n\| . \end{aligned}$$

Since $\|x - y_n\| < 1/n$ and g is a weak* cluster point of $\{f_n\}$, the righthand side can be made arbitrarily small by correctly choosing n, and so $g(x) = 1$. Therefore $g \in D(x)$. However, since g is a weak* cluster point of $\{f_n\}$ and U is a weak* neighbourhood of 0, we have $f_n \in g + U \subseteq D(x) + U$ for some n, which is contradictory.

LEMMA 3. *Let $T \in G(X)$, and let the scalar field have its usual topology. Then the mapping $x \to V(T, x)$ is a τ-continuous and usc mapping of $S(X)$ with the norm topology into the set of subsets of the scalar field.*

Proof: Observe that $V(T, x)$, for $x \in S(x)$ and $T \in G(X)$, is compact, because it is the image of the weak* compact set $D(x)$ under the weak* continuous mapping $f \to f(T(x))$. Therefore, by Lemma 1, it suffices to prove that $x \to V(T, x)$ is usc.

Let $x \in S(X)$ and $\varepsilon > 0$, and let $U = \{g \in X' : |g(T(x))| < \varepsilon/2\}$. Then U is a weak* neighbourhood of 0, and so, by Lemma 2 and the continuity of T, we may choose $\delta > 0$ such that for every $y \in S(X)$ with $\|x - y\| < \delta$ it follows that $\|T(x) - T(y)\| < \varepsilon/2$ and $D(y) \subseteq D(x) + U$. So if $y \in S(X)$, $\|x - y\| < \delta$, and $f \in D(y)$ then $f = g + u$ for some $g \in D(x)$ and $u \in U$. Since $g(T(x)) \in V(T, x)$ the distance from $f(T(y))$ to $V(T, x)$ is at most $|f(T(y)) - g(T(x))|$ and $|f(T(y)) - g(T(x))| \leqslant |f(T(y)) - f(T(x))| + |u(Tx)| < \|T(y) - T(x)\| + \varepsilon/2 < \varepsilon$. But $f(T(y))$ was an arbitrary point of $V(T, y)$, and so $V(T, y) \subseteq V(T, x) + \{t \in \text{scalar field} : |t| < \varepsilon\}$. Thus $x \to V(T, x)$ is usc.

The two proofs of the connectedness of the numerical range which we give use the connectedness of $S(X)$. Since $S(X)$ is disconnected only when X is $\mathbf{R}$, the real numbers, that case is treated separately. In fact when $X = \mathbf{R}$, $V(T) = \{1/\alpha\, T(\alpha)\} \cup \{-1/\alpha\, T(-\alpha)\}$ where $S(X) = \{\alpha, -\alpha\}$. This gives:

PROPOSITION. If $X = \mathbf{R}$, $V(T)$ is connected if and only if $T(-\alpha) = -T(\alpha)$, where $S(X) = \{\alpha, -\alpha\}$. In particular, if T is linear, $V(T)$ is connected.

Both proofs also use the following fact: If $\{L_x\}$ is a family of connected subsets of some topological space and if $G_1 \cup G_2 = \bigcup L_x$ is a decomposition of $\bigcup L_x$ into two non-empty disjoint sets, G_1 and G_2, open in the relative topology, then for each x the entire set L_x lies in either G_1 or G_2.

THEOREM 1. *Let $T \in G(X)$. If $X \neq \mathbf{R}$ then $\mathbf{V}(T)$ is connected.*

Proof: Suppose $X \neq \mathbf{R}$ and $V(T)$ is disconnected. Then $V(T) \subseteq H_1 \cup H_2$, where H_1 and H_2 are open sets giving a decomposition $V(T) = G_1 \cup G_2$ of $V(T)$ into disjoint, non-empty, relatively open sets $G_i = V(T) \cap H_i$ for $i=1$ and 2. By Lemma 3, the mapping $x \to V(T, x)$ is τ-continuous, and therefore, for $i=1$ and 2, the inverse image U_i of the τ-open set $\mathscr{P}(H_i)$ under $x \to V(T, x)$ is an open subset of $S(X)$, the domain of the mapping. For $x \in S(X)$, the set $V(T, x)$ is connected, being the image of the set $D(x)$, which is connected in the norm topology, under the norm continuous mapping $f \to f(Tx)$. Hence, by the sentence preceding the theorem, $V(T, x) \subset G_1 \subseteq H_1$ or $V(T, x) \subseteq G_2 \subseteq H_2$. Thus $x \in U_1$ or $x \in U_2$, but not both. We deduce that $S(X) = U_1 \cup U_2$, where $U_1 \cap U_2 = \emptyset$. But this is impossible, since $S(X)$ is connected, and the theorem is proved.

