Reprinted from JOURNAL OF ALGEBRA, Vol. 1, No. 3, September 1964
 Printed in Belgium

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The Decomposition of Cones in Modules over Ordered Rings*

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Communicated by R. H. Bouck

Received March 9, 1964

I. INTRODUCTION

A proper cone in a module over an ordered ring is prime if it is not the sum of two nonzero proper cones (Section III). Prime cones can easily be found; thus, all proper cones contained in the integers are prime (Section IV). In Section VII we investigate the decomposition of cones into the direct sum of prime cones. There are striking differences from the corresponding results for modules. For example, a 2-dimensional vector space can be expressed as the direct sum of 1-dimensional spaces in many ways. For cones, however, a decomposition into prime cones, if it exists, is always unique (Section VII). This is true even in the infinite case. We show in Sections V and VI that the algebra of direct summands of a proper cone is a Boolean algebra.

If $\mathscr{P}$ is an infinite family of prime cones, the Boolean algebra of all direct sums of subfamilies of $\mathscr{P}$ is isomorphic to the Boolean algebra of all direct products of subfamilies of $\mathscr{P}$ (Section VIII); however, their decompositions are quite distinct. The direct product of the cones in $\mathscr{P}$ has no prime decomposition. If a cone satisfies a chain condition then there is a finite prime decomposition (Section IX).

In Section II we discuss ordered rings, semigroups, and semigroup rings. In Section X, we introduce some topological notions and show, under very general hypotheses, that if a cone has a direct summand, the summand is contained in the crust (boundary). In Section XI we discuss an important class of modules, which satisfy the hypotheses of Section X. In Section XII we apply our results to the cone of positive semidefinite matrices, which is shown to be prime.

* Work on this paper was supported in part by the National Science Foundation under grant N.S.F.G.P.-2273 and by the Mathematics Research Center, United States Army, Madison, Wisconsin, under grant DA-11-022-ORD-2059.

II. Results on Ordered Rings

We use the usual set theoretic notions with $\in$, $\cap$, $\cup$, $\setminus$, $\emptyset$, $U(X)$ denoting membership, intersection, union, relative complementation, the empty set, the set of subsets of X, respectively.

(2.1) Definition. An abelian group A will be called *ordered* if A is a totally ordered set and $\alpha, \beta, \gamma \in A$ and $\alpha \geqslant \gamma$ imply that $\alpha + \beta \geqslant \gamma + \beta$. A (associative) ring A with identity 1 will be called *ordered* if A is an ordered abelian group under addition and $\alpha \geqslant 0$, $\beta \geqslant 0$ imply that $\alpha\beta \geqslant 0$.

Elements of A will be denoted by lower case Greek letters. If $\alpha > 0$ then it follows from the trichotomy law that $n\alpha > 0$, for all positive integers n, and $\alpha + \beta > 0$ if $\beta \geqslant 0$. However, we do not rule out the possibility that A may have zero divisors. For $\alpha \in A$, we denote by $|\alpha|$ the max $\{\alpha, -\alpha\}$.

(2.2) Definition. A semigroup S with identity 1 is called *ordered* if S is totally ordered and if $s, s', t, t' \in S$ with $s \leqslant s'$, $t \leqslant t'$ then $st \leqslant s't'$.

(2.3) Lemma. *If S has a zero it is the minimal element if $1 > 0$ and the maximal element if $1 < 0$.*

Proof. If $1 > 0$ and $x \in S$, then $0x \leqslant 1x$ therefore $0 \leqslant x$.

It is equally trivial to see that if S has no zero, a zero can be adjoined as the minimal element.

In the sequel we suppose that if there is a zero in an ordered semigroup, then it is always the minimal element. A dual theory can be developed for the case in which zero is maximal. The set of (nonzero) *positive* elements of S is denoted by S^+.

(2.4) Definition. If A is a ring and S is a semigroup, the *semigroup ring* of S over A, $A(S)$, is defined to be the set of all formal sums $\alpha = \Sigma^+ \alpha_s s$, where Σ^+ denotes summation over S^+, in which only finitely many α_s are nonzero. If $\beta = \Sigma^+ \beta_s s$ then $(\alpha + \beta)_s = \alpha_s + \beta_s$ and $(\alpha\beta)_s = \Sigma_{wz=s}\, \alpha_w \beta_z + 0$. If A is an ordered ring and S is an ordered semigroup then $\alpha \in A(S)$ is called *positive* if there is an s such that $\alpha_s > 0$ and for $t > s$, $\alpha_t = 0$. If $\alpha, \beta \in A(S)$ then $\alpha > \beta$ in the *natural order on* $A(S)$ if $\alpha - \beta$ is positive.

We note that the natural order on $A(S)$ is the unique total order on $A(S)$ which is preserved under addition and for which the *positive* elements defined above are greater than zero.

It is apparent that $A(S)$ is a ring. Further the 0 of S plays an unimportant role in the construction of $A(S)$. It follows from the existence of the identity elements in A and S, if they are ordered, that both A and S are isomorphically embedded in $A(S)$. For amplifications and generalizations of these remarks, see Conrad [5].

(2.5) THEOREM. *Let A and S be an ordered ring and an ordered semigroup, respectively. Further suppose S has a nonzero element other than 1. Then $A(S)$ is an ordered ring in the natural order if and only if*

(i) *A has no zero divisors,*

and for $s, s', t, t' \in S$

$(\text{ii})_l$ *$st = st' \neq 0$ implies $t = t'$, and*

$(\text{ii})_r$ *$st = s't \neq 0$ implies $s = s'$.*

Further if $A(S)$ is a naturally ordered ring then $A(S)$ has zero-divisors, if and only if S does.

Proof. Suppose $A(S)$ is a naturally ordered ring.

(i) Let $\alpha, \beta \in A$, $\beta > 0$, $\alpha \geqslant 0$ and $\alpha\beta = 0$. By the hypotheses on S there are non-zero elements $s, s' \in S$ with $s > s'$. Consider $\alpha = \alpha \cdot 1 \in A(S)$ and $\beta s - s' \in A(S)$. Both these elements are nonnegative; thus $\alpha(\beta s - s') \geqslant 0$. On the other hand $\alpha(\beta s - s') = - \alpha s' \leqslant 0$; hence $\alpha = 0$. Therefore there are no zero divisors in A.

$(\text{ii})_l$ Let $s, t, t' \in S$ with $st = st'$ and $t > t'$. Consider the positive elements s, $t - 2t'$ of $A(S)$. We see that $s(t - 2t') \geqslant 0$, but $s(t - 2t') = - st \leqslant 0$; whence $st = 0$.

The proof of $(\text{ii})_r$ is similar.

Suppose now that (i), $(\text{ii})_l$, and $(\text{ii})_r$ hold. From $(\text{ii})_l$ and $(\text{ii})_r$ we see that if $s \geqslant s'$, $t \geqslant t'$ then $st \geqslant s't'$ and equality holds if and only if either $st = s't' = 0$ or $s = s'$ and $t = t'$.

Let $\alpha, \beta \in A(S)$ with $\alpha > 0, \beta > 0$. We already know $A(S)$ is totally ordered by the natural order, $A(S)$ is a ring, and positivity is preserved under addition; thus we need only show that $\alpha\beta \geqslant 0$. There are elements $s, t \in S^+$ such that $\alpha_s > 0$, $\beta_t > 0$ and if $u > s$ and $v > t$, $\alpha_u = \beta_v = 0$. In view of the above remark either $st = 0$ or $(\alpha\beta)_{st} = \alpha_s\beta_t$ which is positive by (i). Furthermore if $u > st$, $(\alpha\beta)_u = 0$, whence in the first case $\alpha\beta = 0$, by the minimality of 0 in S, and $\alpha\beta > 0$ in the second case. Thus $A(S)$ is an ordered ring in the natural order.

Examining the second case in the above paragraph, we see that if S has no zero divisors, then neither does $A(S)$. If S has zero divisors, so does $A(S)$ since S is embedded in $A(S)$.

We now use the theorem (2.5) to construct some interesting examples of ordered rings. We make some additional definitions for this purpose.

(2.6) DEFINITION. Let A be an ordered ring. A nonzero element $\alpha \in A$ is (strongly) *infinitesimal* if $|\lambda\alpha| < 1$, for all $\lambda \in A$. An element $\alpha \in A$ is (strongly) *infinite* if $|\lambda\alpha| > 1$, for all $\lambda \in A$, $\lambda \neq 0$.

(2.7) DEFINITION. We recall that an *ideal* of a semigroup S is a nonempty subset T for which $ST \cup TS \subseteq T$. The *Rees quotient* S/T [4, p. 17] of S by an ideal T consists of $S \setminus T \cup \{0\}$, where for $s, s' \in S \setminus T$ the product ss' is redefined to be 0 if $ss' \in T$.

(2.8) *Examples.* Let T be the semigroup of nonnegative powers of t with ordering $1 = t^0 < t^1 < t^2 < \cdots$. Let A be an ordered ring without zero-divisors. Then the semigroup ring $A_0 = A(T)$ is simply the polynomial ring $A[t]$. Clearly by (2.5), A_0 is a naturally ordered ring, and a polynomial is positive if its leading coefficient is positive. We note that any nonconstant polynomial is infinite in A_0.

