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Group Rings, Semigroup Rings and Their Radicals

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1. INTRODUCTION

In [*1*] Amitsur proved that if R is a ring with no nil ideals, then the polynomial ring $R[t]$ is semiprimitive, i.e., has $\{0\}$ Jacobson radical. In [*3*] Bovdi proved that if R has no zero divisors and G is an SN group with a normal system whose factors are Abelian torsion-free, then the group ring RG is semiprimitive. We obtain both of these results as corollaries of a more general theorem concerning semigroup rings. Let D be a semigroup with two properties: the Γ property (Definition 2.4) and the 2Ω property (Definition 2.17). We call such a semigroup a $2\Omega\Gamma$ semigroup, and we prove in Section 3 that if R is a ring with no nil ideals and D is $2\Omega\Gamma$ with identity element, then the semigroup ring, RD, is semiprimitive (Lemma 3.6).

In Section 2 we investigate the 2Ω and Γ properties. We show that the class of oriented semigroups (Definition 2.6), which includes all directed groups (Proposition 2.7), are Γ (Proposition 2.12). Every strict, fully ordered semigroup with more than one element is $2\Omega\Gamma$ (Theorem 2.22), and every SN group with a normal system whose factors are Abelian torsion-free is also $2\Omega\Gamma$ (Theorem 2.23). In [*7*] Kemperman conjectured that every torsion-free group is 2Ω. It is conceivable that every torsion-free group is also Γ.

In Section 4 we prove that if R is a ring with no nil ideals and G is a group such that the order of every element of finite order in G is cancelable in R (4.3) then RG has no nil ideals (Theorem 4.4). As corollaries we obtain two theorems of Passman [*10*]. We then combine our two main theorems (3.6 and 4.4) to the special case of Abelian groups with at least one element of infinite order. We prove that if the upper nil radical of R is $\{0\}$ and if every integer n

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such that G has an element of order n is cancelable in R, then RG is semiprimitive. When R is commutative, this yields part of a theorem of Connell's [*4*].

2. Semigroups

Throughout this section, D will denote a semigroup. If $A \subseteq D$ we denote by $\langle A \rangle$ the smallest subsemigroup of D containing A. The cardinality of A is denoted by $|A|$.

Definition 2.1. A nonempty subset G of D is *grouplike* (in D) if

(a) $a, \; b \in G$ imply $ab \in G$,
(b) $a, ab \in G$ imply $b \in G$.

Definition 2.2. If $A \subseteq D$, $A \neq \phi$, then $\langle\!\langle A \rangle\!\rangle = \cap \{G \mid A \subseteq G \subseteq D$ and G is grouplike$\}$.

Remarks. The intersection of a collection of grouplike subsets is grouplike. The set $\langle\!\langle A \rangle\!\rangle$ is the smallest grouplike set in D containing A. If D has identity 1 then $1 \in \langle\!\langle A \rangle\!\rangle$ by Definition 2.1(b). If D is a group, a subset A of D is grouplike if and only if it is a subgroup, and $\langle\!\langle A \rangle\!\rangle$ is just the subgroup generated by A. The following result will be needed later.

Proposition 2.3. *Suppose φ is a homomorphism taking the semigroup D into a semigroup E. Then*

(i) *If H is grouplike in E, then $\varphi^{-1}(H)$ is grouplike in D.*
(ii) *If $A \subseteq D$, then $\varphi(\langle\!\langle A \rangle\!\rangle) \subseteq \langle\!\langle \varphi(A) \rangle\!\rangle$.*

Proof. (i) If $a, \; b \in \varphi^{-1}(H)$, then $\varphi(a), \; \varphi(b) \in H$. Hence $\varphi(ab) = \varphi(a)\varphi(b) \in H$, since H is grouplike, so $ab \in \varphi^{-1}(H)$. Thus Definition 2.1(a) is proved. Further, if $a, ab \in \varphi^{-1}(H)$, then $\varphi(a), \varphi(a)\varphi(b) \in H$ whence $\varphi(b) \in H$, since H is grouplike. Hence $b \in \varphi^{-1}(H)$ and Definition 2.1(b) is proved.

(ii) By (i), $K = \varphi^{-1}(\langle\!\langle \varphi(A) \rangle\!\rangle)$ is grouplike and clearly $A \subseteq K$, whence $\langle\!\langle A \rangle\!\rangle \subseteq K$. It follows that $\langle\!\langle \varphi(A) \rangle\!\rangle = \varphi(K) \supseteq \varphi(\langle\!\langle A \rangle\!\rangle)$.

Remark. The additive semigroup $D = Z \oplus P$ where Z denotes the integers and P the non-negative integers furnishes an example for which there is proper containment in Proposition 2.3(ii). Just let $E = Z$ and define an epimorphism $\varphi : D \to E$ by $\varphi(n, p) = n + p$. Then if $A = \{0\} \oplus P$, $\varphi(\langle\!\langle A \rangle\!\rangle) = P \neq Z = \langle\!\langle \varphi(A) \rangle\!\rangle$. Also, A is an example of a grouplike set for which $\varphi(A)$ is not grouplike.

The following definition is motivated by Herstein's proof ([*6*], p. 33) of Amitsur's theorem [*1*].

DEFINITION 2.4. A semigroup D is said to be a *Γ-semigroup* if and only if it satisfies:

(Γ) For all nonempty finite sets A contained in D, there exists g in D, such that for every finite set B contained in $\langle\langle Ag \rangle\rangle$ there exists an integer $n = n(B)$, for which $(Ag)^n B \cap B = \phi$.

Remark. If D is Γ then D satisfies:

(Γ') For all finite sets A contained in D there exists $g \in D$, such that for every finite set B_0 contained in $\langle\langle Ag \rangle\rangle$ there exists an integer $n = n(B_0)$, such that $(Ag)^n \cap B_0 = \phi$.

Just let $B = (Ag) \cup B_0$ and the proof is immediate.

