2:

OHM, J., and H. SCHNEIDER
Math. Zeitschr. 85, 373—381 (1964)

Matrices similar on a Zariski-open set*

By
JACK OHM and HANS SCHNEIDER

1. Introduction

(1.1) Let A, B be $n \times n$ matrices whose elements are functions holomorphic in a connected open subset V of the complex plane. The matrices A, B are called *pointwise similar on* V if for each $x \in V$ there exists a non-singular $n \times n$ matrix C_x with complex elements such that $B(x) = C_x^{-1} A(x)\, C_x$. They are *holomorphically similar on* V if there exists a matrix C of functions holomorphic on V for which $B(x) = C(x)^{-1} A(x)\, C(x)$, for all $x \in V$. WASOW [5] gives two related criteria (one due to OSTROWSKI) to determine when pointwise similarity on a neighborhood V of x_0 implies holomorphic similarity on a (possibly smaller) neighborhood of x_0.

(1.2) Here we shall generalize these theorems to the case when A and B are matrices with entries in an arbitrary commutative integral domain D with identity. Our chief tool for this purpose is the Zariski topology. The proof of WASOW's criterion goes through virtually unchanged. It is indeed possible to prove a slightly stronger theorem. When D is a unique factorization domain, we can give a simple proof of the generalization of OSTROWSKI's criterion. There are some difficulties, due to the absence of GAUSS' Lemma, when D is an arbitrary domain.

We have tried to make the paper readable to a person not familiar with the terminology of algebraic geometry, going into detail wherever possible and giving references when the details become lengthy. In particular we make explicit in Section 5 how one obtains WASOW's original theorem from our algebraic version of the theorem.

2. Terminology and notation

(2.1) Integral domain will mean commutative ring without zero divisors and with identity. Let D be an integral domain, and let Spec(D) denote the set of prime ideals in D, including (0) but excluding D itself. If E is an arbitrary non-empty subset of D, we denote by $Z(E)$ the set of prime ideals in Spec(D) which contain E. The sets $Z(E)$ constitute the closed sets of the Zariski topology on Spec(D). We shall write $Z^*(E) = \text{Spec}(D) - Z(E)$ for the open sets of Spec(D). For these notions see, for instance, GROTHENDIECK [3] p. 80.

* The research of the authors was supported by the Office of Naval Research under contract NONR-1202(20), by the National Science Foundation under grant NSF GP-2273, and by the Mathematics Research Center, United States Army, Madison, Wisconsin under Contract DA-11-022-ORD-2059.

From now on, all our topological terms will refer to this topology on Spec(D); for example, a neighborhood of $\mathfrak{p} \in \operatorname{Spec}(D)$ is an open subset of Spec(D) which contains $\mathfrak{p}$.

(2.2) If S is any non-empty subset of Spec(D) then $M = D - \bigcup \mathfrak{p}_\alpha$, $\mathfrak{p}_\alpha \in S$, is a multiplicative system in D; and we denote the quotient ring of D with respect to M by D_S. Thus D_S consists of all $b^{-1}a$, $a \in D$, $b \in M$. For information on quotient rings, we refer the reader to ZARISKI-SAMUEL [*6*].

If $\mathfrak{p} \in \operatorname{Spec}(D)$, we shall write $D(\mathfrak{p}) = D_\mathfrak{p}/\mathfrak{p} D_\mathfrak{p}$. Note that $D(\mathfrak{p})$ is a field which is canonically isomorphic to the quotient field of $D/\mathfrak{p}$.

If $a \in D_\mathfrak{p}$, then $a(\mathfrak{p})$ will denote the image of a under the canonical homomorphism $D_\mathfrak{p} \to D_\mathfrak{p}/\mathfrak{p} D_\mathfrak{p} = D(\mathfrak{p})$. Identifying $D(0)$ with the quotient field K of D, we shall write simply a rather than $a(0)$. More generally, if f is in the polynomial ring $D_\mathfrak{p}[t]$, then $f(\mathfrak{p})$ will denote the image of f under the canonical homomorphism $D_\mathfrak{p}[t] \to D(\mathfrak{p})[t]$. Observe also that for $a \in D$, the $\mathfrak{p}$ with $a(\mathfrak{p}) \neq 0$ form an open set.