We next consider the semigroup S_0 of nonnegative powers of t with ordering $1 = t^0 > t^1 > t^2 > \cdots$. Then the semigroup ring $B_0 = A(S_0)$ is again a naturally ordered ring and every polynomial without constant term is infinitesimal. Note that $S_0 t^i$ is an ideal in S_0, and, for $i \geqslant 1$, define $S_i = S_0/S_0 t^i$, the Rees quotient. We now order S_i by

$$1 = t^0 > t^1 > \cdots > t^{i-1} > t^i = 0.$$

By the theorem (2.5), B_i is naturally ordered. Polynomials without constant term are zero divisors and infinitesimals.

The above rings will be commutative or noncommutative as the ring A is taken to be commutative or noncommutative respectively. The example B_0 shows that there are rings with infinitesimals but no zero divisors.

The following remark shows that zero divisors are always infinitesimals.

(2.9) *Remark.* Let A be an ordered ring and let $\alpha > 0$. If $\lambda\alpha > 0$ and $\mu\alpha = 0$ then for all $\beta, \gamma \in A$, where $\beta > 0$ either $\beta\lambda\alpha = 0$ or $\beta\lambda > \gamma\mu$. In particular μ is infinitesimal.

For proof, note that $(\beta\lambda - \gamma\mu)\,\alpha = \beta(\lambda\alpha) \geqslant 0$. Hence, if $\beta\lambda\alpha \neq 0$, the positivity of α implies $\beta\lambda - \gamma\mu > 0$. If $\beta = \lambda = 1$, we obtain that $1 > \gamma\mu$, for all $\gamma \in A$, and the remark follows.

Further information on ordered rings may be found in Fuchs [*10*] and Neumann [*22*].

(2.10) Definition. Let E be a unitary left module over an ordered ring A. Then E is called an *ordered module* if E is an ordered abelian group under addition and if $x \in E$, $\alpha \in A$ are both nonnegative, then αx is nonnegative.

(2.11) Definition. Let X be a totally ordered set containing a distinguished element, say 0. A nonempty subset Y of X is called an initial subset of X if (i) $Y \neq X$ and (ii) $y \in Y$ and either $0 \leqslant x \leqslant y$ or $y \leqslant x \leqslant 0$ imply $x \in Y$. If X is an ordered algebraic system (e.g., ring, module, etc.), then an initial set Y of X which is an ideal, submodule, etc., will be called an *initial ideal, initial submodule*, etc.

The standard term for initial (left) ideal is *convex* (left) ideal [*10*]. However the authors reserve the term convex for the more general situation defined in (3.1).

The following proposition indicates one reason why initial submodules are important in the theory. Further reasons will be forthcoming in Section XI.

(2.12) PROPOSITION. Let E be an ordered module, and F a submodule of E. Then E/F is an ordered module in the relation induced by the order on E if and only if F is an initial submodule.

The proof of this proposition is analogous to the proof of Fuchs (see [*10*], Chap. II, Theorem 7), and will therefore be omitted.

For completeness we repeat some standard definitions (see Jacobson [*17*], Chap. 8).

(2.13) DEFINITIONS. An element $\alpha \in A$ is *nilpotent*, if $\alpha^n = 0$, for some integer n. Let N be an *ideal* in A. We call N *nilpotent* if $N^n = \{0\}$, for some n, *locally nilpotent* if every finitely generated ideal contained in N is nilpotent, and *nil* if all its elements are nilpotent. The ideal N is a *nil radical* of A if N is nil and A/N contains no nonzero nilpotent ideals.

The following theorem is slightly more complete than Fuchs (see [*10*], Chapter VIII, Theorem 6).

(2.14) THEOREM. *Let A be an ordered ring and let N be the set of all nilpotent elements in A. Then:*

(i) *N is an initial ideal and A/N is an ordered ring in the naturally induced order,*
(ii) *A/N has no zero divisors,*
(iii) *N is a locally nilpotent ideal, and*
(iv) *N is the unique nil radical of A.*

Proof. Assertions (i) and (ii) follow immediately from (2.12) and the theorem of Fuchs quoted above. To prove (iii), let C be the ideal generated by the elements $\nu_1, \cdots, \nu_m$ of N. Let $\nu = \max\{|\nu_i| : i = 1, \cdots, m\}$. Then, for some $k \geqslant 0$, $\nu^k = 0$. It follows that $0 \leqslant |\nu_i^k| \leqslant \nu^k = 0$, whence $\nu_i^k = 0$, $i = 1, \cdots, m$. Since the set N_k of all μ such that $\mu^k = 0$ is a nilpotent ideal in A [*10*, p. 130] and since the $\nu_i \in N_k$, it follows that $C \subseteq N_k$. Hence C is nilpotent.

We now prove (iv). Let M be any nil radical. By definition $M \subseteq N$. On the other hand every nilpotent ideal is contained in M; whence

$$M \supseteq \bigcup_1^\infty N_k = N,$$

where N_k are the ideals defined in the proof of (iii). Thus $M = N$, and the theorem is established.

Rings A containing a nil ideal M such that A/M has no zero divisors, and group rings over such rings, have recently been studied by Rudin and Schneider [*24*].

(2.15) *Examples.* The rings B_i of (2.8) show that there exist rings with nilpotent elements of index up to any given integer, but no higher. We now construct an example of an ordered ring which has simultaneously elements of every index of nilpotency. Let S_α be the semigroup of reals $[0, \alpha]$, where $0 \leqslant \alpha \leqslant 1$. Let $S'_{\beta,\alpha} = [\beta, \alpha]$, $0 \leqslant \beta \leqslant \alpha \leqslant 1$, with multiplication defined by $s \circ t = \max(st, \beta)$. Thus $S'_{\beta,\alpha}$ is the Rees quotient S_α/S_β. Let $S_{\beta,1} = S'_{\beta,1}$, and if $\alpha < 1$, let $S_{\beta,\alpha} = S'_{\beta,\alpha} \cup \{1\}$, where 1 is the maximal element. Let A be an ordered ring without zero divisors. Then the semigroup ring $A(S_{\beta,\alpha})$ is an ordered ring. If $0 < \beta < \alpha \leqslant 1$, then $A(S_{\beta,1})$ has nilpotent elements of all indices, and $A(S_{\beta,\alpha})$ has nilpotent elements of all orders up to n, where $\sqrt[n-1]{\alpha} < \beta < \sqrt[n]{\alpha}$.

In all the previous examples of ordered rings, every zero divisor has been nilpotent, and indeed, in every ordered ring if $\alpha\beta = 0$ then it follows that either $\alpha^2 = 0$ or $\beta^2 = 0$. We now give an example of an ordered ring which has nonnilpotent zero divisors. Let N be any nonempty ordered set with minimal element, say 0^*. Let $T_\alpha = N \cup S_{0,\alpha}$, and define every element of N to be less than every element of $S_{0,\alpha}$. Define multiplication thus: if $t \neq 1$, then $Nt = 0^*$, 1 is the identity of T_α and multiplication in $S_{0,\alpha}$ is as before. Then every element $t \neq 1$ is a zero divisor, but only elements in N are nilpotent. If $\alpha, \beta \in A(T_\alpha)$ where $\alpha_t = 0$ for all $t \in T_\alpha \setminus N$ while, for some $t \in T_\alpha \setminus N$, $\beta_t \neq 0$ and $\beta_1 = 0$,then $\alpha\beta = 0$ but $\beta^n \neq 0$ for any n.

Through their considerations of the above example, the authors were lead to the following theorem, which, intuitively says that the operations of forming Rees quotients and semigroup rings commute.

(2.16) THEOREM. *Let S be a semigroup and A be a ring such that $A^2 \neq 0$. Let $T \subseteq S$. Then:*

(i) *T is an ideal of S if and only if $A(T)$ is an ideal in $A(S)$.*

(ii) *If T is an ideal of S then the projection from $A(S)$ to $A(S/T)$, defined by $\varphi(\Sigma_{S^+} \alpha_s s) = \Sigma_{(S/T)^+} \alpha_s s$, is an epimorphism with kernel $A(T)$. Thus $A(S)/A(T) \cong A(S/T)$.*

(iii) *If S is an ordered semigroup, T is an initial ideal of S, then S/T is an ordered semigroup in the induced order.*

(iv) *If $A(S)$ is naturally ordered semigroup ring, then T is an initial ideal of S if and only if $A(T)$ is an initial ideal of $A(S)$.*

(v) *If $A(S)$ is a naturally ordered semigroup ring, and T is an initial ideal of S, then $A(S/T)$ is a naturally ordered semigroup ring and the mappings of (ii) preserve order.*

The proof is straightforward and will be omitted. G. O. Losey has informed us that in [21], he has a more general theorem which yields the algebraic part of (2.16). Related results can be found in Clifford-Preston (see [4], Section 5.2) who give references to the literature.

The results in this section hold, with minor modifications. for rings and semigroups which do not necessarily have an identity, for details see Fuchs [10]. In the sequel there will be places in which the presence of an identity is essential.

III. Preliminary Results on Cones

Conventions. The letter A will denote an ordered ring and E will denote a unitary left A-module. If P, P', P_i, Q, etc., are subsets of E, then lower case p, p', p_i, q etc., will denote elements of P, P', P_i, Q, respectively, unless otherwise indicated. If $P \subseteq E$, then $-P$ consists of all elements $-p$. More generally, αP, for $\alpha \in A$, consists of all αp. If also $Q \subseteq E$, then $P + Q$ consists of all $p + q$. By $[P]$ we shall denote the submodule of E generated by P.

