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Inertia Theorems for Matrices: The Semidefinite Case

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I. INTRODUCTION

1a. The *inertia* of an $n \times n$ matrix A with complex elements is defined to be the integer triple $\operatorname{In} A = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ $\{\nu(A)\}$ is the number of eigenvalues of A in the open right $\{$left$\}$ half-plane, and $\delta(A)$ is the number of eigenvalues on the imaginary axis. The best-known classical theorem on inertias is that of Sylvester [1, I p. 296; 2], which may be stated as:

If $P > 0$ (positive definite), and H is Hermitian, then $\operatorname{In} PH = \operatorname{In} H$.
Lyapunov's theorem [3, p. 245; 1, II p. 187; 4, 5] is less well-known:
For a given A, there exists an $H > 0$ such that

$$\mathscr{R}(AH) = \tfrac{1}{2}(AH + HA^*) > 0$$

if and only if $\operatorname{In} A = (n, 0, 0)$.

Both classical theorems are contained in a generalization due to Taussky [6] and to Ostrowski and Schneider [7], which we shall call the

MAIN INERTIA THEOREM. *For a given A, there exists a Hermitian H such that $\mathscr{R}(AH) > 0$ if and only if $\delta(A) = 0$. If $\mathscr{R}(AH) > 0$, then* $\operatorname{In} A = \operatorname{In} H$.

1b. In this paper we shall discuss the situation when we require only that $\mathscr{R}(AH) \geq 0$ (positive semidefinite). In this case the relation of $\operatorname{In} H$ to $\operatorname{In} A$ may be very complex and we shall here, in Section 2, solve the problem only in two special cases; first, under the assumption

(1.1) *All elementary divisors of imaginary roots (if any) of A are linear,* and second when A consists of just one Jordan block belonging to one imaginary root.

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We deduce two general existence theorems from these special cases:

COROLLARY II. 1. *For any A, there exists a nonsingular H such that* $\mathscr{R}(AH) \geq 0$.

COROLLARY III. 1. *For given A, there exists an $H > 0$ for which* $\mathscr{R}(AH) \geq 0$ *if and only if (1.1) holds and* $\nu(A) = 0$.

Corollary II. 1 was proved by Givens [8] for matrices satisfying (1.1).

In Section 3, we discuss the relationships between the null-spaces $\mathscr{N}(H)$ and $\mathscr{N}(\mathscr{R}(AH))$ and given subspaces $\mathscr{N}$ of the (column) space on which these matrices act. Given a matrix A and subspace $\mathscr{N}$ of the space of all columns, Theorem V gives a necessary and sufficient condition for the existence of an H such that $\mathscr{R}(AH) \geq 0$ and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}(H) = \mathscr{N}$. In this case, $\operatorname{In} H \leq \operatorname{In} A$. Here we define

$$\operatorname{In} B \leq \operatorname{In} A: \quad \pi(B) \leq \pi(A), \qquad \nu(B) \leq \nu(A).$$

This is a generalization of the Main Theorem, which is obtained from Theorem V by setting $\mathscr{N} = (0)$. Again, given a matrix A and subspace $\mathscr{N}$, Theorem VI answers the question: when does $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}$ imply that $\mathscr{N}(H) \supseteq \mathscr{N}$. This theorem generalizes the well-known result that $\mathscr{R}(AH) = 0$ implies that $H = 0$ if and only if $\alpha_i + \bar{\alpha}_j \neq 0$ for all pairs of eigenvalues α_i, α_j of A.

A matrix A is called H-stable if, for each Hermitian H, $\operatorname{In} AH = (n, 0, 0)$ if and only if $H > 0$ (cf. [7]). In [7], a necessary and sufficient condition was found for H-stability, but this was of the nature of an existence theorem. In Section 4 we provide an effective test for H-stability. Necessary conditions include that A be nonsingular, and $\mathscr{R}(A) \geq 0$. Under these assumptions, we may determine H-stability by block-diagonalizing $\mathscr{R}(A)$ by a (complex) congruence transformation and examining the effect of this particular transformation on $\mathscr{I}(A) = (1/2i)(A - A^*)$.

Throughout our paper, we will assume that all matrices are $n \times n$ with complex elements, and matrices denoted by H and K are Hermitian. All triples $\omega = (\pi, \nu, \delta)$ will have nonnegative integers as elements, and satisfy $\pi + \nu + \delta = n$, the order of A. We shall call such triples *inertia triples*.