We note that $a \in D_S$ is a unit in that ring if and only if $a(\mathfrak{q}) \neq 0$ for all $\mathfrak{q} \in S$; for $a = d/m$ with $d \in D$, $m \in M$, and $a^{-1} \in D_S$ if and only if $d \in M$, which is equivalent to $a(\mathfrak{q}) \neq 0$ for all $\mathfrak{q} \in S$. The subset S of Spec(D) is *dense* in Spec(D) if and only if $0 = \bigcap \mathfrak{p}_\alpha$, $\mathfrak{p}_\alpha \in S$. Hence if S is dense in Spec(D) and $a \in D_S$, then $a(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in S$ implies $a = a(0) = 0$.

(2.3) We shall denote the set of $m \times n$ matrices with elements in D by $D_{m \times n}$. If $A = (a_{ij}) \in D_{m \times n}$, then we define $A(\mathfrak{p}) = (a_{ij}(\mathfrak{p})) \in D(\mathfrak{p})_{m \times n}$. Note that if $S \subseteq \operatorname{Spec}(D)$ and $A \in (D_S)_{n \times n}$, then A has an inverse in $(D_S)_{n \times n}$ if and only if $d = \det A$ is a unit in D_S; and this is true if and only if $d(\mathfrak{q}) \neq 0$ for all $\mathfrak{q} \in S$. Thus A is invertible in $(D_S)_{n \times n}$ if and only if $A(\mathfrak{q})$ is non-singular for all $\mathfrak{q} \in S$, in which case $A(\mathfrak{q})^{-1} = A^{-1}(\mathfrak{q})$.

Let $A, B \in D_{n \times n}$ and let U be a non-empty subset of Spec(D).

(2.4) *Definition.* The matrix *A is similar to B on U* if there exists a $C \in (D_U)_{n \times n}$ such that

(a) C is invertible in $(D_U)_{n \times n}$;

(b) $B = C^{-1} A C$.

(2.5) *Remark.* Here (a) is equivalent to

(a′) $C(\mathfrak{q})$ is non-singular for all $\mathfrak{q} \in U$,

while (b) implies

(b′) $B(\mathfrak{q}) = C(\mathfrak{q})^{-1} A(\mathfrak{q})\, C(\mathfrak{q})$ for all $\mathfrak{q} \in U$.

If moreover U is dense in Spec(D), then (b′) is equivalent to (b).

(2.6) *Definition.* The matrix *A is pointwise similar to B on U*, if, for every $\mathfrak{q} \in U$, there exists a non-singular matrix $C_\mathfrak{q}$ in $D(\mathfrak{q})_{n \times n}$ such that $B(\mathfrak{q}) = C_\mathfrak{q}^{-1} A(\mathfrak{q}) C_\mathfrak{q}$.

If A is similar to B on U, then it follows from (b′) that A is pointwise similar to B on U. Our efforts will now be directed to determining criteria under which the converse implication holds.

3. Wasow's criterion

(3.1) **Lemma.** *Let* $A \in D_{m\times n}$ *and let*

$$W = \{\mathfrak{p} \in \operatorname{Spec}(D) \mid \operatorname{rank} A(\mathfrak{p}) = \operatorname{rank} A(0)\}.$$

Then W *is open, and for all* $\mathfrak{q} \in \operatorname{Spec}(D)$, $\operatorname{rank} A(\mathfrak{q}) \leqq \operatorname{rank} A(0)$.

Proof. Identify A and $A(0)$, and let $\operatorname{rank} A(0) = r$. If a is any (determinant of) minor of order s, $s > r$, then $a = 0$, hence also $a(\mathfrak{q}) = 0$, for all $\mathfrak{q} \in \operatorname{Spec}(D)$. Hence $\operatorname{rank} A(\mathfrak{q}) \leqq \operatorname{rank} A$. Some minor of order r of A does not vanish, so let $d_1, \ldots, d_k$ be the non-vanishing minors of order r. Clearly $\operatorname{rank} A(\mathfrak{p}) = r$ if and only if $d_i(\mathfrak{p}) \neq 0$ (i.e. $\mathfrak{p} \in \boldsymbol{Z}^*(d_i)$) for some i. Thus $\operatorname{rank} A(\mathfrak{p}) = r$ if and only if

$$\mathfrak{p} \in \bigcup_{i=1}^{k} \boldsymbol{Z}^*(d_i) = W.$$

Hence W is open.