(3.1) Definition. A subset K of E is *convex* if $k_i \in K$ and $\alpha_i \in A$, $i = 1, \cdots, n$, with $0 \leqslant \alpha_i$ and $\sum \alpha_i = 1$ imply $\sum \alpha_i k_i \in K$.

We note that in any ordered ring every left ideal is convex.

(3.2) Definition. A subset P of E will be called a *cone* in E if

(a) $0 \in P$, and

(b) $P + P \subseteq P$.

(c) $\alpha P \subseteq P$, for all $\alpha \geqslant 0$.

A *cone* P is *proper* if it is nonzero and

(d) $P \cap (-P) = \{0\}$, i.e., $p \in P$ and $-p \in P$ imply $p = 0$.

A cone that is not proper is said to be *improper*.

If X is any subset of E, $\langle X \rangle$ will denote the cone generated by X. Henceforth, P, P', P_i, Q, R, etc., will always denote cones in E. Note that $[P]$ consists of all $p - p'$, $p, p' \in P$.

(3.3) Proposition. *If P is a cone then P is convex.*

Proof. Let $p_i \in P$ and $\alpha_i \in A$ such that $0 \leqslant \alpha_i$ and $\sum \alpha_i = 1$, then $\alpha_i p_i \in P$. Thus $\sum \alpha_i p_i \in P$.

(3.4) PROPOSITION. *If P is a cone in a module E, then $P \cap (-P) = M$ is the maximal submodule of E contained in P. The cone P/M is proper in E/M, and conversely if N is a submodule in E such that P/N is proper in E/N then N contains M.*

Proof. Clearly $P \cap (-P) = M$ is a submodule of E, and every submodule contained in P is contained in M. If Q is a cone and $M \subseteq Q$, then $\bar{q} \in Q/M$ implies that $\bar{q} \subseteq Q$. Hence if $\bar{p} \in P/M \cap (-P/M)$ then $\bar{p} \subseteq P \cap (-P) = M$, whence $\bar{p} = \bar{0}$. Thus P/M is proper. Conversely, if $(P/N) \cap (-P/N) = \bar{0}$, then $M/N = \bar{0}$, whence $M \subseteq N$.

The following corollary is now obvious.

(3.5) COROLLARY. *If P is a cone in E then P is proper if and only if P contains no nonzero submodules.*

In view of the above proposition we shall restrict our main efforts to the study of proper cones.

(3.6) DEFINITION. Let $\{P_i\}\ i \in I$ be a family of cones in E. Then $\Sigma_I P_i$ is the cone consisting of all $\Sigma_I p_i$, where $p_i \neq 0$ for only a finite number of $i \in I$. The sum $R = \Sigma_I P_i$ is *direct* ($R = \oplus_I P_i$) if each r can be written uniquely $r = \Sigma_I p_i$. If $\mathscr{P} = \{P_i : i \in I\}$ then we sometimes use $\Sigma\mathscr{P}$ for $\Sigma_I P_i$ and $\oplus\mathscr{P}$ for $\oplus_I P_i$.

(3.7) PROPOSITION. *The sum $\Sigma_I P_i$ is direct if and only if $\Sigma_I [P_i]$ is direct.*

Proof. Let $\Sigma_I p_i = \Sigma_I p_i'$, $p_i, p_i' \in P_i$. If $\Sigma_I [P_i]$ is direct then $\Sigma_I (p_i - p_i') = 0$ implies $p_i - p_i' = 0$ and so $p_i = p_i'$. Conversely, let $\Sigma_I P_i$ be direct. If $x \in [P_i] \cap \Sigma_J [P_j]$, $J = I \setminus \{i\}$, then

$$x = p_i - p_i' = \sum_J (p_j' - p_j)$$

whence $\Sigma_I p_i = \Sigma_I p_i'$. Hence $p_i = p_i'$ and $x = 0$.

(3.8) PROPOSITION. *Let $R = \oplus_I P_i$. Then the P_i are proper or zero if and only if R is proper or zero.*

Proof. Let $R = \oplus_I P_i$ and suppose $r = \Sigma_I p_i$ and $-r \in R$. Then, if $-r = \Sigma_I p_i'$, $\Sigma_I (p_i + p_i') = 0$ whence, for all i, $p_i + p_i' = 0$, and as P_i is proper or zero, $p_i = 0$. Thus $r = 0$. The converse is obvious.

(3.9) DEFINITION. A cone R is called *composite* if there exist proper cones, P, Q such that $R = P \oplus Q$. A proper cone R is called *prime* if R is not composite.

Note: Composite cones are proper, by (3.8).

IV. Cones in the Ring of Integers

It is interesting to determine all cones contained in the ring Z of integers. We require two easy number theoretic lemmas. We shall use some results in number theory which can be found in many standard textbooks (e.g., Hardy-Wright [*15*], Chap. 5), without further reference.

In the ring Z we shall define for $\delta, \mu \in Z$,

$$\langle \delta : \mu \rangle = \{\alpha \in [\delta] : \alpha \geqslant \mu\} \cup \{0\};$$

here $[\delta]$ is the ideal generated by $\delta \in Z$.

(4.1) Lemma. *In Z, let $\alpha > 0$, $\beta < 0$, with g.c.d. δ (positive greatest common divisor). Then there exist nonnegative integers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that*

$$\lambda_1\alpha + \lambda_2\beta = \delta, \qquad \lambda_3\alpha + \lambda_4\beta = -\delta.$$

Proof. We can solve the congruence $\lambda_1\alpha \equiv \delta \pmod{\beta}$ for $1 \leqslant \lambda_1 \leqslant |\beta|$. Thus, for suitable λ_2, $\lambda_1\alpha + \lambda_2\beta = \delta$, and since $\lambda_1\alpha \geqslant \delta$, and $\beta < 0$, it follows that $\lambda_2 \geqslant 0$. The proof for $\lambda_3\alpha + \lambda_4\beta = -\delta$ is similar.

(4.2) Lemma. *In Z, let $\alpha_i > 0$, $i = 1, \cdots, s$, and let $\delta =$ g.c.d. $(\alpha_1, \cdots, \alpha_s)$, $\mu =$ l.c.m. $(\alpha_1, \cdots, \alpha_s)$ (positive least common multiple). Then*

$$\langle \delta : (s-1)(\mu - 1) \rangle \subseteq \langle \alpha_1, \cdots, \alpha_s \rangle.$$

Proof. Let $\kappa \in \langle \delta : s(\mu - 1) \rangle$, then $\kappa \in \langle \delta \rangle \subseteq [\alpha_1, \alpha_2, \cdots, \alpha_n]$. Thus there exist λ_i satisfying $\sum \lambda_i\alpha_i = \kappa$. On replacing λ_i by $(\lambda_i - \mu(n_i/\alpha_i))$ with n_i arbitrary integers we still obtain a solution $\sum \lambda_i\alpha_i \equiv \kappa \pmod{\mu}$, and can pick n_i such that $\mu/\alpha_1 \leqslant \lambda_1 < 0$ and $0 \leqslant \lambda_i < \mu/\alpha_i$, $i = 2, \cdots, s$. Thus $\sum \lambda_i\alpha_i < (s-1)(\mu - 1)$. Hence if $\kappa \geqslant (s-1)(\mu - 1)$,

$$\kappa = \sum \lambda_i\alpha_i + \rho\mu = \left(\lambda_1 + \mu\left(\frac{\rho}{\alpha_1}\right)\right)\alpha_1 + \sum_2^s \lambda_i\alpha_i$$

in which the coefficients of the α_i are all nonnegative, since $\rho \geqslant 1$.

Related inequalities have been studied in connection with the index of primitivity of a nonnegative matrix (e.g. Wielandt [28] and Varga [27]).

(4.3) Theorem. *Let P be a subset of the ring of integers Z. Then P is an improper cone if and only if P is an ideal in Z. The subset P is a proper cone if and only if, for some positive $\delta, \nu \in Z$*

$$\pm P = S \cup \langle \delta : \nu \rangle \tag{*}$$

where S is a finite subset of $\langle \delta : \delta \rangle$ such that $S + S \subseteq \pm P$. All proper cones in Z are prime.

Proof. Obviously every ideal is an improper cone. Let P be a nonzero improper cone in Z. Then P contains both positive and negative elements; so let α be the least positive element, β the largest negative element in P. Let $\delta = \text{g.c.d.}(\alpha, \beta)$, then by (4.1), both $\delta \in P$ and $-\delta \in P$. Hence $[\delta] \subseteq P$. If $\kappa \in P$, then, for some $r \geqslant 0$, $0 \leqslant \kappa + r(\pm\delta) < \delta \leqslant \alpha$ whence $\kappa + r(\pm\delta) = 0$ and $\kappa \in [\delta]$. Thus $P = [\delta]$.

It is again easy to check that if P satisfies (*) with $S + S \subseteq \pm P$, then P is a proper cone. Conversely, let P be a proper cone. If P contains both positive and negative elements, then, by the argument used for improper cones, $P = [\delta]$, for $\delta > 0$, and is improper. Hence $P \setminus \{0\}$ contains only positive or only negative elements, say all elements of $Q = \pm P$ are nonnegative. Let δ be the g.c.d. of all elements in Q. Then there exist $\alpha_1, \cdots, \alpha_s$ in Q, $\alpha_i > 0$, such that $\delta = \text{g.c.d.}(\alpha_1, \cdots, \alpha_s)$. By (4.2), $Q \supseteq \langle \delta : \nu \rangle$, for some $\nu > 0$. Since $Q \subseteq [\delta]$ $S = Q \setminus \langle \delta : \nu \rangle \subseteq \langle \delta : \delta \rangle$ and is finite, and obviously $S + S \subseteq Q$.