Here we begin our second proof of the connectedness of $V(T)$. Let P denote $\{(x, f) \in X \times X' : x \in S(X), f \in D(x)\}$. We shall first prove the following general theorem:

THEOREM 2. *Let $X \neq \mathbf{R}$, and let X have the norm topology. Let X' have a topology satisfying*

(a) *X' is a topological linear space,*

(b) *$x \to D(x)$ is usc for all $x \in S(X)$,*

(c) *$D(x)$ is compact, for all $x \in S(X)$.*

Then P is connected as a subset of $X \times X'$ with the product topology.

We shall then deduce:

COROLLARY. Let $X \neq \mathbf{R}$, and let $X \times X'$ be topologized by the product of the norm topology and the weak* topology. Then P is connected, as a subset of $X \times X'$.

Finally we shall show that this implies Theorem 1.

Proof of Theorem 2: We shall first show that every sequence $\{f_i\}$ in X' which is eventually in every neighbourhood of $D(x)$ has a limit point g in $D(x)$. For suppose this is false, then $D(x)$ has an open covering by sets each containing only finitely many f_i's. Since $D(x)$ is compact it has a finite covering by such sets. But the union of the sets in this finite covering is a neighbourhood of $D(x)$ containing only finitely many f_i's. Since this is a contradiction, the f_i's must have a limit point g in $D(x)$.

Let $\pi : P \to S(X)$ denote the projection mapping $(x, f) \to x$. Then π is a closed mapping. For suppose K is a closed subset of P and x is a limit point of $\pi(K)$. Then there exists $\{(x_i, f_i)\} \subset K$ such that $x = \lim x_i = \lim \pi(x_i, f_i)$. Since $t \to D(t)$ is usc, $\{D(x_i)\}$ is eventually within each neighbourhood of $D(x)$. Since $f_i \in D(x_i)$, $\{f_i\}$ is eventually in each neighbourhood of $D(x)$ and hence has a limit point g in $D(x)$. Thus $(x, g) \in \overline{\{(x_i, f_i)\}} \subseteq K$, whence $x \in \pi(K)$, and so $\pi(K)$ is closed.

Suppose that $P=G_1\cup G_2$ where the G_i's are non-empty, disjoint, open and closed subsets of P. For each $x\in S(X)$ the set $\pi^{-1}(x)$ is connected because it is homeomorphic to $D(x)$. The set $\pi^{-1}(x)$ must thus be a subset of either G_1 or G_2. It follows that $\pi(G_1)\cap\pi(G_2)=\emptyset$. Since $\pi(G_1)$ and $\pi(G_2)$ are closed and cover $S(X)$, this contradicts the connectedness of $S(X)$. Hence no such G_i's can exist and therefore P is connected.

The Corollary now follows immediately from Theorem 2, in view of Lemma 2 and the compactness of $D(x)$ in the weak* topology.

Let $X\times X'$ have the product topology formed from the norm topology of X and the weak* topology of X'. If X is infinite dimensional, it can be shown that the mapping $(x,f)\to f(x)$ is not continuous on $X\times X'$, because it is unbounded on every open subset of $X\times X'$. However:

LEMMA 4. *Let F be a norm-bounded subset of X'. Let $X\times X'$ have the product topology formed from the norm topology of X and the weak* topology of X'. Then the mapping $(x,f)\to f(x)$ defined on $X\times X'$ is a continuous mapping of the set $X\times F$ with the relative topology.*

Proof: Suppose that F is contained in a ball of radius r centered at the origin of X'. If (x_i,f_i) is a net in $X\times F$,

$$\begin{aligned}|f_i(x_i)-f(x)| &\leqslant |f_i(x-x_i)|+|f_i(x)-f(x)|\\ &\leqslant r\,\|x-x_i\|+|f_i(x)-f(x)|\,.\end{aligned}$$

Thus $f_i(x_i)$ will converge to $f(x)$ if both $x_i\to x$ in the norm topology and $f_i\to f$ in the weak* topology, i.e. if $(x_i,f_i)\to(x,f)$ in $X\times X'$.

Second Proof of Theorem 1. Let $X\times X'$ be topologized as it was in the Corollary to Theorem 2 and Lemma 4. We view the mapping $(x,f)\to\ [T(x)]$ defined on P as the composition of the continuous functions $(x,f)\to(T(x),f)\to f[T(x)]$. (Lemma 4 shows the continuity at the final step.) Since by the Corollary to Theorem 2, P is connected, it follows that $V(T)$ must be connected, as it is the image of P under a continuous function.