We shall say rank A is constant on a subset U of $\operatorname{Spec}(D)$ if $\operatorname{rank} A(\mathfrak{p}) = \operatorname{rank} A(\mathfrak{q})$ for all $\mathfrak{p}, \mathfrak{q} \in U$.

(3.2) **Lemma.** *Let* $A \in D_{m\times n}$, *and suppose rank* A *is constant on a dense subset* V *of* $\operatorname{Spec}(D)$. *Suppose* $\mathfrak{p} \in V$ *and that the vector* $x_\mathfrak{p} \in D(\mathfrak{p})_{n\times 1}$ *satisfies* $A(\mathfrak{p})\, x_\mathfrak{p} = 0$. *Then there exists a neighborhood* U *of* $\mathfrak{p}$ *and a vector* $x \in (D_U)_{n\times 1}$ *such that* $x(\mathfrak{p}) = x_\mathfrak{p}$ *and* $Ax = 0$.

Proof. Let W be the open set of lemma 3.1 on which rank A is maximal. Since an open set and a dense set have a non-empty intersection, $V \cap W \neq \Phi$; whence $\operatorname{rank} A(\mathfrak{p}) = \operatorname{rank} A = r$ on V. There exists therefore an $r\times r$ submatrix A_1 of rank r such that $d = \det A_1$ has $d(\mathfrak{p}) \neq 0$.

We may suppose that A_1 is the leading $r\times r$ submatrix of A, and let B be the leading $(r+1)\times(r+1)$ submatrix of A. For $i = 1, \ldots, r+1$ let x_i^1 be the cofactor of $a_{r+1,i}$ in B, and put $x_i^1 = 0$, $i = r+2, \ldots, n$. If $x^1 = (x_1^1, \ldots, x_n^1) \in D_{n\times 1}$, then $Ax^1 = 0$ and $x_{r+1}^1 = \pm d$. Similarly there exist $x^2, \ldots, x^{n-r} \in D_{n\times 1}$ for which $Ax^i = 0$ and $x_{r+i}^i = \pm d$, but $x_j^i = 0$, $j \neq i$, $j = r+1, \ldots, n$. Since $d(\mathfrak{p}) \neq 0$, the vectors $x^1(\mathfrak{p}), \ldots, x^{n-r}(\mathfrak{p}) \in D(\mathfrak{p})_{n\times 1}$ are linearly independent, and as $\operatorname{rank} A(\mathfrak{p}) = r$, there must exist $\bar{c}_i \in D(\mathfrak{p})$, $i = 1, \ldots, n-r$, such that

$$x_\mathfrak{p} = \sum_1^{n-r} \bar{c}_i\, x^i(\mathfrak{p}).$$

Choose $c_i \in D_\mathfrak{p}$ so that $c_i(\mathfrak{p}) = \bar{c}_i$, and put

$$x = \sum_1^{n-r} c_i\, x^i.$$

Then $Ax = 0$ and $x(\mathfrak{p}) = x_\mathfrak{p}$.

Finally, since $c_i \in D_\mathfrak{p}$, we can write $c_i = c_i'/d_i$, where $c_i', d_i \in D$, $d_i \notin \mathfrak{p}$. Let then U be the neighborhood of $\mathfrak{p}$ consisting of all primes in $\operatorname{Spec}(D)$ which do not contain the set $\{d_1, \ldots, d_{n-r}\}$. Then $x \in (D_U)_{n\times 1}$.

(3.3) Let A be an $n\times n$ matrix with elements in a field, say $A \in K_{n\times n}$. Let $g_r \in K[t]$ be the g.c.d. (greatest common divisor) of minors of order r of $A - tI$,

t an indeterminate. The invariant factors q_r of A (really $A-tI$) are the quotients $q_r = g_{n-r+1}/g_{n-r}$, $r=1, \ldots, n (g_0=1)$. If ρ_r is the degree of q_r, then $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$, and

$$\sum_{i=1}^{n} \rho_i = n .$$

If B is similar to A over K, then it is known that the solution space for $C \in K_{n \times n}$ of $AC - CB = 0$ has dimension

(3.4) $$\tau\langle A\rangle = \tau = \rho_1 + 3\rho_2 + \cdots + (2n-1)\rho_n = n + 2(\sigma_1 + \sigma_2 + \cdots + \sigma_{n-1}),$$

where

(3.5) $$\sigma_r = \deg g_r = \sum_{i=n-r+1}^{n} \rho_i$$

(e.g. MACDUFFEE [4], p. 92, 93, GANTMACHER [2], p. 222).