LEMMA 2.5. *If D is a group, then D is Γ if and only if D is Γ'.*

Proof. By the above remark, it suffices to prove that Γ' implies Γ if D is a group. Let $(Ag)^n B \cap B \neq \phi$ for all integers n. Put $B_0 = BB^{-1}$. Then $B_0 \subseteq \langle\langle Ag \rangle\rangle$ and we have $(Ag)^n \cap B_0 \neq \phi$ for all integers n, a contradiction.

Our aim is to show that a certain class of partially ordered (p.o.) semigroups consists of Γ-semigroups. If in a p.o. semigroup D, $a < b$ implies $ac < bc$ and $ca < cb$ for all c in D, we say that the partial order of D is *strict* or D is a *strict* p.o. *semigroup*. (Fuchs [5], p. 153). We write $a \leqslant A$ if a is a lower bound for the set A and $A \leqslant b$ if b is an upper bound for the set A. Following Fuchs ([5], p. 154), we set

$$P_l = \{a \in D \mid ax \geqslant x \text{ for all } x \in D\},$$
$$P_r = \{a \in D \mid xa \geqslant x \text{ for all } x \in D\}$$

and

$$N_r = \{a \in D \mid xa \leqslant x \text{ for all } x \in D\},$$
$$N_l = \{a \in D \mid ax \leqslant x \text{ for all } x \in D\}.$$

Also P_r^*, P_l^*, N_r^*, N_l^* are defined similarly with the inequality being strict and $P = P_l \cap P_r$, $N = N_l \cap N_r$, $P^* = P_l^* \cap P_r^*$, $N^* = N_l^* \cap N_r^*$. A partially ordered semigroup(group) which is a directed set is called a *directed semigroup(group)*. However, for us a useful generalization of directed groups to semigroups is given by:

DEFINITION 2.6.

(i) A *positively oriented* semigroup D is a p.o. semigroup with at least two elements, such that for all nonempty finite sets A contained in D there exists u, x in D for which $A \leqslant u$ and $Ax \subseteq P_l$.

(ii) A *negatively oriented* semigroup D is a p.o. semigroup with at least two elements, such that for all nonempty finite sets A contained in D there exist v, y in D for which $v \leqslant A$ and $Ay \subseteq N_l$.

(iii) A semigroup D is *oriented* if it is either positively or negatively oriented.

PROPOSITION 2.7. *Let G be a group and let $|G| > 1$. Then G is directed if and only if G is oriented.*

Proof. Let G be directed and suppose $\phi \neq A \subseteq G$, where A is finite. There exist v, u in G such that $v \leqslant A \leqslant u$. Hence $Av^{-1} \geqslant 1$ and so $Av^{-1} \subseteq P_l$, whence G is oriented. The converse is trivial.

Remark. An example of a strict oriented semigroup which is not directed is furnished by the additive half-plane $\{(x_1, x_2) : x_1 \geqslant 0\}$ with $(x_1, x_2) < (y_1, y_2)$ if and only if $x_1 < x_2$ and $y_1 < y_2$. Next let D be the upper left quadrant of the plane under addition, $D = \{(x_1, x_2) \mid x_1 \leqslant 0, x_2 \geqslant 0\}$ with $(x_1, x_2) \leqslant (y_1, y_2)$ if and only if $x_1 \leqslant y_1$ and $x_2 \leqslant y_2$. Then $P = \{(0, x_2)\}$, $N = \{(x_1, 0)\}$ and D is a commutative cancelative (hence strict) partially ordered semigroup with identity element $(0, 0)$ which is directed but not oriented. This last example is due to Charles Holland.

In a strict fully ordered semigroup D, $P_l = P_r = P$, $N_l = N_r = N$, and $D = P \cup N^* = N \cup P^*$ (Fuchs [5], p. 159).

PROPOSITION 2.8. *Every strict fully ordered semigroup with more than one element is oriented.*

Proof. Let D be strict fully ordered and suppose D is not oriented. We shall derive a contradiction. Since Definition 2.6(i) is false, there exists a finite subset A of D, such that, for all $x \in D$, $Ax \cap N \neq \phi$. Let $a = \min A$. Then for all $x \in D$, $ax \in N^*$. Similarly, by negating Definition 2.6(ii), there exists b in D such that for all y in D, $by \in P^*$. It follows that $ab \in N^*$ and $ba \in P^*$, whence $bab < b$ and $bab > b$—a contradiction.

DEFINITION 2.9. If D is a p.o. semigroup, let $\mathscr{K}(A) = \{x \in D \mid$ there exist $c, c' \in \langle A \rangle$ with $x \leqslant c$ and $c'x \in P_l\}$.

LEMMA 2.10. *Let A be a nonempty subset of the* p.o. *semigroup D.*

(i) *If $\mathscr{K}(A) \neq \phi$, then $\mathscr{K}(A)$ is grouplike in D.*
(ii) *If $A \subseteq P_l$, then $\langle\langle A \rangle\rangle \subseteq \mathscr{K}(A)$.*

Proof. (i) If $x, y \in \mathscr{K}(A)$, then there exist c, c', d, d' in $\langle A \rangle$ such that $x \leqslant c$, $c'x \in P_l$ and $y \leqslant d$, $d'y \in P_l$. Hence $d'c'xy \geqslant d'y$, whence $d'c'xy \in P_l$.

Also $xy \leqslant cd$ and hence, since $d'c'$ and cd are in $\langle A\rangle$, $xy \in \mathscr{K}(A)$. Thus, Definition 2.1(a) is satisfied. If x, $xy \in \mathscr{K}(A)$, then there exist $c'c'$, f, f' in $\langle A\rangle$ such that $x \leqslant c$, $c'x \in P_l$, and $xy \leqslant f$, $f'xy \in \mathscr{P}_l$. Hence $f'cy \geqslant f'xy$, whence $f'cy \in P_l$. Also $y \leqslant c'xy \leqslant c'f$ and hence, since $c'f$ and $f'c$ are in $\langle A\rangle$, $y \in \mathscr{K}(A)$. Thus Definition 2.1(b) is satisfied and $\mathscr{K}(A)$ is grouplike.