If P and R are proper cones in Z, then $[P] \cap [R] \neq \{0\}$. Hence by (3.7), all proper cones in Z are prime. This completes the proof.

(4.4) *Remark.* Let A be an ordered ring. Then all proper cones in A are prime if and only if any two nonzero left ideals in A have a nonzero intersection. Following Goldie [*12*] we call a ring A with this property *left uniform.* Thus A is left uniform if and only if

(C.1). For all non-zero $\alpha, \beta \in A$ there exist $\lambda, \mu \in A$ such that $\lambda\alpha = \mu\beta \neq 0$.

This condition is related to, but not equivalent to

(C.2). For all $\alpha, \beta \in A$, with β regular (not a zero divisor) there exist $\lambda, \mu \in A$, with λ regular such that $\lambda\alpha = \mu\beta$.

For rings without zero divisors (C.1) and (C.2) are equivalent. It is known that a ring A without zerodivisors satisfies (C.2) if and only if A has a *left quotient division ring* (see [*16*], p. 118 and [*12, 20*]). Thus in an ordered ring A with left quotient division ring all proper cones are prime.

V. The Boolean Algebra of Cones

The next lemma is of great importance. Propositions (5.2) and (5.3) below depend on it; the lemma and the Proposition (5.2) will be used heavily in the proof of our structure theorems. The lemma has no analogue in the case of module direct summands.

(5.1) Lemma. *Let P be a proper cone, and let $P \oplus P' = R$. If $r_1 + r_2 \in P'$, then $r_1, r_2 \in P'$.*

Proof. Let $r_1 = p_1 + p_1'$, $r_2 = p_2 + p_2'$ and $p' = r_1 + r_2$, then

$$p' = (p_1 + p_2) + (p_1' + p_2').$$

Since the sum is direct and $p_1 + p_2 \in P$, we see that $p_1 + p_2 = 0$. But P is proper, whence $p_1 = p_2 = 0$. Hence $r_1 = p_1'$, $r_2 = p_2'$.

(5.2) PROPOSITION. *If P is proper and $P \oplus P' = \oplus_I Q_i$, then*

$$P' = \oplus_I (P' \cap Q_i).$$

Proof. Evidently $\Sigma_I (P' \cap Q_i)$ is direct and $P' \supseteq \oplus_I (P' \cap Q_i)$. To prove the reverse inclusion, let $p' = \Sigma_I q_i = q_j + q'$, where $q' = \Sigma_L q_l$, $L = I \setminus \{j\}$. By (5.1), $q_j \in P'$, whence for each j, $q_j \in P' \cap Q_i$. It follows that $P' \subseteq \oplus_I (P' \cap Q_i)$.

(5.3) PROPOSITION. *Let P be a proper cone, and suppose that $P \supseteq Q$. Then $P \oplus P' = Q \oplus Q'$ implies that $P' \subseteq Q'$. In particular, if $P = Q$, then $P' = Q'$.*

Proof. By (5.2),

$$P' = (P' \cap Q) \oplus (P' \cap Q') = P' \cap Q' \subseteq Q'.$$

The algebraic properties of direct summation of subcones of a cone can best be characterized by the following theorem. For terminology, see Halmos [13] or Sikorski [26]. We recall that an atom is a minimal nonzero element of a Boolean algebra.

(5.4) THEOREM. *Let S be a proper cone, $\mathscr{S}$ the set of all subcones of S which are direct summands of S. Then, under the operations of intersection and addition, $\mathscr{S}$ is a Boolean algebra in which the prime cones correspond to atoms.*

Proof. It is clear that the operations are well-defined, associative, commutative, and idempotent. If P is a direct summand of S, by definition, there exists a complement P' such that

$$P \cap P' = \{0\} \qquad \text{and} \qquad P \oplus P' = P + P' = S;$$

furthermore it follows from (5.3) that P' is unique. Thus $\mathscr{S}$ is uniquely complemented. It is immediate that $\{0\}$ and S are the zero and unit. The prime cones are clearly atoms.

We show now that $\mathscr{S}$ is closed under the operations $\cap$ and $+$. Let $P \oplus P' = Q \oplus Q' = S$, then by (5.2), $P = (P \cap Q) \oplus (P \cap Q')$ and thus

$$(P \cap Q) \oplus ((P \cap Q') \oplus P') = S. \tag{*}$$

Hence, $P \cap Q$ is a direct summand and $\mathscr{S}$ is closed under intersection. To show that $\mathscr{S}$ is closed under addition, we show that

$$(P \cap Q)' = P' + Q'. \tag{**}$$

From (5.3), we have both

$$P' \subseteq (P \cap Q)' \quad \text{and} \quad Q' \subseteq (P \cap Q)',$$

hence

$$P' + Q' \subseteq (P \cap Q)'.$$

On the other hand, we have from (*) above that

$$(P \cap Q)' = (P \cap Q') + P'$$

which yields $(P \cap Q)' \subseteq Q' + P'$, and equality follows.

In the presence of unique complementation and (**), it is sufficient to prove one distributive law. We show that

$$(P \cap (Q + R)) = (P \cap Q) + (P \cap R).$$

It is obvious that

$$P \supseteq (P \cap Q) + (P \cap R) \quad \text{and} \quad Q + R \supseteq (P \cap Q) + (P \cap R)$$

hence

$$P \cap (Q + R) \supseteq (P \cap Q) + (P \cap R).$$

The reverse containment follows from (5.1). For, let $p \in P \cap (Q + R)$, then $p = q + r$, and since P' is proper, this yields that $q, r \in P$. It follows that

$$p \in (P \cap Q) + (P \cap R).$$

We remark that a result stronger than (5.3), namely (5.6) may be proved. We shall not use it in the sequel, but we feel it is of sufficient interest to include it.

(5.5) Lemma. *Let Q be a proper cone, and let*

$$Q \oplus Q' = R. \quad \textit{Then} \quad R \cap (-Q) = \{0\}.$$

Proof. Suppose $-q \in R$. Then $q - q = 0 \in Q'$, whence by (5.1) $q, -q \in Q$. Since Q is proper, $q = 0$.

(5.6) Proposition. *Let Q be a proper cone, and suppose that $P \supseteq Q$. Then $P \oplus P' = Q \oplus Q'$ implies that $P' \subseteq Q'$.*

Proof. Let $p' = q + q'$, $q' = p_1 + p_1'$. Then $p' = (q + p_1) + p_1'$ whence $q + p_1 = 0$, and $-q \in P \oplus P'$. By (5.5), it follows that $q = 0$, and so $p' = q' \in Q'$.

VI. Some Results on Boolean Algebras

If $\mathscr{B}$ is an abstract Boolean algebra, we shall denote its operations by $\wedge$, $\vee$, and $'$. A Boolean algebra is a partially ordered set in a natural way: $P \leqslant Q$

if and only if $P \wedge Q = P$. If $\mathscr{P}$ is any subset of $\mathscr{B}$, then we shall denote the supremum (least upper bound), if it exists, by $\bigvee \mathscr{P}$ or for $\mathscr{P} = (P_i)_I$ by $\bigvee_I P_i$ (cf. Sikorski [*26*, Chap. 2]).

(6.1) LEMMA. *Let $\mathscr{B}$ be a Boolean algebra, and let $\mathscr{P}$ be a set of atoms in $\mathscr{B}$. Let $\mathscr{P}(Q) = \{P \in \mathscr{P} : P \leqslant Q\}$. Then $Q \to \mathscr{P}(Q)$ is a homomorphism of $\mathscr{B}$ into the Boolean algebra $U(\mathscr{P})$ of subsets of $\mathscr{P}$ under the normal set theoretic operations.*

Proof. Let $P \in \mathscr{P}$. Note that $P \leqslant Q \wedge R$ if and only if $P \leqslant Q$ and $P \leqslant R$, whence

$$\mathscr{P}(Q \wedge R) = \mathscr{P}(Q) \cap \mathscr{P}(R).$$

We now show

$$\mathscr{P}(Q \vee R) = \mathscr{P}(Q) \cup \mathscr{P}(R).$$

If $P \leqslant Q$ or $P \leqslant R$ then $P \leqslant Q \vee R$. Conversely, suppose $P \leqslant Q \vee R$. Then

$$P = P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R).$$

Since P is an atom $P \wedge Q$ is 0 or P, and if $P \wedge Q = 0$ then $P \wedge R = P$. Hence $P \leqslant Q \vee R$ if and only if $P \leqslant Q$ or $P \leqslant R$, whence

$$\mathscr{P}(Q \vee R) = \mathscr{P}(Q) \cup \mathscr{P}(R).$$

Putting $R = Q'$, we obtain

$$\mathscr{P}(Q) \wedge \mathscr{P}(Q') = \mathscr{P}(Q \wedge Q') = \mathscr{P}(0) = 0,$$

and

$$\mathscr{P}(Q) \cup \mathscr{P}(Q') = \mathscr{P}(Q \vee Q') = \mathscr{P}(1) = \mathscr{P},$$

whence $\mathscr{P}(Q') = \mathscr{P}(Q')$, and the lemma is proved.