II. THE GENERAL INERTIA PROBLEM UNDER SPECIAL ASSUMPTIONS

2a. If $\mathscr{R}(AH) \geq 0$ and $B = SAS^{-1}$, $K = SHS^*$, then

$$\mathscr{R}(BK) = S\mathscr{R}(AH)S^* \geq 0,$$

and

$$\operatorname{In} A = \operatorname{In} B, \qquad \operatorname{In} H = \operatorname{In} K, \qquad \text{and} \qquad \operatorname{In} \mathscr{R}(AH) = \operatorname{In} \mathscr{R}(BK).$$

Thus we may often replace A by a matrix B similar to A, and H by a matrix K (complex) congruent to H. In particular, in proving many results it is convenient to assume that A is in some variant of the Jordan canonical form.

An inequality obtained in this section, which relates the inertias {or ranks} of matrices A and H, will be said to be "*best possible*" if it satisfies the following: For a given A satisfying the conditions of the theorem, and any given inertia triple ω {or, nonnegative integer r} satisfying the particular inequality, there exists a Hermitian H for which $\operatorname{In} H = \omega$ {or, rank $H = r$} satisfying the conditions of the theorem. As an example, see Lemma 1.

2b. LEMMA 1. *Let A be a matrix with $\delta(A) = 0$. If H is Hermitian and $\mathcal{R}(AH) \geq 0$, then* $\operatorname{In} H \leq \operatorname{In} A$. *This inequality is best possible, i.e., for any inertia triple $\omega \leq \operatorname{In} A$ there exists an H for which* $\operatorname{In} H = \omega$ *and* $\mathcal{R}(AH) \geq 0$.

PROOF. By the Main Inertia Theorem, there exists a Hermitian H_1 such that $\mathcal{R}(AH_1) > 0$, and $\operatorname{In} H_1 = \operatorname{In} A$. Set $H_t = H + tH_1$. For all $t > 0$, $\mathcal{R}(AH_t) = \mathcal{R}(AH) + t\mathcal{R}(AH_1) > 0$, whence, again by the Main Theorem, $\operatorname{In} H_t = \operatorname{In} A$. By the continuity of the eigenvalues of H_t as functions of t, we must have $\operatorname{In} H \leq \operatorname{In} A$.

To prove the "best possible" part of the lemma, let $\omega = (\pi, \nu, \delta)$ be an inertia triple for which $\omega \leq \operatorname{In} A$. By the remark at the beginning of this section we may (as in the proof of the Main Theorem in [7]), assume A to be in the form

$$A = \sum_{i=1}^{l} \oplus (\lambda_i I_i + \epsilon U_i),$$

where U_i is a matrix with ones in the first superdiagonal and zeros elsewhere, and ϵ is any positive number. Let $H = \Sigma_{i=1}^{l} \oplus J_i$, where

$$J_i = \operatorname{sign} \mathcal{R}(\lambda_i) \cdot \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}$$

has j_i ones on the diagonal, and

$$\pi = \sum_{\mathcal{R}(\lambda_i) > 0} j_i, \qquad \nu = \sum_{\mathcal{R}(\lambda_i) < 0} j_i.$$

It is easily verified that $\mathscr{R}(AH) \geq 0$ for sufficiently small ϵ, that rank $\mathscr{R}(AH) = \pi + \nu$, and that $\operatorname{In} H = \omega$. This completes the proof.

THEOREM I. *Let A be a given matrix for which all elementary divisors of imaginary eigenvalues are linear. If H is a Hermitian matrix for which $\mathscr{R}(AH) \geq 0$, then*

$$\pi(H) \leq \pi(A) + \delta(A), \qquad \nu(H) \leq \nu(A) + \delta(A) \tag{2.1}$$

and also we have

$$\operatorname{rank} \mathscr{R}(AH) \leq \pi(A) + \nu(A). \tag{2.2}$$

If equality holds in (2.2), then (2.1) can be strengthened to

$$\pi(A) \leq \pi(H) \leq \pi(A) + \delta(A), \qquad \nu(A) \leq \nu(H) \leq \nu(A) + \delta(A), \tag{2.3}$$

Inequalities (2.1), (2.2) and (2.3) are each best possible.