(ii) Let $x \in A \subseteq P_l$. Then $c = x$ and $c' = x$ satisfy the requirements of Definition 2.9. Hence $A \subseteq \mathscr{K}(A)$ and so by (i), $\langle\!\langle A\rangle\!\rangle \subseteq \mathscr{K}(A)$.

Remark. *If* D *is a* p.o. *group and* $A \geqslant 1$ *then* $\mathscr{K}(A)$ *is the convex subgroup generated by* A.

LEMMA 2.11. *Let D be a strict* p.o. *semigroup. If D is positively oriented, then for all finite subsets A of D there exists an x in D such that $Ax \subseteq P_l^*$. If D is negatively oriented, then there exists an x in D such that $Ax \subseteq N_l^*$.*

Proof. Assume D is positively oriented. We first prove that $P_l^* \neq \phi$. There exist two distinct elements a, b in D. Either $a < b$ or $b < a$ or a is incomparable to b, in which case, by definition, there exists c such that $a < c$. Hence there exist two comparable elements g, h; say $g < h$. There exists x such that $gx \in P_l$. For all $y \in D$ $(hx)y > (gx)y \geqslant y$, whence $q = hx \in P_l^*$.

Now let A be a finite subset of D. For some z, $Az \subseteq P_l$. Hence $Azq \geqslant q$ and so $Azq \subseteq P_l^*$.

A similar argument proves the statement concerning negatively oriented semigroups.

PROPOSITION 2.12. *An oriented semigroup D with a strict partial order is a Γ semigroup.*

Proof. Assume D is positively oriented. Let A be any finite subset of D. By Lemma 2.11 there exist x, u in D such that $A \leqslant u$ and $Ax \subseteq P_l^*$. Then $ux \geqslant Ax$ so $ux \in P_l^*$ and also $ux < Axux \leqslant (ux)^2$. Letting $g = xux$, $a = ux$ we obtain $a < Ag \leqslant a^2$, where $Ag \subseteq P_l^*$ since $a \in P_l^*$. Note that $a^n < (Ag)^n \leqslant a^{2n}$.

Now suppose $B \subseteq \langle\!\langle Ag\rangle\!\rangle$ and B is finite. Then by Lemma 2.10(ii) $B \subseteq \mathscr{K}(Ag)$, since $Ag \subseteq P_l^*$. Hence for every $b_i \in B$ there exist c_i, c_i', in $\langle Ag\rangle$ for which $b_i \leqslant c_i$ and $c_i'b_i \in P_l$. The c_i and c_i' are words in elements of Ag, and suppose m is the maximum of the lengths of these words. Then $c_i \leqslant a^{2m}$ and $c_i' \leqslant a^{2m}$, for all $b_i \in B$. Hence $b_i \leqslant c_i \leqslant a^{2m}$, and we have shown that $B \leqslant a^{2m}$.

We next show that, for suitable n, $d \in (Ag)^n B$ implies that $d \nleqslant a^{2m}$. Since $c_i'b_i \in P_l$, and $a^{2m} > c_i'$ for all i, observe that $(Ag)^{2m}B \subseteq P_l$. Now let $n = 4m$, and suppose $d \in (Ag)^n B$. Then $d = hp$, where $h \in (Ag)^{2m}$ and $p \in a^{2m}B \subseteq P_l$. Since $(Ag)^{2m} > a^{2m}$, it follows that $h > a^{2m}$. Suppose $d \leqslant a^{2m}$. Then

$a^{2m}y < hy \leqslant hpy = dy \leqslant a^{2m}y$ for $y \in D$, and this is absurd. Hence $d \nleqslant a^{2m}$. Since $B \leqslant a^{2m}$, we conclude that $(Ag)^n B \cap B \neq \phi$.

COROLLARY 2.13. *A directed group with more than one element is a Γ semigroup.*

Proof. Obvious by Proposition 2.7.

LEMMA 2.14. *Let D be a semigroup such that for every finite nonempty subset of A there exists an element h, such that for some semigroup D' with $\langle Ah \rangle \subseteq D' \subseteq D$, there is a homomorphism φ, taking D' onto a Γ semigroup. Then D is a Γ semigroup.*

Proof. Let A be a finite nonempty subset of D. Let φ, h, and D' be as in the hypotheses. Let F be the epimorphic image of D'. There exists $\bar{f}$ in F such that for all finite sets $\bar{B}$ contained in $\langle\langle \varphi(Ah)\bar{f} \rangle\rangle$ there exists n, such that $(\varphi(Ah)\bar{f})^n \bar{B} \cap \bar{B} = \phi$. Now φ is onto, so there exists $f \in D'$ with $\varphi(f) = \bar{f}$. Let $g = hf$. Then if B is a finite set such that $B \subseteq \langle\langle Ag \rangle\rangle$, we have, by Proposition 2.3, $\varphi(B) \subseteq \varphi(\langle\langle Ag \rangle\rangle) \subseteq \langle\langle \varphi(Ag) \rangle\rangle = \langle\langle \varphi(Ah)\bar{f} \rangle\rangle$. Hence for some n, $(\varphi(Ag))^n \varphi(B) \cap \varphi(B) = \phi$, and therefore $(Ag)^n B \cap B = \phi$.

COROLLARY 2.15. *The inverse image of a Γ-semigroup is Γ.*

COROLLARY 2.16. *If every finitely generated subsemigroup (grouplike subset) of D is the inverse image of a Γ-semigroup, then D is Γ.*

Remark. There exist Γ groups with subgroups which are not Γ, e.g., $Z \oplus Z_2$, where Z are the integers and Z_2 the integers (mod 2) under addition.