(6.2) PROPOSITION. *Let $\mathscr{B}$ be a Boolean algebra, and let $\mathscr{P}$ be a set of atoms in $\mathscr{B}$. If $1 = \bigvee \mathscr{P}$, then for all $Q \in \mathscr{B}$, $Q = \bigvee \mathscr{P}(Q)$, the mapping $Q \to \mathscr{P}(Q)$ is a monomorphism into $U(\mathscr{P})$, $\mathscr{B}$ is atomic, and $\mathscr{P}$ is the set of all atoms in $\mathscr{B}$.*

Proof. Let R be an upper bound of $\mathscr{P}(Q)$ in $\mathscr{B}$. Then $\mathscr{P}(R) \supseteq \mathscr{P}(Q)$, whence by (6.1)

$$\mathscr{P}(R \vee Q') = \mathscr{P}(R) \cup \mathscr{P}(Q') = \mathscr{P} = \mathscr{P}(1).$$

Since $Q \to \mathscr{P}(Q)$ is a homomorphism, $R \vee Q' = 1$, whence $R \geqslant Q$. Since Q is clearly an upper bound of $\mathscr{P}(Q)$, it follows that $Q = \bigvee \mathscr{P}(Q)$. Thus $\mathscr{P}(Q_1) = \mathscr{P}(Q_2)$ implies $Q_1 = Q_2$, and $Q \to \mathscr{P}(Q)$ is a monomorphism. Furthermore, for all $Q \neq 0$, there exists $P \in \mathscr{P}$ such that $P \leqslant Q$. Hence $\mathscr{B}$ is atomic. In particular, if P_1 is an atom in $\mathscr{B}$, then $P \leqslant P_1$, for some $P \in \mathscr{P}$, whence $P_1 = P \in \mathscr{P}$ and $\mathscr{P}$ is the set of all atoms in $\mathscr{B}$. The proposition is proved.

VII. Prime Decompositions of Cones

(7.1) Definition. If P_i are prime cones, $R = \oplus_I P_i$ will be called a *prime decomposition* of R.

(7.2) Theorem. *Let $\mathscr{P} = (P_i)_I$ be a family of prime cones and let $R = \oplus_I P_i$. Then the Boolean algebra $\mathscr{R}$ of direct summands of R is isomorphic to $U(\mathscr{P})$; $\mathscr{P}$ consists of all prime summands of R and every direct summand Q of R has a unique prime decomposition: $Q = \oplus\mathscr{P}(Q)$, where*

$$\mathscr{P}(Q) = \{P \in \mathscr{P} : P \leqslant Q\}.$$

Proof. Let $\mathscr{Q}$ be a subset of $\mathscr{P}$. It is clear that every upper bound of $\mathscr{Q}$ in $\mathscr{R}$ contains $\oplus\mathscr{Q}$. Since $\oplus\mathscr{Q}$ is a direct summand of $\mathscr{R}$, $\oplus\mathscr{Q} = \bigvee \mathscr{Q}$, and the operations $\oplus$ and $\bigvee$ coincide, even in the case of infinite subsets $\mathscr{Q}$ of $\mathscr{P}$. In particular, $\bigvee \mathscr{P} = R = 1$, and since $\mathscr{P}$ consists of atoms, the assertions of the theorem follow immediately from (6.2), except that we must still prove that $Q \rightarrow \mathscr{P}(Q)$ is onto $U(\mathscr{P})$. If $\mathscr{Q} \subseteq \mathscr{P}$ and $\mathscr{Q}' = \mathscr{P} \setminus \mathscr{Q}$, then

$$\mathscr{P}(\oplus\mathscr{Q}) \supseteq \mathscr{Q} \qquad \text{and} \qquad \mathscr{P}(\oplus\mathscr{Q}') \supseteq \mathscr{Q}'.$$

But $\mathscr{P}(\oplus\mathscr{Q}) \wedge \mathscr{P}(\oplus\mathscr{Q}') = \phi$ since $\oplus\mathscr{Q}'$ is the complement of $\oplus\mathscr{Q}$ in $\mathscr{R}$. Hence $\mathscr{P}(\oplus\mathscr{Q}) = \mathscr{Q}$, and the proof is complete.

(7.3) Corollary. *Let E be a module over an ordered ring, and let R be a cone in E with a prime decomposition. Then the prime decomposition is unique.*

As mentioned in the introduction, there are no comparable uniqueness theorem for decompositions of modules or vector spaces.

(7.4) Corollary. *Let $(P_i)_I$ be a family of prime cones and suppose $R \supseteq \oplus_I (P_i)$. For each i, let $P_i \oplus P_i' = R$ and suppose $\cap_I P_i' = \{0\}$. Then R has a prime decomposition if and only if $R = \oplus_I P_i$.*

Proof. Suppose R has a prime decomposition. Let Q be a prime direct summand of R, $Q \neq P_i$ for all i. By (5.2),

$$Q = (Q \cap P_i) \oplus (Q \cap P_i') = Q \cap P_i'$$

whence $Q \subseteq P_i'$ for all i. Hence $Q \subseteq \cap_I P_i' = \{0\}$. But this is a contradiction. Hence $R = \oplus_I P_i$. The converse is obvious.

VIII. Direct Product of Cones

(8.1) Definition. As usual, the *direct product* $\Pi_I P_i$ of cones P_i in E_i will consist of all (generalized) sequences $(p_i)_I$, with $p_i \in P_i$. This product is contained in the direct product $\Pi_I E_i$. If we make the natural identification of $p_j \in P_j$ with the sequence $(q_i)_I$, $q_i = p_j$, $i = j$, $q_i = 0$ otherwise, then as usual $\oplus_I P_i$ is the subcone of $\Pi_I P_i$ consisting of all $(p_i)_I$ with only a finite number of nonzero p_i. If $\mathscr{P} = (P_i)_I$, we may also write $\Pi\mathscr{P}$ for $\Pi_I P_i$, when convenient.

(8.2) LEMMA. *Let $(P_i)_I$ be a family of proper cones. If*

$$\oplus_I P_i \subseteq R \subseteq \Pi_I P_i = S,$$

and R is a direct summand of S, then $R = S$.

Proof. Let $P_i'' = \Pi_{j \neq i} P_j$. Then $S = P_i \oplus P_i''$. Suppose $S = R \oplus R''$. By (5.2) $R = P_i \oplus P_i'$, where $P_i' = R \cap P_i''$. Therefore $S = P_i \oplus P_i' \oplus R''$, whence $R'' \subseteq P_i''$, for each i. Hence $R'' \subseteq \cap_I P_i'' = \{0\}$, and so $R = S$.

(8.3) THEOREM. *Let $\mathscr{P}$ be a family of prime cones and let $S = \Pi\mathscr{P}$. Then the Boolean algebra $\mathscr{S}$ of direct summands of S is isomorphic to the Boolean algebra $U(\mathscr{P})$; $\mathscr{P}$ consists of all prime cones in $\mathscr{S}$, and every direct summand Q of S can be expressed uniquely as the direct product of prime cones: $Q = \Pi\,\mathscr{P}(Q)$, where $\mathscr{P}(Q) = \{P \in \mathscr{P} : P \leqslant Q\}$.*

Proof. Let $\mathscr{Q} \subseteq \mathscr{P}$. We show now that $\vee\mathscr{Q}$ exists, and $\vee\mathscr{Q} = \Pi\mathscr{Q}$. Clearly $\Pi\mathscr{Q} \in \mathscr{S}$, and $\Pi\mathscr{Q}$ is an upper bound for $\mathscr{Q}$. Let $R \in \mathscr{S}$ and suppose $R \subseteq \Pi\mathscr{Q}$ and R is an upper bound for $\mathscr{Q}$. Since R is closed under addition, $R \supseteq \oplus\mathscr{Q}$. Hence, by (8.2) (applied to $\mathscr{Q}$) $R = \Pi\mathscr{Q}$ and $\vee\mathscr{Q} = \Pi\mathscr{Q}$. In particular $S = \vee\mathscr{P}$. Thus (6.2) applies and the proof that $Q \to \mathscr{P}(Q)$ is onto is similar to that of the theorem (7.2).

If R and S are *cones*, a 1–1 mapping φ of R onto S is called an *isomorphism* if φ preserves addition and positive scalar multiplication.

(8.4) COROLLARY. *Let $(P_i)_I$, $(Q_j)_J$ be families of prime cones, and let $R = \Pi_I P_i$, $S = \Pi_J Q_j$. Let φ be an isomorphism of R onto S. Then, for a suitable reindexing of the Q_j, $J = I$, and $\varphi(P_i) = Q_i$.*

Proof. It is easy to show that φ induces an isomorphism between the Boolean algebras $\mathscr{R}$ and $\mathscr{S}$ of direct summands of R and S, and hence take the family of all atoms in $\mathscr{R}$ onto the family of all atoms *in* $\mathscr{S}$ in a 1–1 fashion.