We shall use in this theorem and others the well-known result that all principal minors of $H \geq 0$ are nonnegative; thus if $h_{ii} = 0$, then $h_{ij} = h_{ji} = 0$ for all $j \neq i$; or if $H_{ii} = 0$, then $H_{ij} = H_{ji} = 0$ for any partitioning of H.

PROOF OF THEOREM I. Since the elementary divisors of imaginary eigenvalues are linear, we may suppose that

$$A = \left(\sum_{i=1}^{q} \oplus \lambda_i I_i\right) \oplus A_{qq},$$

where the λ_i are distinct and imaginary and $\delta(A_{qq}) = 0$. Let H and $\mathscr{R}(AH)$ be partitioned conformably with A. Then, for $1 \leq i \leq q - 1$,

$$\mathscr{R}(AH)_{ii} = \tfrac{1}{2}(\lambda_i H_{ii} - \lambda_i H_{ii}) = 0,$$

and thus, since $\mathscr{R}(AH) \geq 0$, $\mathscr{R}(AH)_{ij} = 0$ for all distinct i, j. But for $1 \leq j \leq q - 1$ and $j < i \leq q$, $\mathscr{R}(AH)_{ij} = \tfrac{1}{2}(A_{ii} - \lambda_j I_i) H_{ij}$, and as $A_{ii} - \lambda_j I_i$ is nonsingular, we must have $H_{ij} = H_{ji} = 0$. Thus we have

$$H = \sum_{i=1}^{q} \oplus H_{ii} \tag{2.4}$$

and

$$\mathscr{R}(AH) = O \oplus \mathscr{R}(A_{qq} H_{qq}). \tag{2.5}$$

We note that (2.5) immediately yields (2.2), as order $A_{qq} = \pi(A) + \nu(A)$. Since $\delta(A_{qq}) = 0$, and $\mathscr{R}(A_{qq} H_{qq}) \geq 0$, we may apply Lemma 1 and obtain that $\operatorname{In} H_{qq} \leq \operatorname{In} A_{qq}$. From (2.4) and order $(\sum_{i=1}^{q-1} \oplus H_{ii}) = \delta(A)$, we deduce that (2.1) holds.

If equality holds in (2.2), then clearly, $\mathscr{R}(A_{qq}H_{qq}) > 0$, by (2.5), and by the Main Theorem, In $H_{qq} =$ In A_{qq}. Now (2.3) is obvious.

We shall prove that (2.1) is best possible. (Proofs for (2.2) and (2.3) are similar.) Let $\omega = (\pi, \nu, \delta)$ be an inertia triple for which

$$\pi \leq \pi(A) + \delta(A), \qquad \nu \leq \nu(A) + \delta(A).$$

We may write $\pi = \pi_1 + \pi_2$, $\nu = \nu_1 + \nu_2$ where $\pi_2 \leq \pi(A)$, $\nu_2 \leq \nu(A)$. By Lemma 1, there is an H_{qq} for which $\mathscr{R}(A_{qq}H_{qq}) \geq 0$, and $\pi(H_{qq}) = \pi_2$, $\nu(H_{qq}) = \nu_2$. Since, for $i < q$, $\mathscr{R}(A_{ii}H_{ii}) = 0$ for any H_{ii}, we choose $H_0 = \Sigma_{i=1}^{q-1} H_{ii}$ as a diagonal matrix so that $\pi(H_0) = \pi_1$, $\nu(H_0) = \nu_1$ and then for $H = H_0 \oplus H_{qq}$, In $H = \omega$ and $\mathscr{R}(AH) \geq 0$.

2c. Our next theorem concerns a matrix consisting of a single Jordan block $A = \lambda I + U$. where λ is imaginary, and U is the matrix with ones on the first superdiagonal and zeros elsewhere (if A has order 1, we take $U = 0$).

THEOREM II. *Let $A = \lambda I + U$. If H is a Hermitian matrix for which $K = \mathscr{R}(AH) \geq 0$, and if rank $H = r$, rank $K = s$, then*

$$2s \leq r \tag{2.6}$$

$$|\pi(H) - \nu(H)| \leq 1 \tag{2.7}$$

$$h_{ij} = 0 \quad \text{if} \quad i + j > r + 1 \tag{2.8}$$

$$k_{ij} = \tfrac{1}{2}(h_{i,j+1} + h_{i+1,j}); \qquad k_{ij} = 0 \quad \text{if} \quad i > r/2 \quad \text{or} \quad j > r/2, \tag{2.9}$$

where $h_{i,n+1} = h_{n+1,j} = 0$.