DEFINITION 2.17. A semigroup D is said to be a 2Ω-*semigroup* if and only if for all pairs of finite nonempty subsets A, B of D with $|A| + |B| \geqslant 3$, there exist at least two elements c in AB which admit exactly one representation $c = ab$, with $a \in A$, $b \in B$. We say that such $c \in AB$ is uniquely expressible with respect to A, B.

Remark. In [11], Rudin and Schneider defined an Ω-group to be a group such that for all pairs of finite nonempty sets A, B there exists at least one element in AB uniquely expressible with respect to A, B. Earlier, Kemperman [7], conjectured that every torsion-free group is 2Ω. If this is so, then the three concepts coincide for groups, for a 2Ω-group is certainly Ω, and it is easily seen that Ω implies torsion-free. We also note that every 2Ω-semigroup D is cancelative and $n \neq m$ and $a^n = a^m$ imply a is an idempotent. If D has 1, then 1 is the only idempotent in D. In addition one can show that every fully right ordered cancelative semigroup is 2Ω.

In particular, a fully right ordered group is 2Ω. We do not know if such a group is Γ.

LEMMA 2.18. *Let D be a cancelative semigroup. Suppose that for all pairs of finite nonempty subsets A, B of D with $|A| + |B| \geqslant 3$ there exist elements h, k, and a homomorphism φ taking $\langle hA, Bk\rangle$ into a 2Ω-semigroup such that $|\varphi(hA)| + |\varphi(Bk)| \geqslant 3$. Then D is a 2Ω-semigroup.*

Proof. Let A and B be finite sets with $n = |A| + |B| \geqslant 3$. We must show that there exist two elements in AB uniquely expressible with respect to A, B. We proceed by induction on n.

If $n = 3$, the cancellation laws guarantee that the two elements in AB are uniquely expressible. Suppose $n > 3$. Let h, k be as in the hypotheses. Let $E = hA$, $F = Bk$. Since the cancellation laws hold, it is sufficient to prove that there exist two elements in EF uniquely expressible with respect to E, F. By assumption, $|\varphi(E)| + |\varphi(F)| \geqslant 3$; so, for $i = 1, 2$, there exist $r_i \in \varphi(E)$, $s_i \in \varphi(F)$ such that $r_1s_1 \neq r_2s_2$ and r_is_i are uniquely expressible with respect to $\varphi(E)$, $\varphi(F)$. Clearly, either $r_1 \neq r_2$ or $s_1 \neq s_2$. Suppose $r_1 \neq r_2$. Let $E_i = \varphi^{-1}(r_i) \cap E$, $F_i = \varphi^{-1}(s_i) \cap F$, for $i = 1, 2$. Then $E_i \neq E$ and $|E_i| + |F_i| < n$, for $i = 1, 2$. Either $|E_i| = |F_i| = 1$ or $|E_i| + |F_i| \geqslant 3$ and the induction hypothesis applies. In either case there exist $e_i \in E_i$, $f_i \in F_i$ such that $e_1f_1 \neq e_2f_2$ and e_if_i is uniquely expressible with respect to E_i, F_i for $i = 1, 2$. But then e_if_i is uniquely expressible with respect to E, F. For, if $ef = e_if_i$, then $\varphi(e)\varphi(f) = \varphi(e_i)\varphi(f_i) = r_is_i$ and hence $\varphi(e) = r_i$ and $\varphi(f) = s_i$. Thus $e \in E_i$, $f \in F_i$ and therefore $e = e_i$, $f = f_i$.

DEFINITION 2.19. The semigroup D is a *$2\Omega\Gamma$-semigroup* if it is both 2Ω and Γ.

By combining Lemmas 2.14 and 2.18 we have the following.

PROPOSITION 2.20. *Let D be a cancelative semigroup. Suppose that for all pairs of finite nonempty subsets A, B of D with $|A| + |B| \geqslant 3$ there exist elements h, k, a subsemigroup D' of D containing $\langle hA, Bk\rangle$, and a homomorphism taking D' onto a $2\Omega\Gamma$-semigroup, such that $|\varphi(hA)| + |\varphi(Bk)| \geqslant 3$. Then D is a $2\Omega\Gamma$-semigroup.*

COROLLARY 2.21. *Let G be a group and suppose every finitely generated subgroup of G can be mapped homomorphically onto a $2\Omega\Gamma$-group. Then G is a $2\Omega\Gamma$-group.*

Proof. Let A, B be finite nonempty subsets of G with $|A| + |B| \geqslant 3$. Choose $h^{-1} \in A$, $k^{-1} \in B$. Then $1 \in hA$, $1 \in Bk$. Let φ be a homomorphism taking $\langle\langle hA, Bk\rangle\rangle$ onto a $2\Omega\Gamma$-group. Then $\varphi(\langle\langle hA, Bk\rangle\rangle) \neq \{1\}$ so there exists x in $hA \cup Bk$ such that $\varphi(x) \neq 1$. Hence $|\varphi(hA)| + |\varphi(Bk)| \geqslant 3$and the corollary follows from Proposition 2.20.

THEOREM 2.22. *Every strict fully ordered semigroup D with more than one element is a $2\Omega\Gamma$-semigroup.*

Proof. By Proposition 2.8, D is oriented, and hence by Proposition 2.12 it is Γ. Let A, B be nonempty finite subsets of D and let $a_1 = \max A$, $b_1 = \max B$, $a_2 = \min A$, $b_2 = \min B$. Then a_1b_1, a_2b_2 are uniquely expressible with respect to A, B, whence D is 2Ω.

We now show that a rather wide class of groups consists of $2\Omega\Gamma$-semigroups. A *normal system* for a group is a complete ordered system of subgroups $\{N_\alpha\}$ such that, whenever α has successor $\alpha + 1$, N_α is normal in $N_{\alpha+1}$. A group G is an *SN-group* if the factors $N_{\alpha+1}/N_\alpha$ are Abelian (cf. Kurosh [8], pp. 171 and 182).