(8.5) COROLLARY. *The Boolean algebras of direct summands of $\oplus_I P_i$ and $\Pi_I P_i$ are isomorphic. If I is infinite, the cone $\Pi_I P_i$ has no prime decomposition. Moreover, $\oplus_I P_i$ cannot be expressed as the direct product of prime cones.*

Proof. The first statement follows immediately from the theorems (7.2) and (8.3). If $S = \Pi_I P_i$ had a prime decomposition, it would have to be

$\oplus_I P_i$, since $(P_i)_I$ consists of all prime cones in S. But, for infinite I, $\oplus_I P_i \subset \Pi_I P_i$, the $\subset$ indicating proper containment. The proof of the last assertion concerning $\oplus_I P_i$ is similar.

(8.6) *Remark.* Let A be a left uniform ordered ring (see (4.4)), so that all proper cones in A are prime. Let $E = \Pi_I A_i$, $A_i = A$; let P_i be a proper cone in A_i. Let $S = \Pi_I P_i$, where I is an infinite set. Let $\mathscr{E} = \{x_k\}_K$, $x_k \in S$, be any free subset of S : that is $\sum_K \beta_k x_k = 0$ implies that all $\beta_k = 0$. Let $Q = \langle \mathscr{E} \rangle$. Then it follows from (8.3) that Q is properly contained in S, for $Q = \oplus_K \langle x_k \rangle$, and the $\langle x_k \rangle$ are prime. In particular, let $\mathscr{E}$ be a basis for the space of all (countable) real sequences, and suppose that each basis element has non-negative entries. Then the cone generated by $\mathscr{E}$ is not the cone of all sequences with nonnegative entries.

(8.7) *Remark.* We can show that for every left uniform ordered ring A there exists a countably generated A-module E which contains a proper cone without prime decomposition. Let $\mathscr{E}$ be the set of all countable sequences $x = (\alpha_1, \alpha_2, \alpha_3, \cdots)$ such that $\alpha_i = 0$ or $\alpha_i = 1$ and the α_i are periodic from some point on; i.e., there is an integer $m = m(x)$ such that for all $n > n_0(x)$, $\alpha_{m+n} = \alpha_n$. The set $\mathscr{E}$ is countable, since the set of all sequences not ending in all 1's corresponds, by the base 2 "decimal" expansion, to the set of all rationals in (0,1); while the sequences ending in all 1's correspond to the set of rationals in [0, 1] of the form $k/2^n$. Let $S = [\mathscr{E}]$, where $\mathscr{E}$ is considered as a subset of $\Pi_1^\infty A_i$, where $A_i = A$.

Let $P_i \subseteq S$ be the subcone of all sequences with $\beta_i \geqslant 0$ and $\beta_j = 0$ if $j \neq i$. We can use the arguments from the proof of (8.2) to show that $\sum_{i=1}^\infty P_i$ is not a direct summand of S. It follows, in a fashion analogous to the proof of the theorem (8.3) that S has no prime decomposition.

(8.8) *Examples.* Let S be the cone of nonnegative valued real functions of a real variable. Clearly S is isomorphic to an infinite direct product of nonnegative reals, and, as we have already pointed out, S has therefore no prime decomposition. Let α be real and suppose now R consists of all $f \in S$ for which $f(x) \to 0$ as $x \to \alpha$. Then R satisfies the condition of (7.4), and hence R has no prime decomposition. A similar conclusion holds for the cone of all bounded functions in S.

IX. Finite Prime Decompositions

For the existence of prime decompositions we require some finiteness conditions.

(9.1) DEFINITION. A cone R satisfies the descending summand condition (DSC) if every chain $R = P_0 \supseteq P_1 \supseteq P_2 \supseteq P_3 \cdots$, where P_i is a direct summand of R, stops: i.e., $P_s = P_{s+1} = \cdots$ for some s. The cone R satisfies the *ascending summand condition* (ASC) if every chain

$$\langle 0 \rangle = P_0' \subseteq P_1' \subseteq P_2' \subseteq \cdots,$$

where each P_i is a direct summand of R, stops.

Since an ascending chain of summands corresponds to a descending chain of complements, it is clear that ASC and DSC are equivalent conditions on proper cones. It is known, but more difficult to prove, that if A is a division ring ASC and DSC applied to vector spaces over A are also equivalent [*1*, *2*].

(9.2) LEMMA. Let R be a proper cone. There exists a finite prime decomposition of $R : R = P_1 \oplus \cdots \oplus P_s$, if and only if R has ASC (or DSC).

Proof. Let $\mathscr{R}$ be the Boolean algebra of direct summands of R. If $Q \in \mathscr{R}$ and Q is not prime, then $Q = Q_1 \oplus Q_2$, where Q_1 and Q_2 are proper. Hence, by DSC, every $Q \in \mathscr{R}$ contains a prime direct summand. Let $\mathscr{P}$ be the set of prime cones in R. If $P_i \in \mathscr{P}$, and $P_1 \oplus \cdots \oplus P_s \subset R$ then, since $\mathscr{R}$ is a Boolean algebra, $P_1 \oplus \cdots \oplus P_s \oplus Q' = R$ and so, for some prime cone P_{s+1}, $P_{s+1} \subseteq Q'$. Hence $P_1 \oplus \cdots \oplus P_s \subset P_1 \oplus \cdots \oplus P_{s+1}$. By ASC this process must stop, whence $R = P_1 \oplus \cdots \oplus P_s$, for some s. The converse is obvious.

(9.3) THEOREM. *Let A be an ordered ring with ascending chain condition on left ideals, and let E be a finitely generated A module. Then every proper cone R in E has a finite prime decomposition. In particular, this is true if (i) E is a finitely generated abelian group, or (ii) E is a finite dimensional vector space over an ordered division ring.*

Proof. Let R be a proper cone in E, and let $\langle 0 \rangle = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$ be an ascending chain of direct summands of R. Since $\mathscr{R}$ is a Boolean algebra, $P_s = Q_1 \oplus \cdots \oplus Q_s$ where $Q_i = P_i \cap P_{i-1}'$. Hence, by (3.7),

$$[P_s] = [Q_1] \oplus \cdots \oplus [Q_s].$$

But E has ascending chain condition on submodules [*6*, p. 56], hence $[P_s] = [P_{s+1}] = \cdots$. Hence $\langle 0 \rangle = Q_{s+1} = Q_{s+2} = \cdots$ and $P_s = P_{s+1} = \cdots$. Thus R has ASC, and the theorem follows from (9.2).

X. SOME TOPOLOGICAL NOTIONS

(10.1) DEFINITION. *Let X be a subset of a module E over an ordered* ring A. The *core* of X consists of all x, such that for every $y \in E$ there exists such a $\kappa = \kappa(x, y) > 0$ that $x + \alpha y \in X$ for all α, $0 \leqslant \alpha < \kappa$. We shall denote the core of X by $\iota(X)$. The *expanse* of X is defined to be

$$\epsilon(X) = E \setminus \iota(E \setminus X),$$

and the *crust* of X is $\partial(X) = \epsilon(X) \setminus \iota(X)$. The *relative core* (expanse, crust) of X consists of the core (expanse, crust) of X considered as a subset of the module $[X]$ and will be denoted by $\iota'(X)\,(\epsilon'(X), \partial'(X))$. If $\iota(X) = X$, we call X a *core set*.

It is evident that $\iota(X) \cap \partial(X) = \phi$, $\iota(X) \subseteq X$ and $\iota(X) \cup \partial(X) = \epsilon(X)$. If E is a real vector space, then our term core which has been considered by many authors, e.g., Klee [*19*], coincides with the terminology of Day ([*7*], p. 10.) In their book Dunford-Schwartz ([*8*], p. 410) call a core point an "internal point" and a crust point a "bounding point."

Since the core operator ι defined above is not idempotent, the notion of core does not immediately give rise to a topology. Study of such operators has been revived by the work of Hammer [*14*] on extended topologies. There is, however, a natural topology which is the unique strongest topology in which the topological boundary of every set contains its crust. In a real normed linear space, for example, the boundary of any set in any of the following topologies, norm topology, convex core topology ([*7*], p. 16), E^{+} topology ([*8*], p. 419), etc., contains the crust of the set. In the sequel we shall prove that various cones are in the crust of a given cone. In view of the above remark, the reader may substitute boundary for crust and obtain a weaker, but more conventional result.

The notion of crust and boundary coincide for convex sets in the case of finite dimensional Euclidean spaces with the usual topology. This can easily be proved with the aid of the standard theorems on separation and relative topology of convex sets (cf. Eggleston [*9*], Chap. 1 and Klee [*19*]).

The authors elucidate more fully the above comments in their forthcoming paper [*3*].

The order on A is called *discrete* if for some $\epsilon > 0$, $0 \leqslant \alpha < \epsilon$ implies that $\alpha = 0$, otherwise A is *nondiscrete*. If A is discrete, then every subset X of E is a core set, i.e., $\iota(X) = X$.

Following standard terminology an element $p \in E$ is called *free* if $\alpha p = 0$ implies $\alpha = 0$.

(10.2) THEOREM. *Let R be a cone in a module over a nondiscrete ordered ring, and let $P \oplus P' = R$. If P is proper and contains a free element, then P' is contained in the relative crust of R; in fact $P' \subseteq \partial'(R) \cap R$.*

Proof. For suppose $p' \in \iota'(R) \cap P'$. Let p be a free element in P. Then for some $\alpha > 0$, $r = p' - \alpha p \in R$. Hence $p' = r + \alpha p$, $\alpha p \in P$, and $\alpha p \neq 0$. This contradicts (5.1). Thus $P' \subseteq \partial'(R)$ and obviously $P' \subseteq R$.