If $A(\mathfrak{p}) \in D(\mathfrak{p})$, denote the corresponding quantities by $\rho_r(\mathfrak{p})$, $\sigma_r(\mathfrak{p})$, $\tau(\mathfrak{p})$. We shall say that τ is constant on $U \subseteq \operatorname{Spec}(D)$ if for all $\mathfrak{p}, \mathfrak{q} \in U$, $\tau(\mathfrak{p}) = \tau(\mathfrak{q})$.

(3.6) **Theorem.** *(Generalization of Wasow's Theorem) Let $A, B \in D_{n \times n}$ and let V be a dense subset of* Spec(D). *Suppose that for the matrix A, $\tau\langle A\rangle$ defined by* 3.4 *is constant on V. If A is pointwise similar to B on V, and if $\mathfrak{p} \in V$, then there exists a neighborhood U of $\mathfrak{p}$ such that A is similar to B on U.*

Proof. If $X \in D_{n \times n}$, the linear transformation $L: X \to AX - XB$ over D may be identified with a suitable matrix representative in $D_{n^2 \times n^2}$. For $\mathfrak{q} \in V$ the nullspace of $L(\mathfrak{q})$ has dimension $\tau(\mathfrak{q})$ given by 3.4. By assumption there exists a non-singular $C_{\mathfrak{p}} \in D(\mathfrak{p})_{n \times n}$ such that $AC_{\mathfrak{p}} - C_{\mathfrak{p}}B = 0$. Hence applying Lemma 3.2 to the matrix representative for L, there exists a neighborhood W of $\mathfrak{p}$ and a $C \in (D_W)_{n \times n}$ such that $C(\mathfrak{p}) = C_{\mathfrak{p}}$, and $AC - CB = 0$. Since $d = \det C \in D_W$, $d = b^{-1}a$ where $a, b \in D$ and b is a unit in D_W. Let $U = W \cap \boldsymbol{Z}^*(a)$. Then U is a neighborhood of $\mathfrak{p}$ because $d(\mathfrak{p}) \neq 0$; and also $D_W \subset D_U$, and d is unit of D_U. Thus, C is invertible in $(D_U)_{n \times n}$; and hence B is similar to A on U.

(3.7) *Remark.* If D is a principal ideal domain, a non-empty subset U of Spec(D) is open if and only if $0 \in U$ and Spec$(D) - U$ is finite. The set U is dense if and only if either $0 \in U$ or U is infinite.

(3.8) *Example.* We shall show that in Theorem 3.6 the neighborhood U might well be smaller than V. Let D be the ring of integers, and let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 5 \end{bmatrix}.$$

Then A and B have the same invariant factors, namely $t^2 - 5t$ and 1; so if $V = \operatorname{Spec}(D)$, then A is pointwise similar to B on V, and $\tau = 2$ on V. Also, $D_V = D$. But if $C = (c_{ij}) \in D_{2 \times 2}$ is such that $B = CAC^{-1}$, then by direct computation $c_{21} = 0$, (a): $c_{11} + 5c_{12} = 2c_{22}$, and $c_{11}c_{22} = \pm 1$. Multiplying (a) by (c_{11}) we obtain $c_{11}^2 \equiv \pm 2 \pmod 5$, and since this congruence has no solution, A cannot be similar to B on V. However, V can be covered by the neighborhoods $U_1 = \boldsymbol{Z}^*(3)$ and $U_2 = \boldsymbol{Z}^*(5)$, on which A is similar to B via the respective matrices

$$C_1=\begin{bmatrix}1 & 1\\ 0 & 3\end{bmatrix}, \quad C_2=\begin{bmatrix}5 & 1\\ 0 & 5\end{bmatrix}.$$

4. Ostrowski's criterion

As before, let D be an integral domain with quotient field K. We shall say that the element $b\in D$ is a *greatest common divisor* (g.c.d.) in D of $a_1, \dots, a_s$, $a_i\in D$, if $(b)=(a_1, \dots, a_s)$. Here

$$(a_1, \dots, a_s) \quad \text{is the ideal} \quad \sum_{i=1}^{s} a_i D.$$

It follows immediately that such a b is also a g.c.d. in D' of $a_1, \dots, a_s$ for any domain $D'\supseteq D$.