These inequalities are best possible in the following strong sense:

Given integers r, s, $0 \leq 2s \leq r \leq n$ and an inertia triple $\omega = (\pi, \nu, \delta)$ for which $\pi + \nu = r$ and $|\pi - \nu| \leq 1$, there exists an H for which $\mathscr{R}(AH) \geq 0$, $s =$ rank $\mathscr{R}(AH)$, and $\omega =$ In H.

To make clear the meaning of conditions (2.8) and (2.9) we shall represent by Fig. 1 the most general H and K satisfying (2.8) and (2.9). Matrices of even and odd order differ slightly and therefore we shall illustrate both the case $n = 5$ and $n = 6$. Below, $\cdot$ represents a 0 element, $\times$ an element not necessarily 0, and $\times - \times$ indicates that the sum of the two linked elements is 0.

In the proof of theorem II we shall use a theorem of Cauchy's on the separation of eigenvalues of a Hermitian matrix by the eigenvalues of a principal minor ([9; 10, p. 75; 11, p. 75]). Thus let H be Hermitian of order n and let L be a submatrix of order $n - m$. If the eigenvalues of H and L are ordered in magnitude:

$$\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_n(H) \qquad \text{and} \qquad \lambda_1(L) \geq \lambda_2(L) \geq \cdots \geq \lambda_{n-m}(H)$$

then

$$\lambda_i(H) \geq \lambda_i(L) \geq \lambda_{i+m}(H), \qquad i = 1, \cdots, n - m.$$

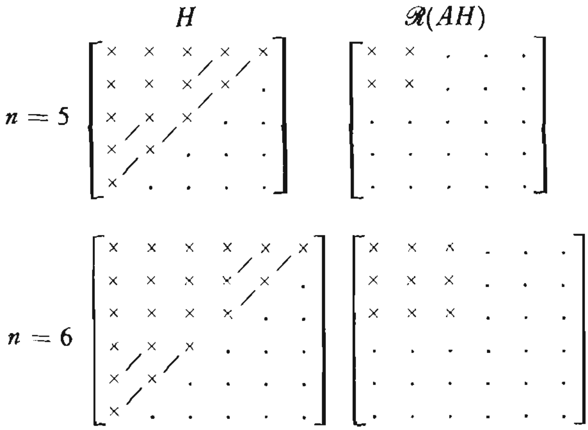

FIG. 1

We shall also note that if $r = 2r'$, then (2.9) asserts that $\pi(H) = \nu(H) = r'$, while if $r = 2r' + 1$ then one of $\pi(H)$, $\nu(H)$ is r' and the other $r' + 1$.

PROOF OF THEOREM II. We first note that

$$K = \mathscr{R}(AH) = \tfrac{1}{2}(\lambda H + UH - \lambda H + HU^*) = \mathscr{R}(UH).$$

To avoid "boundary problems" we define the infinite matrices

$$U' = (u_{ij}), \qquad i, j = 1, 2, \cdots \qquad \text{by} \qquad \begin{cases} u_{i,i+1} = 1 \\ u_{i,j} = 0 \text{ otherwise} \end{cases}$$

and

$$H' = (h'_{ij}), \qquad i, j = 1, 2, \cdots \qquad \text{by} \qquad \begin{cases} h'_{ij} = h_{ij},\ i \le n,\ j \le n \\ h'_{ij} = 0 \ \text{ otherwise} \end{cases}$$

Then $K' = \mathscr{R}(U'H') = \mathscr{R}(UH) \quad 0$. We shall prove the theorem for H' and K', while for convenience we write H for H', K for K'.