The following theorem includes the case of SN-groups with torsion-free Abelian factors since torsion-free Abelian groups can be fully ordered (cf. Fuchs [5], p. 36).

THEOREM 2.23. *If G is a group with a normal system with fully ordered factor groups, then G is a $2\Omega\Gamma$-semigroup.*

Proof. Let H be a finitely generated subgroup of G, say $H = \langle\langle g_1, \ldots, g_n \rangle\rangle$. Let $\{N_\alpha\}$ be a normal system of G with fully ordered factors. Then for $i = 1, \ldots, n$ there exist N_{α_i} such that $g_i \notin N_{\alpha_i}$, but $g_i \in N_{\alpha_i+1}$. Let $\alpha = \max\{\alpha_i \mid i = 1, \ldots, n\}$. Then $N_\alpha \cap H$ is a proper normal subgroup of H and $H \subseteq N_{\alpha+1}$. Then group $H' = H/(N_\alpha \cap H)$ is fully ordered since $H(N_\alpha \cap H) \cong HN_\alpha/N_\alpha \subseteq N_{\alpha+1}/N_\alpha$, and $N_{\alpha+1}/N_\alpha$ is fully ordered. Further H' is nontrivial since $N_\alpha \cap H \neq H$. Hence by Theorem 2.22, H' is $2\Omega\Gamma$. The result now follows from Corollary 2.21.

3. SEMIGROUP RINGS

If D is a semigroup and R is an associative ring, let RD denote the semigroup ring of D over R. Thus RD consists of all functions from D into R which are zero off a finite set. We write elements of RD as finite formal sums $x = \alpha_1 d_1 + \cdots + \alpha_n d_n$, where $x(d_i) = \alpha_i \in R$. Addition is pointwise and multiplication is convolution. Thus, if $x, y \in RD$, then

$$xy(d) = \sum_{ab=d} x(a)y(b).$$

The support of x, written Supp(x), is $\{d \in D \mid x(d) \neq 0\}$. We write Coeff($x$) for the range of x. Note that although the semigroup D may have a zero, $\alpha d = 0$ if and only if $\alpha = 0$.

If A is a subset of the ring R, we write $[A]$ for the subring generated by the elements of A. As usual, $x \circ y = x + y - xy$. An element x of R is right-quasi-invertible if there exists x' in R such that $x \circ x' = 0$, or equivalently where R has identity if and only if $1 - x$ is right-invertible. Let $\mathscr{J}(R)$ denote the Jacobson radical of R (cf. McCoy [9], p. 112). Every element x of $\mathscr{J}(R)$ has a unique right quasi-inverse x', and $x' \in \mathscr{J}(R)$. By $\mathscr{P}(R)$ we denote the prime or McCoy radical [9]. It is well known ([9], p. 70), that $\mathscr{P}(R)$ is a nil ideal. By $\mathscr{U}(R)$ we denote the upper nil radical, namely the union of all nil ideals in R and by $\mathscr{L}(R)$ we denote the Levitzki radical, the union of all locally nilpotent ideals. The following relationships hold:

$$\mathscr{P}(R) \subseteq \mathscr{L}(R) \subseteq \mathscr{U}(R) \subseteq \mathscr{J}(R).$$

LEMMA 3.1. *Let r be a nonzero element in RD. If $r \circ x = 0$, then $r \circ y = 0$ for some y such that* $\operatorname{Supp}(y) \subseteq \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$.

Proof. Define y by

$$y(d) = \begin{cases} x(d) & \text{if} \quad d \in \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle \\ 0 & \text{otherwise.} \end{cases}$$

Case 1. $g \in \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$. Then $y(g) = x(g)$. Let $ab = g$. If $r(a) = 0$, then $r(a)x(b) = 0 = r(a)y(b)$. If $r(a) \neq 0$, then $a, ab \in \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$, whence $b \in \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$ and $y(b) = x(b)$. Hence, in either case $r(a)x(b) = r(a)y(b)$. Thus

$$\begin{aligned} (r \circ y)(g) &= r(g) + y(g) + \sum_{ab=g} r(a)y(b) \\ &= r(g) + x(g) + \sum_{ab=g} r(a)x(b) = (r \circ x)(g) = 0. \end{aligned}$$

Case 2. $g \notin \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$. Then $r(g) = y(g) = 0$. Let $ab = g$. Then either $a \notin \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$ or $b \notin \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$, so $r(a)y(b) = 0$. Hence $r \circ y(g) = 0$.

The following corollary was observed for group algebras by Amitsur [2].

COROLLARY 3.2. *If both R and D have identity and $rx = 1$ then $ry = 1$ for some y such that* $\operatorname{Supp}(y) \subseteq \langle\!\langle \operatorname{Supp}(r) \rangle\!\rangle$.

Proof. The proof follows immediately by letting

$$r'' = 1 - r, \qquad x'' = 1 - x \qquad \text{since} \qquad \langle\!\langle \operatorname{Supp}(1 - z) \rangle\!\rangle = \langle\!\langle \operatorname{Supp} z \rangle\!\rangle \quad \text{for all} \quad z \in RD.$$

LEMMA 3.3. *Let D be a Γ semigroup. Let R be a ring with identity and S a subring of R. If $y \in \mathscr{J}(RD) \cap SD$, then there exists $x \in \mathscr{J}(RD) \cap SD$ such that*

(i) *the quasi-inverse x' of x is in $\mathscr{J}(RD) \cap SD$;*
(ii) *if D has identity* 1, *then* $1 \notin \operatorname{Supp}(x)$ *and;*
(iii) $\operatorname{Coeff}(x) = \operatorname{Coeff}(y)$.

Proof. If $y = 0$, then let $x = 0$. If $y \neq 0$, let $A = \mathrm{Supp}(y)$. Clearly $A \neq \phi$, whence by Definition 2.4 there exists g in D such that, for every finite $B \subseteq \langle\!\langle Ag \rangle\!\rangle$, there exists $n = n(B)$ for which $(Ag)^n B \cap B = \phi$.