However, the numerical range $W(T)$ need not be connected:

EXAMPLE. Let X be $\mathbf{R}^2$ or $\mathbf{C}^2$ with the norm given for each $x=(\xi_1,\xi_2)\in X$ by $\|x\|=\max(|\xi_1|,|\xi_2|)$. Given $a=(\alpha_1,\alpha_2)\in S(X)$, and $x=(\xi_1,\xi_2)\in X$, let $f_a(x)$, T be defined by

$$f_a(x)=\begin{cases}\bar\alpha_1\xi_1 & \text{if } |\alpha_1|=1,\\ \bar\alpha_2\xi_2 & \text{if } |\alpha_1|<1.\end{cases}\qquad Tx=(\xi_1,0).$$

Then $f_a\in D(a)$, and T is a continuous linear operator. Also

$$f_a(Ta)=\begin{cases}1 & \text{if } |\alpha_1|=1\\ 0 & \text{if } |\alpha_1|<1.\end{cases}$$

Therefore the numerical range $W(T)$ in the sense of Lumer for the semi-inner-product space corresponding to the mapping $a \to f_a$ is the set with exactly two elements, 1 and 0.

3. Upper semi-continuity and τ-continuity of the mapping $x \to D(x)$ with the norm topology on X'.

If X has finite dimension, the norm topology coincides with the weak* topology on X', and so, by Lemma 2, the mapping $x \to D(x)$ is upper semi-continuous and τ-continuous with respect to the norm topology on X'. We show that the mapping $x \to D(x)$ is also upper semi-continuous and τ-continuous in this sense when $X = (c_0)$, but not for certain other spaces X including the space (c).

Let $\mathbf{F}$ denote either the real or the complex field, and $\mathbf{P}$ the set of all positive integers. We denote by (m), as usual, the Banach space of all bounded mappings of $\mathbf{P}$ into $\mathbf{F}$ with the *sup norm*

$$\|x\|_\infty = \sup\{|x(n)| : n \in \mathbf{P}\} \qquad (x \in (m)),$$

and by (c) and (c_0) the subspaces of (m) consisting of all sequences that converge and converge to zero respectively. Also, as usual, we denote by (l_1) the Banach space of all mappings x of $\mathbf{P}$ into $\mathbf{F}$ such that $\|x\|_1 = \sum_{n=1}^{\infty} |x(n)| < \infty$, normed by $\|\cdot\|_1$.

THEOREM 3. *Let $X = (c_0)$. Then the mapping $x \to D(x)$ is a usc and τ-continuous mapping of $S(X)$ into subsets of X' with respect to the norm topologies in X and X'.*

Proof: To each element f of X' corresponds a sequence $\{\lambda_k\}$ of elements of $\mathbf{F}$ such that $\sum_{k=1}^{\infty} |\lambda_k| = \|f\|$ and

$$f(x) = \sum_{k=1}^{\infty} \lambda_k x(k) \qquad (x \in X).$$

Given $x \in S(X)$, let $E_x = \{k \in \mathbf{P} : |x(k)| = 1\}$. Then E_x is a non-empty finite set.

Let $a \in S(X)$ and $\varepsilon > 0$. Since the set $\{k \in \mathbf{P} : |a(k)| \geq 1/2\}$ is finite

$$\sup\{|a(k)| : k \in \mathbf{P} \backslash E_a\} = 1 - \eta$$

with $\eta > 0$. Choose δ with $0 < \delta < \min(\varepsilon, \eta)$, let $b \in S(X)$ with $\|b - a\|_\infty < \delta$, and let $f \in D(b)$. The sequence $\{\lambda_k\}$ corresponding to f satisfies

$$\sum_{k=1}^{\infty} |\lambda_k| = \sum_{k=1}^{\infty} \lambda_k b(k) = 1 .$$

Therefore $\lambda_k = 0$ $(k \in \mathbf{P} \backslash E_b)$ and $\lambda_k = |\lambda_k| \, \overline{b(k)}$ for all k. Let $\mu_k = |\lambda_k| \, \overline{a(k)}$ $(k \in \mathbf{P})$, and let g be the element of X' corresponding to the sequence $\{\mu_k\}$. Since $\|b - a\|_\infty < \eta$, we have