(4.1) **Proposition.** *Let D be an integral domain with quotient field K, and suppose D is integrally closed. Let f, g be monic polynomials in the polynomial rings $D[t]$, $K[t]$ respectively. Then $g|f$ in $K[t]$ implies $g\in D[t]$.*

Proof. Zariski-Samuel [*6*], p. 260, thm. 5.

(4.2) **Lemma.** *Let R, R' be commutative rings with identity, let $R\subseteq R'$ and $f, g\in R[t]$. If $f=gh$ for some $h\in R'[t]$ and if the leading coefficient of g is a unit in R, then $h\in R[t]$.*

Proof. Compare coefficients.

(4.3) **Theorem.** *Let K be the quotient field of an integral domain D. Let $f_1, \dots, f_n\in D[t]$, and let g_0 be their unique monic g.c.d. in $K[t]$. Let $g_\mathfrak{p}$ be a g.c.d. of*[1] *$f_1(\mathfrak{p}), \dots, f_n(\mathfrak{p})$, in $D(\mathfrak{p})[t]$. If f_1 is monic, then* $\deg g_0 \leqq \deg g_\mathfrak{p}$.

Proof. Let D_1 be the integral closure of D. By 4.1, $g_0\in D_1[t]$; and moreover by 4.2, $g_0 | f_i$ in $D_1[t]$. There exists a prime ideal $\mathfrak{p}_1$ of D_1 such that $\mathfrak{p}_1\cap D=\mathfrak{p}$ ([*6*], p. 257, Theorem 3), and then $D(\mathfrak{p})$ may be identified with a subfield of $D_1(\mathfrak{p}_1)$. Since $g_\mathfrak{p}$ is a g.c.d. of $f_1(\mathfrak{p}), \dots, f_n(\mathfrak{p})$ in $D(\mathfrak{p})[t]$, $g_\mathfrak{p}$ is also their g.c.d. in $D_1(\mathfrak{p}_1)[t]$. But $g_0 | f_i$ in $D_1[t]$ implies that $g_0(\mathfrak{p}_1) | f_i(\mathfrak{p}_1)$ in $D_1(\mathfrak{p}_1)[t]$. Therefore $g_0(\mathfrak{p}_1) | g_\mathfrak{p}$ in $D_1(\mathfrak{p}_1)[t]$. Thus, $\deg g_\mathfrak{p} \geqq \deg g_0(\mathfrak{p}_1)$. Since g_0 is monic, $\deg g_0=\deg g_0(\mathfrak{p}_1)$; and hence we are done.

(4.4) **Lemma.** *Let $f_1, \dots, f_n\in D[t]$, and let $g_\mathfrak{p}$ be the monic* g.c.d. *of $f_1(\mathfrak{p}), \dots, f_n(\mathfrak{p})$* in *$D(\mathfrak{p})[t]$. Then there exists a neighborhood U of* 0 *such that $g_0\in D_U[t]$ is a* g.c.d. *of $f_1, \dots, f_n$ in $D_U[t]$. Moreover, then for all* $\mathfrak{p}\in U$, $g_0(\mathfrak{p})=g_\mathfrak{p}$ *and* $\deg g_0=\deg g_\mathfrak{p}$.

Proof. The polynomial g_0 is the monic g.c.d. of $f_1, \dots, f_n$ in $K[t]$, K being the quotient field of D. Therefore there exists an $a\in D$ such that

$$g_0=(a_1 f_1+\dots+a_n f_n)/a, \quad a_i\in D;$$

$$f_i=(h_i g_0)/a, \quad h_i\in D[t], \quad i=1, \dots, n.$$

Let $U=\boldsymbol{Z}^*(a)$, and then U has the required properties.

[1] We recall that according to section 2, for any $\mathfrak{p}\in\mathrm{Spec}(D)$, $f_i(\mathfrak{p})$ is the image of f_i under the canonical homomorphism $D_\mathfrak{p}[t]\to(D_\mathfrak{p}/\mathfrak{p}D_\mathfrak{p})[t]$.

The next theorem, coupled with Theorem 3.6, gives the second criterion for similarity.