In view of the above theorem we find it expedient to make the following definition.

(10.3) DEFINITION. A module E over an ordered ring is called a *PF-module* if every proper cone in E contains a free element.

Evidently, if E is a module in which every element which generates a *proper cone* is a free element, then E is a *PF*-module. In the next section we shall show that, conversely, every *PF*-module satisfies this apparently much stronger condition. It is clear that all vector spaces are *PF*-modules. More generally, any module which can be embedded in a free module, and in particular any projective module [*23*, p. 63] is a *PF*-module. It is clear that the product of a family of modules over a given ring is a *PF*-module if and only if each factor is again a *PF*-module. Further, if the ring A is considered as a left module over itself, then it is a *PF*-module if and only if A has no zero divisors.

(10.4) COROLLARY. *If R is a proper cone in a PF-module over a nondiscrete ordered ring, then every direct summand of R is contained in the relative crust of R.*

Proof. The proof is immediate from the Theorem (10.2) and (3.8).

(10.5) COROLLARY. *If R is a cone in a PF-module over a nondiscrete ordered ring such that $R \setminus \{0\}$ is a (relative) core set, then R has no nontrivial direct summands, which are proper cones.*

Proof. The corollary follows from Theorem (10.2), since $\partial'(R) \cap R = \{0\}$.

(10.6) COROLLARY. If R is a proper cone in a *PF*-module over a nondiscrete ordered ring for which $R \setminus \{0\}$ is a (relative) core set, then R is prime.

Proof. Follows from the corollary (10.4), since $\partial'(R) \cap R = \{0\}$.

(10.7) PROPOSITION. *Let R be a cone in a PF-module E. If R has an infinite prime decomposition, then both $\iota(R)$ and $\iota'(R)$ are empty.*

Proof. Let $r \in R$, then $r = \sum_I p_i$, where all but finitely many p_i are 0. Suppose, in particular, $p_1 = 0$. Since E is a *PF*-module there exists a free element $p \in P_1$. Clearly $-p \in [R]$, and for all $\alpha > 0$, we have $r + \alpha(-p) \notin R$. Hence $\iota'(R) = \phi$. Since $\iota'(R) \supseteq \iota(R)$ we see that $\iota(R) = \phi$ also.

(10.8) *Examples.* (i) Let A be an ordered ring, and let P be the cone of nonnegative elements of A, considered as a module over itself. Then it follows from the definitions that $\pi \in \iota(P) = \iota'(P)$ if an only if for every λ

there exists a positive κ such that $\kappa\lambda < \pi$. Upon setting $A = A_0$ (see (2.8)) we obtain a module in which $\iota(P) = P \setminus \{0\}$. Similarly if $E = A \oplus \cdots \oplus A$ and $R = P \oplus \cdots \oplus P$ then for $r = (p_1, \cdots, p_n) \in R$ we have $r \in \iota(R)$ if and only if each p_i is positive. The same situation would arise if A were to be an ordered division ring.

(ii) In (i) above, we take $A = B_0$ (see (2.8)) and define E, P, and R as before. Then $\iota(P) = \phi$, and hence we also obtain $\iota(R) = \phi$. The comparison of (i) and (ii) is even more interesting when we recall that A_0 and B_0 are algebraically isomorphic, but have different orderings.

(iii) Many other cones have no core. Let E be the module of real valued functions of a real variable. Then both the cone of all continuous nonnegative valued functions which vanish outside an open interval and the cone of all nonnegative functions have an empty core.

(iv) Again, let $A = B_0$, and consider the module E of all functions from an infinite set I into A; clearly E is a PF-module. Let R be the cone of all functions such that $f(i) \geqslant 0$, all i. Then $f \in R$ is in $\iota(R)$ if and only if there is a uniform bound on the degree of the polynomials $f(i)$ for all $\iota \in I$. More generally, let A be a ring satisfying the following condition of infinitesimality. For all $\lambda > 0$ there is a $\mu \in A$ such that $\lambda > \mu\gamma$, for all γ. Then $f \in R$ is in $\iota(R)$ if and only if f is bounded away from 0, i.e., there exists $\lambda > 0$ such that, for all i, $f(i) \geqslant \lambda$.

(v) Let R consist of all continuous functions f from the unit interval into the real line such that, for each f, there exists b, $0 \leqslant b \leqslant 1$, for which $f(b) > 0$ and f is nonnegative on the closed interval $[0, b]$. Here the crust $\partial(R)$ consists of all f such that $f(0) = 0$. Suppose $R = P \oplus Q$, and let $f = p + q$. Then by (10.4), $P \subseteq \partial(R)$ and $P' \subseteq \partial(R)$, whence

$$p(0) = q(0) = 0$$

and so $f(0) = 0$. It follows that R is prime.

XI. Results on PF-Modules

(11.1) Definition. If $p \in E$, then the set $B_p = \{\beta : \beta p = 0\}$ will be called the *annihilator* of p in A.

We recall that the term initial left ideal in A was defined in (2.11).

(11.2) Lemma. *Let E be a module over an ordered ring A. If p is in E, then $\langle p \rangle$ is proper if and only if the annihilator B_p of p is an initial left ideal.*

Proof. Let $\langle p \rangle$ be a proper cone, and let $\beta p = 0$, $\beta \geqslant 0$. Suppose $0 \leqslant \alpha \leqslant \beta$. Then

$$\alpha p + (\beta - \alpha) p = \beta p = 0,$$

and so $\alpha p \in \langle p \rangle$, $-\alpha p = (\beta - \alpha) p \in \langle p \rangle$. Hence $\alpha p = 0$. Since B_p is clearly a left ideal and $1 \notin B_p$, it follows that B_p is an initial left ideal. Conversely, let B_p be an initial left ideal. Suppose $q \in \langle p \rangle$, $-q \in \langle p \rangle$. Then for some $\alpha \geqslant 0$, $\gamma \geqslant 0$, $q = \alpha p$, $-q = \gamma p$, whence

$$(\alpha + \gamma) p = q - q = 0.$$

Since $0 \leqslant \alpha \leqslant \alpha + \gamma$ and B_p is initial, $q = \alpha p = 0$. Thus $\langle p \rangle$ is proper.

(11.3) DEFINITION. A nonempty subset C of an ordered ring A is called a *tail* if (i) for every $\alpha \in A$, $\alpha C \subseteq C$, and (ii) there exists $\gamma \geqslant 0$ such that $\alpha \geqslant \gamma$ implies that $\alpha \in C$.

(11.4) PROPOSITION. *Let E be a module over an ordered ring A. Let $p \in E$, and suppose $\langle p \rangle$ is a proper cone with annihilator B_p. Then $\langle p \rangle$ contains a free element if and only if there exists a tail C in A for which $B_p \cap C = \{0\}$.*

Proof. Suppose $\langle p \rangle$ contains the free element γp. Let C be the tail consisting of all $\lambda\alpha$ for $\lambda, \alpha \in A$ with $\alpha \geqslant \gamma$. Suppose $\beta \in B_p \cap C$. Then $\beta = \lambda\alpha$, for $\lambda \in A$, $\alpha \geqslant \gamma$. By (11.2), B_p is an initial left ideal, and so $\lambda\gamma \in B_p$. Hence $(\lambda\gamma) p = \lambda(\gamma p) = 0$. But this implies that $\lambda = 0$, whence $\beta = 0$. Thus $B_p \cap C = \{0\}$.

Conversely, let C be a tail for which $B_p \cap C = \{0\}$. Clearly there exists $\gamma \in C$, $\gamma \geqslant 1$. For such γ, $\lambda\gamma p = 0$ implies that $\lambda\gamma \in B_p \cap C$ whence $\lambda\gamma = 0$, and as γ is regular, $\lambda = 0$. Hence γp is free.

(11.5) LEMMA. *Let A be an ordered ring. Let B be an initial left ideal in A and C a tail such that $B \cap C = \{0\}$. Then $B^2 = \{0\}$, and, unless $B = \{0\}$, B is not a two-sided ideal. If A is commutative, then $B = \{0\}$.*

Proof. Let $\beta \geqslant 0$, $\beta \in B$, and $\gamma \geqslant 1$, $\gamma \in C$. Since $1 \notin B$ and $\gamma\beta \in B$, we obtain $\gamma\beta < 1$. Hence $\gamma\beta\gamma < \gamma$, and it follows that $\beta\gamma < 1$. Thus, if $\beta' \geqslant 0$, $\beta' \in B$, then $\beta'\beta\gamma \leqslant \beta'$ whence $\beta'\beta\gamma \in B \cap C$. Thus $\beta'\beta\gamma = 0$, and as γ is regular, $\beta'\beta = 0$. We deduce that $B^2 = \{0\}$. If $\beta \in B$, and $\beta \neq 0$, then $\beta\gamma \neq 0$ and $\beta\gamma \in C$. So $\beta\gamma \notin B$, and it follows that B is not a two-sided ideal. If A is commutative, every left ideal is two-sided whence $B = \{0\}$.

The propositions (11.2), (11.4) and (11.5) can be partially summarized in the following theorem.

(11.6) THEOREM. *Let E be a module over an ordered ring A. Let $p \in E$. Then $\langle p \rangle$ is a proper cone with a free element if and only if the annihilator B_p is an initial left ideal such that $B_p^2 = \{0\}$, and there exists a tail C for which $B_p \cap C = \{0\}$.*

The following corollary is obvious.