Our argument rests on the easily checked result:

$$k_{ij} = \tfrac{1}{2}(h_{i,j+1} + h_{i+i,j}), \qquad i, j = 1, 2, \cdots. \tag{2.10}$$

If $H = 0$, the theorem is trivial. We suppose then that $H \neq 0$, and let t be the largest integer for which there exists a nonzero h_{ij} with $i + j = t + 1$. By (2.10), $k_{ii} = 0$ if $i > t/2$. Thus, as $K \geq 0$, and, using (2.10) again, we obtain

$$k_{ij} = \tfrac{1}{2}(h_{i+1,j} + h_{i,j+1}) = 0 \qquad \text{if} \qquad i > t/2, \tag{2.11}$$

This implies that $s = \operatorname{rank} K \leq t/2$.

Also, by (2.11),

$$h_{ij} = (-1)^{t-i} h_{t1} \qquad \text{for} \qquad i + j = t + 1, \quad i > t/2.$$

$$h_{ij} = \bar{h}_{ji} = (-1)^{t-j} \bar{h}_{t1} \qquad \text{for} \qquad i + j = t + 1, \quad j > t/2.$$

Thus by our choice of t, $h_{t1} \neq 0$, whence $h_{ij} \neq 0$ for all i, j with $i + j = t + 1$. From this we deduce that $t \leq n$, and $r = \operatorname{rank} H = t$, and (2.6), (2.8), and (2.9) are proved. To prove (2.7) we shall apply the theorem of Cauchy we have quoted before the proof of this theorem to the matrix H and the principal submatrix

$$L = (h_{ij}) \qquad (r + 1)/2 < i, j \leq n.$$

Here $L = (0)$ is of order $n - m$, where $m = [(r + 1)/2]$, the integral part of $(r + 1)/2$.
Thus,

$$\lambda_{n-m}(H) \geq \lambda_{n-m}(L) = 0 = \lambda_1(L) \geq \lambda_{m+1}(H)$$

whence

$$\pi(H) \leq m, \qquad \nu(H) \leq m. \tag{2.12}$$

But

$$\pi(H) + \nu(H) = r \geq 2m - 1,$$

so that

$$\pi(H) \geq m - 1, \qquad \nu(H) \geq m - 1 \tag{2.13}$$

and now (2.12) and (2.13) yield (2.7).

Suppose $0 \leq 2s \leq r \leq n$, and $\omega = (\pi, \nu, \delta)$, with $\pi + \nu = r$ and $|\pi - \nu| \leq 1$. We define an H for which $\mathscr{R}(AH) \geq 0$, $s = \operatorname{rank} \mathscr{R}(AH)$, and $\omega = \operatorname{In} H$. For $1 \leq i \leq j \leq n$ we define $h_{ij} = \bar{h}_{ji} = (-1)^i \epsilon_{i+j-1}$, where

$$\epsilon_k = \begin{cases} (-1)^{k/2} & \text{if } k \text{ is even and } k \leq 2s, \\ \sqrt{-1} & \text{if } k = r,\ r \text{ is even and } r > 2s, \\ (-1)^{(k-1)/2}(\nu - \pi) & \text{if } k = r \text{ and } r \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

2d. COROLLARY II. 1. *For any A, there exists a nonsingular H for which $\mathscr{R}(AH) \geq 0$.*

PROOF. We may assume that

$$A = \sum_{i=1}^{q} \oplus A_{ii}, \tag{2.14}$$

where order $A_{qq} = \pi(A) + \nu(A)$, $\delta(A_{qq}) = 0$, and for $i = 1, \cdots, q-1$,

$$A_{ii} = \lambda_i I_i + U_i,$$

order $A_{ii} = \sigma_i$, and λ_i is imaginary. By the Main Theorem, there exists an H_{qq} for which $\mathscr{R}(A_{qq}H_{qq}) > 0$. For $i = 1, \cdots, q-1$, we may find a non-singular H_{ii} for which $\mathscr{R}(A_{ii}H_{ii}) \geq 0$ by Theorem II. Certainly

$$H = \sum_{i=1}^{q} \oplus H_{ii}$$

will satisfy the conditions of the theorem.

Corollary II. 2. *If $\mathscr{R}(AH) \geq 0$, then*

$$\operatorname{rank} \mathscr{R}(AH) \leq \pi(A) + \nu(A) + \sum_{i=1}^{p} [\sigma_i/2] \tag{2.15}$$

where σ_i, $i = 1, \cdots, p$, are the degrees of the elementary divisors belonging to pure imaginary roots of A. The inequality (2.15) is best possible.