(4.5) **Theorem.** *(Generalization of Ostrowski's theorem.) Let* $A \in D_{n\times n}$, *and let* $\rho_r(\mathfrak{p})$, $r=1, \ldots, n$ *be the degrees of the invariant factors of* $A(\mathfrak{p})$, $\mathfrak{p} \in \operatorname{Spec}(D)$, *arranged in descending order. Let* τ *and* σ_r, $r=0, \ldots, n$ *be given by* (3.4) and (3.5). *If* $\mathfrak{p} \in \operatorname{Spec}(D)$, *then the following are equivalent:*

$$\sigma_r(\mathfrak{p})=\sigma_r(0), \quad r=1, \ldots, n. \tag{4.6}$$

$$\rho_r(\mathfrak{p})=\rho_r(0), \quad r=1, \ldots, n. \tag{4.7}$$

(4.8) *There exists a dense set* V_r *containing* $\mathfrak{p}$ *such that* ρ_r, $r=1, \ldots, n$, *is constant on* V_r.

(4.9) *There exists a dense set* V *containing* $\mathfrak{p}$ *on which* τ *is constant.*

(4.10) τ *is minimal at* $\mathfrak{p}$.

Proof. The leading coefficient of each principal minor of $A-tI$ is a unit. Hence Theorem 4.3 and Lemma 4.4 apply:

$$\sigma_r(\mathfrak{q}) \geqq \sigma_r(0), \quad \text{if} \quad \mathfrak{q} \in \operatorname{Spec}(D). \tag{4.11}$$

(4.12) There exists a neighborhood W_r of 0 such that $\sigma_r(\mathfrak{q})=\sigma_r(0)$, if $\mathfrak{q} \in W_r$.

Let $W=\bigcap_{r=1}^{n} W_r$, a neighborhood of (0).

(4.6) *implies* (4.7), (4.8), (4.9) and (4.10). Set $V_r=W_r \cup (\mathfrak{p})$ in (4.8) and $V=W \cup (\mathfrak{p})$ in (4.9). The results are now immediate consequences of (4.11) and (4.12).

(4.8) *implies* (4.7). Since V_r is dense and W is open, there exists $\mathfrak{q}_r \in V_r \cap W$. Thus

$$\rho_r(\mathfrak{p})=\rho_r(\mathfrak{q}_r)=\sigma_{n-r+1}(\mathfrak{q}_r)-\sigma_{n-r}(\mathfrak{q}_r)=\sigma_{n-r+1}(0)-\sigma_{n-r}(0)=\rho_r(0).$$

(4.9) *implies* (4.6). Again there exists $\mathfrak{q} \in V \cap W$ and $\tau(\mathfrak{p})=\tau(\mathfrak{q})=\tau(0)$ and hence by (3.4) and (4.11) $\sigma_r(\mathfrak{p})=\sigma_r(0)$, $r=1, \ldots, n$.

(4.10) *implies* (4.6). Apply 4.11 and 3.4.

(4.13) **Corollary.** *The set* U *of* $\mathfrak{p} \in \operatorname{Spec}(D)$ *for which any one of the equivalent conditions of Theorem* 4.5 *holds is open.*

Proof. The set on which τ is minimal is open. For, by definition $\tau(\mathfrak{p})$ is the dimension of the null space of a certain linear transformation and hence is minimal where the rank of this transformation is maximal. But by Lemma 3.1 this is an open set.

5. Holomorphic rings of functions on a topological space

In this section we shall apply the results of Section 3 to rings of functions; and in particular, we shall explicitly show how WASOW's original theorem for functions of a complex variable follows from our generalization (3.6) of his theorem.

If X is a topological space, and $S \subseteq X$ then x is an *accumulation point* of S if every neighborhood N of x has non-empty intersection with $S-x$.

(5.1) *Definition.* Let X be a topological space and let H be a ring (with identity) of functions defined on X and taking values in a field K. Then we shall call H a *holomorphic ring* if for every $f \in H, f \neq 0$, the set of zeros of f does not have an accumulation point in X.

(5.2) *Remark.* If X is an open, connected subset of the complex plane, then the ring of functions holomorphic on X is a holomorphic ring (see, for example, AHLFORS [*1*], p. 102; and, of course, the motivation for the definition 5.1 comes from this example. In this special case the field K is the complex field.) We shall denote this particular holomorphic ring by $H(X)$.