Let $x = yg$. Then $x \in \mathscr{J}(RD) \cap SD$. If D has identity 1, then $1 \notin Ag = \mathrm{Supp}(x)$. Whether or not D has 1, $\mathrm{Coeff}(x) = \mathrm{Coeff}(y)$.

We have found an $x \in \mathscr{J}(RD) \cap SD$ satisfying (ii) and (iii). To complete the proof we must show that the unique right quasi-inverse x' of x lies in SD.

By Lemma 3.1, $\mathrm{Supp}(x') \subseteq \langle\!\langle \mathrm{Supp}(x) \rangle\!\rangle = \langle\!\langle Ag \rangle\!\rangle$. Now $x + x' - xx' = 0$, whence $x' = -x + xx'$. Iterating we obtain, $x' = -x - x^2 - \cdots - x^n + x^n x'$. Now letting $B = \mathrm{Supp}(x')$ we see that for $n = n(B)$ we have

$$(\mathrm{Supp}(x))^n (\mathrm{Supp}(x')) \cap \mathrm{Supp}(x') = (Ag)^n B \cap B = \phi.$$

But $\mathrm{Supp}(x^n x') \subseteq (\mathrm{Supp}(x))^n (\mathrm{Supp}\, x')$. Hence $\mathrm{Supp}(x^n x') \cap \mathrm{Supp}\, x' = \phi$. Then if $d \in \mathrm{Supp}(x')$, $x^n x'(d) = 0$, whence $x'(d) = (-x - \cdots - x^n)(d)$ and so $\mathrm{Coeff}(x') \subseteq [\mathrm{Coeff}(x)] = [\mathrm{Coeff}(y)] \subseteq S$. Therefore $x' \in SD$.

LEMMA 3.4. *If the ring R with identity has no zero divisors and the semigroup D with identity is 2Ω, then all the units in RD have one point support (i.e., if x is a unit then $|\mathrm{Supp}(x)| = 1$).*

Proof. Suppose $x, y \in RD$ and $xy = 1$. Let $A = \mathrm{Supp}(x)$, $B = \mathrm{Supp}(y)$. If $|A| > 1$, then $|A| + |B| \geqslant 3$, so that there exists $a \in A$, $b \in B$ such that $ab \neq 1$ and ab is uniquely expressible with respect to A, B. Hence $xy(ab) = x(a)y(b) \neq 0$. This is a contradiction. Hence $|A| = 1$.

LEMMA 3.5. *Let D be a semigroup and R a ring and let R^1 be the canonical ring extension of R having an identity element. Then $\mathscr{J}(R^1D) = \{0\}$ implies $\mathscr{J}(RD) = \{0\}$.*

Proof. Clearly RD is an ideal of R^1D and hence ([9], p. 115) $\mathscr{J}(RD) = RD \cap \mathscr{J}(R^1D)$.

THEOREM 3.6. *Let D be a $2\Omega\Gamma$-semigroup with 1, and let R be a ring. Then $\mathscr{U}(R) = \{0\}$ implies $\mathscr{J}(RD) = \{0\}$.*

Proof. Note that $\mathscr{U}(R) = \{0\}$ implies $\mathscr{U}(R^1) = \{0\}$, so by Lemma 3.5 we may assume R has an identity element. If $\mathscr{J}(RD) \neq \{0\}$, pick a nonzero w in $\mathscr{J}(RD)$ of minimal support. Suppose $w = \gamma_1 g_1 + \cdots + \gamma_n g_n$.

Since $\mathscr{U}(R) = \{0\}$ the ideal generated by γ_1 is not a nil ideal, so putting $y = \sum_i \sigma_i w \tau_i = \alpha_1 g_1 + \cdots + \alpha_n g_n$ with σ_i, τ_i in R, we may assume α_1 is not nilpotent. Also $\alpha_i y - y\alpha_i \in \mathscr{J}(RD)$ and has less support than y. Hence $\alpha_i y - y\alpha_i = 0$. Therefore $\alpha_i \alpha_j = \alpha_j \alpha_i$ for all i, j, and hence $S = [1, \alpha_1, \ldots, \alpha_n]$ is a commutative ring with identity and $y \in \mathscr{J}(RD) \cap SD$. By Lemma 3.3, there exists $x \in \mathscr{J}(RD) \cap SD$ with (i) $x' \in SD$, (ii) $1 \notin \operatorname{Supp} x$, and (iii) $\alpha_1 \in \operatorname{Coeff}(x)$. Since $\mathscr{P}(S)$ is a nil ideal, and α_1 is not nilpotent, there exists a prime ideal P in S such that $\alpha_1 \notin P$. Let $\bar{S} = S/P$ and let $\bar{\alpha}$ denote the image of α under the canonical homomorphism. The mapping taking $z = \beta_1 d_1 + \cdots + \beta_n d_n$ into $\bar{z} = \bar{\beta}_1 d_1 + \cdots + \bar{\beta}_n d_n$ is a homomorphism of SD onto $\bar{S}D$. Hence $(1 - x)(1 - x') = 1$ implies $(\bar{1} - \bar{x})(\bar{1} - \bar{x}') = \bar{1}$. Now since $\bar{\alpha}_1 \neq 0$ and $1 \notin \operatorname{Supp} x$, $\bar{1} - \bar{x}$ has at least a two-point support, contradicting Lemma 3.4, since $\bar{S}$ is an integral domain. Hence $\mathscr{J}(RD) = \{0\}$.

Remark. If either R is commutative or R has no zero divisors, then our proof can be modified to show that the conclusion of Theorem 3.6 holds for all 2Ω-semigroups with 1.