$E_b \subseteq E_a$, and therefore $|a(k)|=1$ whenever $\lambda_k \neq 0$. Therefore

$$\|g\| = \sum_{k=1}^{\infty} |\mu_k| = \sum_{k=1}^{\infty} |\lambda_k| = 1,$$

$$g(a) = \sum_{k=1}^{\infty} \mu_k a(k) = \sum_{k=1}^{\infty} |\lambda_k| = 1.$$

Thus $g \in D(a)$. Also

$$\begin{aligned} \|f-g\| &= \sum_{k=1}^{\infty} |\lambda_k - \mu_k| \\ &= \sum_{k=1}^{\infty} |\lambda_k| \, |\overline{b(k)} - \overline{a(k)}| \leqslant \|b-a\|_\infty < \varepsilon. \end{aligned}$$

Since f is an arbitrary element of $D(b)$, $g \in D(a)$, and $f=g+(f-g)$, this proves that $D(b) \subseteq D(a)+\{h \in X' : \|h\| < \varepsilon\}$. Thus $x \to D(x)$ is usc with respect to the norm topologies in X and X'. Furthermore $x \to D(x)$ will be τ-continuous if each $D(x)$ is compact. For each $x \in S(X)$, $D(x)$ is a subset of $\{f \in S(X') : \text{support}\,(f) \subseteq E_x\}$ which is homeomorphic to the compact set $S(\mathbf{F}^k)$, where k is the order of E_x and $\mathbf{F}^k$ has the norm $\|(t_1, \ldots, t_k)\| = \sum |t_i|$. Therefore since $D(x)$ is closed in the norm topology, it is compact.

That P is connected when $X=(c_0)$ and X' has the norm topology is a special case of Theorem 2.

THEOREM 4. *Let X be a linear subspace of (m) such that $(c) \subseteq X$. Then the mapping $x \to D(x)$ is not upper semi-continuous with respect to the norm topologies in X and X' (and therefore not τ-continuous).*

Proof: Given $n \in P$, let e_n, a, b_n denote the elements of X defined by

$$e_n(k) = \begin{cases} 1 & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n \end{cases}, \qquad a(k) = 1 - \frac{1}{k} \; (k \in \mathbf{P}), \qquad b_n(k) = \begin{cases} a(k) & \text{if} \quad k < n \\ 1 & \text{if} \quad k \geqslant n \end{cases}.$$

Let f_n be the element of X' defined by $f_n(x) = x(n)$ $(x \in X)$. Given $g \in D(a)$, we have

$$g(e_n) = 0 \, (n \in \mathbf{P});$$

for we have $\|a+\xi e_n\|_\infty = 1$ whenever $|\xi| \leqslant 1/n$, and so

$$1 \geqslant |g(a+\xi e_n)| = |1 + \xi g(e_n)| \quad (|\xi| \leqslant 1/n),$$

which is impossible unless $g(e_n)=0$. Therefore, for all $g \in D(a)$,

$$\|f_n - g\| \geqslant |(f_n - g)(e_n)| = 1 \quad (n \in \mathbf{P}).$$

However $f_n \in D(b_n)$ and $\|b_n - a\|_\infty = 1/n$. Thus b_n tends to a, but $D(b_n) \nsubseteq D(a)+U_1$, where $U_1 = \{f \in X' : \|f\| < 1\}$, and the result follows.

THEOREM 5. *Let $X=(l_1)$. Then the mapping $x \to D(x)$ is not upper semi-continuous with respect to the norm topologies in X and X' (and therefore not τ-continuous).*

Proof: Given $n \in \mathbf{P}$, let a, b_n be the elements of $S(X)$ defined by

$$a(k) = 1/2^k \quad (k \in \mathbf{P}), \qquad b_n(k) = \begin{cases} 2^n/(2^n - 1)\,2^k & (k \leqslant n) \\ 0 & (k > n). \end{cases}$$

Then

$$\|a - b_n\|_1 = \frac{1}{2^n - 1} \sum_{k=1}^{n} \frac{1}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}.$$

Let f_n be the functional defined by

$$f_n(x) = \sum_{k=1}^{n} x(k) \quad (x \in X).$$

and let $g \in D(a)$. Since $a(k) > 0\,(k \in \mathbf{P})$, we have

$$g(x) = \sum_{k=1}^{\infty} x(k) \quad (x \in X).$$

Also $f_n \in D(b_n)$ and $\|f_n - g\| = 1$ $(n \in \mathbf{P})$. Thus the mapping $x \to D(x)$ is not upper semi-continuous with respect to the norm topologies in X and X'.

Note added in proof: Our lemma 2 is known, cf. Theorem 4.3, D. F. Cudia: *The Geometry of Banach spaces. Smoothness.* Trans. Amer. Math. Soc. *110*, 284–314 (1964).

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