If $x \in X$, define

(5.3)
$$\mathfrak{p}_x = \{f \in H : f(x) = 0\}.$$

Since K is a field, p_x is a prime ideal of the holomorphic ring H. We therefore have a map

(5.4)
$$\sigma : x \to \mathfrak{p}_x$$

of X onto a subset of $\mathrm{Spec}(H)$ (which need not be $1-1$). Taking $\mathrm{Spec}(H)$, as usual, to be endowed with the Zariski topology, $\sigma(X)$ then also becomes a topological space under the relative topology induced by that of $\mathrm{Spec}(H)$.

If $P \subseteq \mathrm{Spec}(H)$, then the closure $\bar{P}$ of P in $\mathrm{Spec}(H)$ is known to be (GROTHENDIECK [*3*], p. 81)

(5.5)
$$\bar{P} = \mathbf{Z}(\cap \mathfrak{p}), \quad \mathfrak{p} \in P.$$

Thus, for any subset S of X, $\mathfrak{p} \in \overline{\sigma(S)}$ if and only if every $f \in H$ which vanishes on S is also in $\mathfrak{p}$.

(5.6) **Proposition.** *Let H be a holomorphic ring on X, and let V be a subset of X such that V has an accumulation point in X. Then $\sigma(V)$ is dense in* $\mathrm{Spec}(H)$.

Proof. If $f \in H$ and f vanishes on V, then $f=0$ by definition 5.1. Therefore, by the above remarks, $\mathfrak{p} \in \overline{\sigma(V)}$ for all $\mathfrak{p} \in \mathrm{Spec}(H)$.

(5.7) **Proposition.** *Let H be a holomorphic ring on X. Then the map σ of 5.4 is continuous.*

Proof. Suppose T is a closed subset of $\sigma(X)$ [the topology of $\sigma(X)$ being that induced by the topology of $\mathrm{Spec}(H)$], and let $S = \sigma^{-1}(T)$. We must see that S is closed in X. If S has no accumulation point in X, this is immediate. If S does have an accumulation point, then $\sigma(S) = T$ is dense in $\mathrm{Spec}(H)$ by proposition 5.6, and hence is also dense in $\sigma(X)$. Therefore in this case, $\sigma(X) = T$, and $S = X$. Thus, S is again closed.

(5.8) **Proposition.** *Let X be a topological space such that no point of X is open, and let H be a holomorphic ring on X. Then H is an integral domain.*

Proof. Let $f \neq 0, g \neq 0$ be elements of H. For any $x \in X$, by definition 5.1 there exists a neighborhood N of x such that f and g are non-zero on $N-x$. Since x is not itself open, $N-x \neq \Phi$. Hence if $y \in N-x$, $f(y) \neq 0$ and $g(y) \neq 0$. Thus, $f(y) \cdot g(y) \neq 0$ and $f \cdot g \neq 0$.

We shall assume throughout the rest of this paper that the holomorphic ring H *is an integral domain,* and to be explicit we shall call H a holomorphic integral domain. An example is given by the following: Let X be a topological space with an infinity of points in which all points are closed, and let H consist of all functions of X into a field K with a finite number of zeros in X. Then H is a holomorphic integral domain.

(5.9) **Proposition.** *Let H be a holomorphic integral domain on X, and let V be a subset of X with an accumulation point in X. Then $H_{\sigma(V)}$ (as defined in* 2.2) *is isomorphic to a holomorphic integral domain on V, which will be denoted by H_V.*

Proof. For any $\alpha \in H_{\sigma(V)}$, there exist $f, g \in H$ such that $\alpha = f/g$ and $g \notin \cup \mathfrak{p}_x$, $x \in V$. Let α correspond to the function $\alpha(x) = f(x)/g(x)$, $x \in V$. This correspondence is evidently a homomorphism. If $\alpha(x) = 0$ for all $x \in V$, then $f(x) = 0$ for all $x \in V$. Therefore, by definition 5.1, $f = 0$, and hence $\alpha = f/g = 0$. Thus, the correspondence is an isomorphism; and one now checks easily that the resulting function ring H_V satisfies definition 5.1.