COROLLARY 3.7. *If D is a $2\Omega\Gamma$-semigroup and R a ring, then $\mathscr{J}(RD) \subseteq \mathscr{U}(R)D$. Equality holds if $\mathscr{L}(R) = \mathscr{U}(R)$.*

Proof. $RD/\mathscr{U}(R)D$ is isomorphic to $(R/\mathscr{U}(R))D$ which has zero Jacobson radical by Theorem 3.6. Hence $\mathscr{J}(RD) \subseteq \mathscr{U}(R)D$. In general, $\mathscr{L}(R)D \subseteq \mathscr{J}(RD)$, for $\mathscr{L}(R)D$ is a nil ideal in RD. Hence if $\mathscr{L}(R) = \mathscr{U}(R)$, $\mathscr{J}(RD) = \mathscr{U}(R)D$.

COROLLARY 3.8. *If D is a strict fully ordered semigroup with identity and with more than one element, then $\mathscr{U}(R) = \{0\}$ implies $\mathscr{J}(RD) = \{0\}$,*

Proof. Immediate by Theorem 2.22.

COROLLARY 3.9. (Amitsur [*1*]) *$\mathscr{U}(R) = \{0\}$ implies $\mathscr{J}(R[t]) = \{0\}$.*

Proof. $R[t]$ is just the semigroup ring of R over the cyclic semigroup which is strictly ordered and hence the result follows by Corollary 3.8.

COROLLARY 3.10. *If G is an SN group with a normal system whose factors are Abelian torsion-free, then $\mathscr{U}(R) = \{0\}$ implies $\mathscr{J}(RG) = \{0\}$.*

Proof. Immediate by Theorem 2.23.

COROLLARY (Bovdi [*3*]) 3.11. *If G is an SN group with a normal system whose factors are Abelian torsion-free and R a ring without zero divisors, then $\mathscr{J}(RG) = \{0\}$.*

Proof. Obviously $\mathscr{U}(R) = \{0\}$.

COROLLARY 3.12. *If G is a torsion-free Abelian group and $\mathscr{U}(R) = \{0\}$, then $\mathscr{J}(RG) = \{0\}$.*

4. The Upper Nilradical of Group Rings

In this section, we shall show that, under very simple conditions, relating the ring R and orders of the elements of a group G, $\mathcal{U}(R) = \{0\}$ implies $\mathcal{U}(RG) = \{0\}$. An element in a group G is a p-element if it is of order p^k, for some $k > 0$.

Lemma 4.1. (Passman [10]): *Let S be a commutative ring, and let G be a group. Let $q = p^k$, where p is a prime. If $x \in SG$, and if* Supp x *contains no p-element, then $x^q(1) = x(1)^q - p\beta$, where $\beta \in S$.*

Proof. We observe that $x^q(1) = \sum \{x(g_1) \cdots x(g_q) \mid g_1 g_2 \cdots g_q = 1\}$. Now note that $g_1 g_2 \cdots g_q = 1$ implies that any cyclic permutation $g_{j+1} \cdots g_q g_1 \cdots g_j = 1$. It is easily seen that the number of distinct cyclic permutations divides q and hence this number is either 1 or p^l, $l \geqslant 1$. In the first case, all g_i are equal, whence by our assumption on Supp x, $g_i = 1$, $i = 1, \ldots, q$, and $x(g_1) \cdots x(g_q) = x(1)^q$. In the second case, the sum of coefficients over all cyclic permutations of $g_1, \ldots, g_q$ is $p^l x(g_1) \cdots x(g_q)$. The lemma follows.

Lemma 4.2. *Let F be a field, let G be a group, and let $x \in FG$. If* char $F = p > 0$, *suppose that* Supp x *contains no p-element. If x is nilpotent, then $x(1) = 0$.*

(Remark: Note that if char $F = 0$, Supp x may contain elements of any order.)

Proof. Case 1: char $F = p > 0$. Let $q = p^k$, and suppose q is sufficiently large so that $x^q = 0$. By Lemma 4.1, $0 = x^q(1) = x(1)^q$ since $p\beta = 0$, whence $x(1) = 0$.

Case 2. char $F = 0$. Let Q be the rational field, and let K be the extension of Q given by $K = Q(\{x(g) \mid g \in G\}) = Q(\{x(g) \mid g \in \operatorname{Supp} x\})$. Since K is finitely generated over Q, it follows from standard field-theoretic results ([12], Vol. I, Chap. 5; [13], Vol. I, Chap. 2) that $K = K'(\beta)$ where $K' = Q(\alpha_1, \ldots, \alpha_m)$, the α_i are algebraically independent over Q, and β is algebraic over K'. Let $I' = Z[\alpha_1, \ldots, \alpha_m]$ where Z denotes the rational integers, and let I be the integral closure of I' in K. Clearly K' is the quotient field of I' and hence, for each $g \in G$ there is an $\sigma \in I'$, $\sigma \neq 0$, such that $\sigma x(g) \in I$ ([12], Vol I, p. 78). Thus there is a nonzero $\rho \in I'$ such that $y = \rho x \in IG$. If x is nilpotent, then so is y, and it is enough to prove that $y(1) = 0$.

Let $y(1) = \tau$, and let p be any prime larger than the order of every element of finite order in Supp y, and such that $y^p = 0$. By Lemma 4.1, $0 = y^p(1) = \tau^p - p\beta$, where $\beta \in I$, whence $\tau^p = p\beta$. For $\eta \in K$, denote by $N(\eta)$ the norm of η in K over K'. Then since I' is a unique factorization domain and therefore integrally closed in its quotient field, it follows that $N(\eta) \in I'$ whenever $\eta \in I$

([13], Vol. I, pp. 260, 261). Hence $N(\tau)^p = N(\tau^p) = N(p\beta) = N(p)N(\beta) = p^t N(\beta)$ where t is the degree of K over K'. Since I' is isomorphic to a polynomial ring in n indeterminates over Z, and p is prime in I', it follows that p divides $N(\tau)$ in I'. It follows that $N(\tau)$ is divisible by an infinity of primes. Since I' is a unique factorization domain, we deduce that $N(\tau) = 0$. We conclude that $y(1) = \tau = 0$, whence also $x(1) = 0$.