Proof. We assume that A is in the form (2.14), with $p + 1 = q$, and we partition H conformably. Since $\mathscr{R}(A_{ii}H_{ii}) = \mathscr{R}(AH)_{ii} \geq 0$, $i = 1, \cdots, p$, by Theorem II. $\mathscr{R}(A_{ii}H_{ii})$ has at most $[\sigma_i/2]$ nonzero rows. Thus $\mathscr{R}(AH)$ has at most $\pi(A) + \nu(A) + \Sigma_{i=1}^{p} [\sigma_i/2]$ nonzero rows; hence (2.15) is proved.

That (2.15) is best possible is obvious from the best possible inequalities (2.2) and (2.6), using block diagonal H.

2e. Theorem III. *Let A be given. If $H \geq 0$ and $\mathscr{R}(AH) \geq 0$, then*

$$\operatorname{rank} H \leq \pi(A) + p(A), \tag{2.16}$$

where $p(A)$ is the number of elementary divisors of imaginary roots of A. The inequality (2.16) is best possible for $H \geq 0$.

Proof. We again assume that A is in form (2.14), and we partition H conformably. We note that both $H_{ii} \geq 0$ and $\mathscr{R}(A_{ii}H_{ii}) \geq 0$, $i = 1, \cdots, q$. For $i = 1, \cdots, p = q - 1$, this implies that rank $H_{ii} = 1$, by (2.7). By Lemma 1, rank $H_{qq} = \pi(H_{qq}) \leq \pi(A_{qq}) = \pi(A)$, as $\delta(A_{qq}) = 0$. As $p(A) = p$, it follows that

$$\sum_{i=1}^{q} \operatorname{rank} H_{ii} \leq \pi(A) + p(A). \tag{2.17}$$

We complete the proof by showing that

$$\operatorname{rank} H \leq \sum_{i=1}^{q} \operatorname{rank} H_{ii} = r. \tag{2.18}$$

As $H_{ii} \geq 0$, $i = 1, \cdots, q$, there exist unitary U_{ii} such that $U_{ii}^* H_{ii} U_{ii}$ is diagonal. We set $U = \Sigma_{i=1}^{q} \oplus U_{ii}$ and note that U^*HU has at most r nonzero diagonal elements, and hence, as $U^*HU \geq 0$, at most r nonzero rows. This proves (2.18), and with (2.17), we have proved (2.16).

That (2.16) is best possible is obvious from Lemma 1, Theorem II, and Corollary II. 1.

2f. COROLLARY III. 1. *Let A be given. There exists an $H > 0$ for which $\mathscr{R}(AH) \geq 0$ if and only if $\nu(A) = 0$ and (1.1) holds.*

PROOF. In general, $p(A) \leq \delta(A)$; (1.1) is equivalent to $p(A) = \delta(A)$. It follows that $\pi(A) + p(A) = n = \pi(A) + \nu(A) + \delta(A)$ if and only if $\nu(A) = 0$ and (1.1) holds. The corollary now follows from Theorem III, the sufficiency following from the best possible property of (2.16).

COROLLARY III. 2. *If $\mathscr{R}(A) \geq 0$ and $H > 0$, then all elementary divisors of imaginary eigenvalues (if any) of AH are linear. In particular, this is true of A itself.*

PROOF. Let $B = AH$ and $K = H^{-1} > 0$, then $A = BK$ and this corollary follows from Corollary III. 1 applied to B and K.

For $AI = A$, this result is part of Theorem 2 of [7].

2g. We shall use Lemma 2 in Section III. It is interesting to compare Lemmas 1 and 2.

LEMMA 2. *Let A be any matrix and H be a nonsingular Hermitian matrix for which $\mathscr{R}(AH) \geq 0$, then* $\operatorname{In} A \leq \operatorname{In} H$. *If, in addition, $\delta(A) = 0$, then* $\operatorname{In} A = \operatorname{In} H$.

PROOF. We let $B = AH$, $K = H^{-1}$, and apply Corollary 4 to Theorem 1 of [7] to B and K. We have $\operatorname{In} A = \operatorname{In} BK \leq \operatorname{In} K = \operatorname{In} H^{-1} = \operatorname{In} H$. (We see by the proof of Theorem I that this inequality is best possible if all elementary divisors of imaginary eigenvalues are linear.) The statement of equality when $\delta(A) = 0$ follows from Lemma 1, and what has already been proved.