(5.10) Fixing an $x \in X$, the map $\tau_x : f \to f(x)$ is a homomorphism of H into K having kernel $\mathfrak{p}_x$. The image $\tau_x(H)$ is then an integral domain whose quotient field in K is naturally isomorphic to $H(\mathfrak{p}_x)$ (see 2.2). If we identify corresponding quantities under this isomorphism, we may write $f(x) = f(\mathfrak{p}_x)$ [where $f(\mathfrak{p}_x)$ is defined in 2.2]. Thus, for instance, to say that a function f is constant on a subset S of X is equivalent to saying f is constant on $\sigma(S)$. If H contains all the constant functions of X into K, then τ_x maps H *onto* K. Hence, in this case $H(\mathfrak{p}_x) \cong K$, for all $x \in X$.

(5.11) Let H again be a holomorphic integral domain on X, let S be a subset of X, and let A, B be matrices in $H_{n \times n}$. We can extend definitions 2.4 and 2.6 in the following way: The matrix A is *similar to B on S* if A is similar to B on $\sigma(S)$. If S has an accumulation point, it follows from proposition 5.9, that A is similar to B on S if and only if there exists an invertible matrix C in $(H_V)_{n \times n}$ such that $B(x) = C^{-1}(x)\, A(x)\, C(x)$, for all $x \in V$. The matrix A is called *pointwise similar to B on S*, if A is pointwise similar to B on $\sigma(S)$. In the special case when H contains all constant functions, it follows from the last sentence in 5.10 that A is pointwise similar to B on S if and only if for each $x \in S$ there exists a non-singular $C_x \in K_{n \times n}$, such that $B(x) = C_x^{-1} A(x)\, C_x$. We have now restated our principal definitions for rings of functions and, thus, we can now give a function ring version of Theorem 3.6. As a corollary we obtain WASOW's original statement of the theorem for functions of a complex variable.

(5.12) **Theorem** (*analogue of* 3.6). *Let H be a holomorphic integral domain on X. Let $A, B \in H_{n \times n}$, and let V be a subset of X with an accumulation point in X. Suppose that for the matrix A, $\tau\langle A \rangle$ defined by* 3.4 *is constant on V. If A is pointwise similar to B on V, and if $x \in V$, then there exists a neighborhood U of x such that A is similar to B on U.*

Proof. By proposition 5.6, $\sigma(V)$ is dense in Spec(H). Keeping in mind the definitions of 5.11 and applying theorem 3.6, there exists a neighborhood U' of $\mathfrak{p}_x$ such that A is similar to B on U'. By proposition 5.7, the map σ of 5.4 is continuous, whence $U=\sigma^{-1}(U')$ is an open subset of X and hence is the required neighborhood.

(5.13) **Corollary** *(Wasow's original theorem). Let $H(X)$ be the ring of functions holomorphic on a connected neighborhood X of x_0 in the complex plane, and let $A, B\in H(X)_{n\times n}$. Suppose that for the matrix A, $\tau\langle A\rangle$ is constant on a neighborhood V of x_0. If A is pointwise similar to B on V (in the sense of* 1.1) *then there exists a neighborhood U of x_0 such that A is holomorphically similar to B on U (again in the sense of* 1.1).

Proof. By 5.8 the ring $H(X)$ is an integral domain. Since $H(X)$ contains all constant functions of X into the complex field K, the definitions of 1.1 and 5.11 of pointwise similarity coincide in this case. Since the open set U has a point of accumulation, similarity on U in the sense of 5.11 will imply holomorphic similarity in the sense of 1.1, provided we can show that $H(X)_U\subseteq H(U)$, the ring of all functions holomorphic on U. But if f, g are holomorphic on X, and g has no zeros on U, then the function defined by f/g is holomorphic on U, and the required inclusion follows.

The corollary now follows from Theorem 5.12.

References

[1] Ahlfors, L. V.: Complex Analysis. New York: McGraw-Hill 1953.

[2] Gantmacher, F. R.: Matrix Theory, Vol. I. New York: Chelsea 1959.

[3] Grothendieck, A., et J. Dieudonné: Éléments de Géométrie Algébrique, I., Inst. des Hautes Études Sci., 4 (1960).

[4] MacDuffee, C. C.: Theory of Matrices. New York: Chelsea 1956.

[5] Wasow, W.: On holomorphically similar matrices. J. Math. Analysis and Applications 4, 202—206 (1962).

[6] Zariski, O., and P. Samuel: Commutative Algebra, Vol. 1. Princeton: van Nostrand 1958.

Dept. of Math., University of Wisconsin, Madison 6, Wisconsin (U.S.A.)

(Received March 15, 1964)

Druck der Universitätsdruckerei H. Stürtz AG., Würzburg