DEFINITION 4.3. (i) An integer n is called *cancelable* in the ring R if, for $\alpha \in R$, $n\alpha = 0$ implies $\alpha = 0$.

(ii) A group G is *cancelable with respect to* R if and only if every integer n such that G has an element of order n is cancelable in R.

THEOREM 4.4. *Let R be a ring, and G a group. Suppose that G is cancelable with respect to R. If $\mathscr{U}(R) = \{0\}$ then $\mathscr{U}(RG) = \{0\}$.*

Proof. Suppose $\mathscr{U}(R) = \{0\}$ but $\mathscr{U}(RG) \neq \{0\}$. As in the proof of Theorem (3.6) we produce a commutative ring S such that $x' = \alpha_1 g_1 + \cdots + \alpha_n g_n \in \mathscr{U}(RG) \cap SG$ and such that α_1 is not nilpotent. Then $x = x'g_1^{-1} \in \mathscr{U}(RG) \cap SG$ and $x(1)$ is not nilpotent. Since S is commutative, $\mathscr{P}(S)$ consists of all nilpotent elements of S, and $S/\mathscr{P}(S)$ is a subdirect sum of integral domains $I_i : S/\mathscr{P}(S) \cong \sum_s I_i$. ([9], pp. 70–72). Let $\alpha \to \bar{\alpha}$ be the natural homomorphism of S onto $S/\mathscr{P}(S)$, and $\bar{\alpha} \to \bar{\alpha}_i$ the natural projection of $S/\mathscr{P}(S)$ onto I_i, and let K_i be the quotient field of I_i. If Supp x has no element of prime power order, let $n = 1$; otherwise let n be the product of all primes q such that, for some $g \in \operatorname{Supp} x$, g is a q-element. The product of cancelable integers is cancelable, and so by our assumption n is cancelable in R and therefore in S. Set $\tau = x(1)$. Since τ is non-nilpotent, we have $n\tau \notin \mathscr{P}(S)$, and therefore there is an index j such that $n\bar{\tau}_j \neq 0$. If char $K_j = 0$, Lemma 4.2 applies to $F = K_j$. Suppose char $K_j = p > 0$. If $n = 1$, clearly Supp x contains no p-element. If $n > 1$, then p is prime to n, and again Supp x contains no p-element. Thus in every case, Lemma 4.2 applies to $\bar{x}_j = \sum \overline{x(g)}_j g$, which is nilpotent since $\sum z(g)g \to \sum \overline{z(g)}_j g$ is a homomorphism. Hence $\bar{\tau}_j = 0$, but this is a contradiction. We deduce that $\mathscr{U}(RG) = \{0\}$. The theorem is proved.

Remark. If a group G contains an element of order n, then G contains a p-element for every p dividing n. Hence Theorem 4.4 is unchanged if we merely suppose that all primes p for which G has a p-element are cancelable.

COROLLARY 4.5 (Passman [10]). *Let R be a commutative ring having no nonzero nilpotent elements. Suppose that* char $R = m \neq 0$ *and that G has no p-elements for any prime p dividing m. Then $\mathscr{U}(RG) = \{0\}$.*

Proof. If G has an element of order n, then the hypothesis implies that n and m are relatively prime. Hence $kn + lm = 1$ for some integers k, l. Thus $na = 0$ implies $a = 0$. Thus n is cancelable and we can apply Theorem 4.4.

COROLLARY 4.6 (Passman [10]). *Let R be a commutative ring without nonzero nilpotent elements. Suppose the additive group of R is torsion-free. Then, for any group G, $\mathscr{U}(RG) = \{0\}$.*

Proof. All integers are cancelable in R.

We conclude this paper with a theorem which improves a result of Connell [4].

THEOREM 4.7. *Let R be a ring and let G be an Abelian group with at least one element of infinite order. If $\mathscr{U}(R) = \{0\}$ and G is cancelable with respect to R, then $\mathscr{J}(RG) = \{0\}$.*

Proof. Suppose g is an element of infinite order in G and x is a nonzero element in $\mathscr{J}(RG)$. Let $H = \langle\langle \mathrm{Supp}(x) \cup \{g\} \rangle\rangle$. Then $x \in \mathscr{J}(RG) \cap RH \subseteq \mathscr{J}(RH)$ by Connell [4], Eq. (24). We will show that $\mathscr{J}(RH) = \{0\}$ and thereby obtain a contradiction.

The group H is finitely generated and contains an element of infinite order. Hence $H = A \times B$ where A is a finite Abelian group and B is a torsion-free Abelian group. Since RH is isomorphic to $(RA)B$ ([11], Theorem 1.4) it is enough to prove $\mathscr{J}((RA)B) = \{0\}$. Clearly A is cancelable with respect to R, and so by Theorem 4.4, $\mathscr{U}(RA) = \{0\}$. By Corollary 3.12, $\mathscr{J}((RA)B) = \{0\}$. This is the required contradiction, and hence $\mathscr{J}(RG) = \{0\}$.

COROLLARY 4.8. *Let R and G be commutative and suppose G has at least one element of infinite order. If $\mathscr{P}(R) = \{0\}$ and G is cancelable with respect to R then $\mathscr{J}(RG) = \{0\}$.*

Proof. If R is commutative, $\mathscr{P}(R) = \mathscr{U}(R)$.

Corollary 4.8 coincides with one direction of Connell [4], Theorem 6(ii).

COROLLARY 4.9. *Let G be an Abelian group with at least one element of infinite order and suppose R is a ring with $\mathscr{U}(R) = \{0\}$. Then RG is semiprimitive if and only if G is cancelable with respect to R.*

Proof. If G is cancelable with respect to R then RG is semiprimitive by Theorem 4.7. Conversely, $\mathscr{J}(RG) = \{0\}$ implies $\mathscr{P}(RG) = \{0\}$ and the rest of the proof coincides with lines 2–6 of Connell's Theorem 5 [4].

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