(11.7) COROLLARY. *Let E be a module over an ordered ring A without zero divisors. Then* $\langle p\rangle$ *is a proper cone with a free element if and only if p is itself free.*

The corollary could have been proved under the apparently weaker hypothesis on A that A has no nilpotent left ideals. But it follows from the theorem (2.14) that an ordered ring without nilpotent left ideals has no zero divisors.

We recall that *PF*-module was defined in (10.3).

(11.8) THEOREM. *Let E be a module over an ordered ring A. Then E is a PF-module if and only if every* $p \in E$ *such that* $\langle p\rangle$ *is proper is a free element.*

Proof. If every $p \in E$ that generates a proper cone is free, then clearly E is a *PF*-module. Suppose that $\langle p\rangle$ is a proper cone and that the annihilator B_p is nonzero. We shall show that E is not a *PF*-module. If $\langle p\rangle$ contains no free element, there is no more to prove. So suppose that for some $\gamma > 0$, γp is free. By the theorem (11.6), $B_p^2 = \{0\}$. Let $\beta \in B_p$, $\beta > 0$. Set $q = \beta\gamma p$.

Clearly $q \neq 0$, and as $\langle q\rangle \subseteq \langle p\rangle$, the cone $\langle q\rangle$ is proper. If $\alpha > 0$, then $\beta\alpha q = \beta\alpha\beta\gamma p = 0$ since $\beta\alpha\beta \in B_p^2 = \{0\}$. Hence, there does not exist an α such that αq is free, and the theorem is proved.

(11.9) THEOREM. *Let A be an ordered ring. Then the following conditions are equivalent.*

(i) *Every A module is a PF-module,*
(ii) *The only initial left ideal in A is* $\{0\}$.
(iii) *The only initial ideal in A is* $\{0\}$.

Proof. The equivalence of (ii) and (iii) is due to Johnson [*18*] (cf. Fuchs [*10*], Chap. 8, Theorem 9). We need therefore only prove the equivalence of (i) and (ii). So suppose that (ii) holds. Let E be an A-module and let $p \in E$ generate a proper cone. Then $B_p = \{0\}$, since B_p is an initial left ideal. Thus p is free, and (i) follows.

To prove the converse, suppose there exists an initial left ideal B in A. Consider the A-module A/B. The annihilator of $1 + B$ is exactly B. Hence, by (11.2), $\langle 1 + B\rangle$ is a proper cone. But $1 + B$ is not free. Hence by (11.8), A/B is not a *PF*-module. The theorem follows.

(11.10) COROLLARY. *If every A-module is a PF-module, then A has no zero divisors.*

Proof. By (2.14), an ordered ring with zero divisors has a nonzero initial ideal.

(11.11) *Example.* We shall give an example of an ordered module E which is not a PF-module, in which every $p \in E$, $p \neq 0$, generates a proper cone, and which contains elements of the following types: (i) p is free, (ii) p is not free, but $\langle p \rangle$ contains a free element, (iii) $\langle p \rangle$ does not contain any free element.

Let S be the semigroup with 0 of all words in s and t, containing at most one t. The multiplication of two words is by juxtaposition unless both words contain t, in which case the product is 0. We order S thus (a) If $m > m'$, $s^m > s^{m'}$, (b) $s^m > s^n t s^p$, (c) $s^n t s^p > s^{n'} t s^{p'}$ if $p > p'$ or $p = p'$ and $n > n'$. Now let A be an ordered ring without zero divisors, say the integers. Let $A(S)$ be the semigroup ring of S over A. Let B consist of all α which involve only terms from S of form $s^n t s^p$; then B is an initial ideal in $A(S)$. We now define a family of initial left ideals in $A(S)$: Let B_i consist of all α which involve only terms of form $s^n t s^p$, with $p \leqslant i$.

For some $i \geqslant 0$, consider the A-module $E = A/B_i$, which is ordered by (2.12). For $\alpha \in A$, denote $\alpha + B_i$ by $\bar{\alpha}$.

(i) If α involves a term s^m, with $m > i$, then $\bar{\alpha}$ is free.

(ii) If α involves a term s^m, $m \leqslant i$, but no term s^n, $n > i$, then $B_{\bar{\alpha}} = B_{(i-m)}$. Hence $\langle \bar{\alpha} \rangle$ is proper, by (11.2). Further, for $j > (i - m)$ $s^j \bar{\alpha}$ is in $\langle \bar{\alpha} \rangle$ and is a free element as remarked above.

(iii) If $\alpha \in B$, then $B_{\bar{\alpha}} = B$. Hence, by (11.2), $\bar{\alpha}$ is proper, but since B is a nonzero two-sided ideal in A, it follows from (11.4) and (11.5) that $\langle \bar{\alpha} \rangle$ does not contain a free element.

Every element in A/B_i is of one of the three types considered.

XII. Application to Semidefinite Hermitian Matrices

Now let A be the real field, E the space of $n \times n$ Hermitian matrices. If R is the cone of positive semidefinite Hermitian matrices, then $\partial(R)$ consists of all singular matrices in R.

(12.1) Lemma. *Let P be a cone of singular, positive semidefinite Hermitian matrices. Then for some vector x, $x \neq 0$, we have $Hx = 0$, for all $H \in P$.*

Proof. Since $[P]$ is finite dimensional, there exists a finite set $H_1, \cdots, H_s$, $H_i \in P$, of generators for $[P]$ over the reals. Since $H = H_1 + \cdots + H_s$ is again singular, $Hx = 0$, for some $x \neq 0$; hence

$$x^*Hx = x^*H_1x + \cdots + x^*H_sx = 0$$

and since the H_i are semi-definite, $x^*H_ix = 0$, $i = 1, \cdots, s$. But this implies $H_ix = 0$, $i = 1, \cdots, s$ [9, Vol. 2, p. 322]. Since all elements of P are real linear combinations of $H_1, \cdots, H_s$, the result follows.

(12.2) THEOREM. *The cone R of positive semidefinite Hermitian matrices is prime.*

Proof. Let $R = P \oplus Q$, where P, Q are nonzero. Since R is proper, (10.2) applies; thus, $P \subseteq \partial(R)$, $Q \subseteq \partial(R)$. By (12.1), there exist nonzero vectors x, y such that $Hx = 0$ for all $H \in P$, and $Ky = 0$ for all $K \in Q$. Hence for all $L \in R$, we may write $L = H + K$, and so

$$y^*Lx = y^*Hx + y^*Kx = 0.$$

But we shall show that this is impossible. Normalize x, y so that

$$x^*x = y^*y = 1 \qquad \text{and} \qquad \beta = y^*x \geqslant 0.$$

Then

$$M = (x + y)(x + y)^* \in R \qquad \text{and} \qquad y^*Mx = (1 + \beta)^2 \neq 0.$$

The theorem is proved.

Let E, F be two left A-modules over an ordered ring A, and let γ be a homomorphism of E into F. If R is a prime cone in E, then $\gamma(R)$ need not be prime. As an example, take the prime cone R of positive semidefinite matrices in the real space of (2×2) complex matrices. For $H = (h_{ij})$, $i, j = 1, 2$, define $\gamma(H) = \operatorname{diag}(h_{11}, h_{22})$. Then $\gamma(R)$ is not prime. On the other hand, it is easy to check that if γ is an isomorphism, then γ preserves direct summands, and hence preserves prime decompositions. We shall apply this remark to cones of matrices. If M is a complex $n \times n$ matrix, we can write $M = H + iK$, where H, K are Hermitian and further this representation is unique since $H = \frac{1}{2}(M + M^*)$, $K = (1/2i)(M - M^*)$. Hence if R is as above, the sum $S = R + iR$ is direct. Thus S consists of all $H + iK$, where both H and K are positive semidefinite Hermitian.

The authors believe the following theorem on Hermitian matrices is new.

(12.3) THEOREM. *Let γ be a real linear transformation of the real space E of all complex $(n \times n)$ matrices into itself. Let R be the cone of positive semidefinite Hermitian matrices, and let $S = R + iR$. If $\gamma(S) = S$ then either $\gamma(R) = R$ and $\gamma(iR) = iR$ or $\gamma(R) = iR$ and $\gamma(iR) = R$.*

Proof. Since $M \to iM$ is a real isomorphism on E, it follows from the theorem (12.2) that iR is prime as well as R. Hence $R + iR$ is a prime decomposition of S. Since $[S] = E$, and $\gamma(S) = S$, γ is an isomorphism. Hence $S = \gamma(S) = \gamma(R) + \gamma(iR)$ is a prime decomposition also. The result now follows from the uniqueness of prime decompositions (7.3).

In [*25*] it is shown that $\gamma(R) = R$ if and only if, for some matrix X either $\gamma(H) = XHX^*$, for all H, or $\gamma(H) = XH^tX$, for all H.

The question arises if there is a similar result for $R \oplus iT$, where T consists of all Hermitian matrices. Since T is not proper, its complement need not be unique, e.g., the cone of matrices $(1 + i) H$, $H \in R$, is another complement. However, it is very easy to prove that if γ is a real linear transformation taking $R + iT$ onto itself, then $\gamma(T) = T$.

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PRINTED IN BRUGES, BELGIUM, BY THE ST CATHERINE PRESS LTD.