III. THE INERTIA PROBLEM WITH PRESCRIBED NULLSPACES

3a. Let $\mathscr{W}$ be a subspace of $\mathscr{V}$ (the n-dimensional space of all columns); then $\mathscr{W}^{\perp} = \{y : y^*x = 0 \text{ for all } x \text{ in } \mathscr{W}\}$. We may find an orthonormal basis $u_1, \cdots, u_n$ of $\mathscr{V}_n$ so that $u_1, \cdots, u_r$ is a basis of $\mathscr{W}^{\perp}$ and $u_{r+1}, \cdots, u_n$ is a basis of $\mathscr{W}$. Let $U = [u_1, \cdots, u_n]$, a unitary matrix. For any matrix B, $B' = U^*BU$

with respect to the basis $u_1, \cdots, u_n$ is the same transformation as B with respect to the usual basis $e_1, \cdots, e_n$. It is clear that in the proofs below we may simultaneously replace all matrices B by U^*BU, and then $\mathscr{W}^\perp$ is spanned by the first r unit vectors and $\mathscr{W}$ the last $n-r$ unit vectors.

We will say that B is in form (3.1) with respect to $\mathscr{W}$ when we have replaced B by $B' = U^*BU$, and partitioned B (actually B') in the following way:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \qquad \text{where} \qquad B_{11} = [b_{ij}], \quad 1 \leq i, j \leq r, \quad \text{etc.} \tag{3.1}$$

We will, in several proofs, let $\mathscr{W} = \mathscr{N}(B)$, where $\mathscr{N}(B)$ is the nullspace of B. Then for B in form (3.1), we have

$$B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & 0 \end{bmatrix}, \tag{3.2}$$

where the rank of $\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$ is full. If B is Hermitian, then also $B_{21} = 0$, and B_{11} is nonsingular. For B in form (3.2), $\mathscr{N}(B) \subseteq \mathscr{N}(C)$ is equivalent to $C_{12} = 0$ and $C_{22} = 0$.

In other proofs, B will map a subspace $\mathscr{W}^\perp$ into itself: $B\mathscr{W}^\perp \subseteq \mathscr{W}^\perp$; then we shall use $(B \mid \mathscr{W}^\perp)$ to denote the restriction of B to $\mathscr{W}^\perp$. When B is in form (3.1) with respect to $\mathscr{W}$, $B\mathscr{W}^\perp \subseteq \mathscr{W}^\perp$ is equivalent to $B_{21} = 0$, and to $B^*\mathscr{W} \subseteq \mathscr{W}$ and then $(B \mid \mathscr{W}^\perp) = B_{11}$, $(B^* \mid \mathscr{W}) = B_{22}$. Thus

$$\operatorname{In} B = \operatorname{In} B_{11} + \operatorname{In} B_{22} = \operatorname{In} (B \mid \mathscr{W}^\perp) + \operatorname{In} (B^* \mid \mathscr{W}), \tag{3.3}$$

since $\operatorname{In} B_{22} = \operatorname{In} B_{22}^*$.

It seems preferable to us to state our theorems in terms of subspaces, thus avoiding the dilemma of either constantly referring to unitary similarity transformations or else restricting ourselves to special cases. In the proofs, however, we shall usually go to a pure matrix form of our theorems.

For example, Theorem IV below is equivalent to:

If $\mathscr{R}(AH) \geq 0$ and $H = H_{11} \oplus 0$, H_{11} nonsingular, then for conformably partitioned A,

$$\mathscr{R}(AH) = \mathscr{R}(A_{11}H_{11}) \oplus 0 \tag{3.4}$$

$$A_{21} = 0 \tag{3.5}$$

$$\operatorname{In} A_{11} \leq \operatorname{In} H. \tag{3.6}$$

This is the form of the proposition we shall prove.

Our definition of $\operatorname{In} A \leq \operatorname{In} B$ will be carried over to the case where A and B are square matrices of perhaps different orders. For such matrices we define

$$\operatorname{In} A \leq \operatorname{In} B \quad \text{if} \quad \pi(A) \leq \nu(A), \quad \nu(B) \leq \nu(B).$$

and

$$\operatorname{In} A = \operatorname{In} B \quad \text{if} \quad \pi(A) = \pi(B), \quad \nu(A) = \nu(B).$$

3b. THEOREM IV. *If* $\mathscr{R}(AH) \geq 0$ *then*

$$\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}(H), \tag{3.4}$$

$$A\mathscr{N}(H)^{\perp} \subseteq \mathscr{N}(H)^{\perp}, \tag{3.5}$$

and

$$\mathrm{In}\,(A \,|\mathscr{N}(H)^{\perp}) \leq \mathrm{In}\, H. \tag{3.6}$$

PROOF. We may assume that all matrices are in form (3.1) with respect to $\mathscr{N}(H)$; thus $H = H_{11} \oplus 0$, where H_{11} is nonsingular. Then

$$\mathscr{R}(AH) = \begin{bmatrix} \mathscr{R}(A_{11}H_{11}) & \frac{1}{2}(H_{11}A_{21}^*) \\ \frac{1}{2}(A_{21}H_{11}) & 0 \end{bmatrix}$$

so, since $\mathscr{R}(AH) \geq 0$, $A_{21}H_{11} = 0$ and $A_{21} = 0$ follows. Thus (3.5) is proved.

For (3.4) we simply note that $\mathscr{R}(AH) = \mathscr{R}(A_{11}H_{11}) \oplus 0$, and to prove (3.6) we note that $\mathscr{R}(A_{11}H_{11}) \geq 0$, whence, by Lemma 2,

$$\mathrm{In}\, A_{11} \leq \mathrm{In}\, H_{11} = \mathrm{In}\, H.$$

We remark that (3.4) is also a consequence of the following easily-proved assertions (with $B = AH$):

$$\mathscr{N}(AH) \supseteq \mathscr{N}(H), \tag{3.7}$$

$$\mathscr{N}(\mathscr{R}(B)) \supseteq \mathscr{N}(B), \qquad \text{if} \qquad \mathscr{R}(B) \geq 0. \tag{3.8}$$

The first corollary generalizes Lemma 2.

COROLLARY IV. 1. *If* $\mathscr{R}(AH) \geq 0$ *and* $\mathrm{In}\,(A^* \,|\mathscr{N}(H)) = (0, 0, \delta)$, *then* $\mathrm{In}\, A = \mathrm{In}\,(A \,|\mathscr{N}(H)^{\perp}) \leq \mathrm{In}\, H$.

PROOF. The proof follows immediately from (3.3) (with $\mathscr{W} = \mathscr{N}(H)$) and Theorem IV.

COROLLARY IV. 2. *If* $\mathscr{R}(AH) \geq 0$ *and rank* $A =$ *rank* $\mathscr{R}(AH)$, *then* $\mathrm{In}\, A = \mathrm{In}\,(A \,|\mathscr{N}(H)_{\perp}) \leq \mathrm{In}\, H$.

PROOF. We may assume $H = H_{11} \oplus 0$ and $A_{21} = 0$. Then, using (3.8), rank $A \geq$ rank $A_{11} \geq$ rank $A_{11}H_{11} \geq$ rank $\mathscr{R}(A_{11}H_{11}) =$ rank $\mathscr{R}(AH)$, and by our hypotheses we must have equality throughout. Rank $A =$ rank A_{11} implies that $A_{22} = 0$, and hence $\mathrm{In}\,(A^* \,|\mathscr{N}(H)) = (0, 0, \delta)$. We apply Corollary IV. 1 to complete the proof.

The following lemma reduces to Lemma 1 when $\mathscr{N}(H) = 0$.

LEMMA 3. *Suppose that* $\mathscr{R}(AH) \geq 0$ *and* $\delta(A \,|\mathscr{N}(H)^{\perp}) = 0$. *Then* $\mathrm{In}\, H = \mathrm{In}\,(A \,|\mathscr{N}(H)^{\perp}) \leq \mathrm{In}\, A$.

PROOF. We assume as usual that $H = H_{11} \oplus 0$, with H_{11} nonsingular. Then

$$\mathscr{R}(AH) = \mathscr{R}(A_{11}H_{11}) \oplus 0.$$

As H_{11} is nonsingular and

$$\delta(A \mid \mathscr{N}(H)^{\perp}) = \delta(A_{11}) = 0,$$

we must have

$$\operatorname{In} H = \operatorname{In} H_{11} = \operatorname{In} (A \mid \mathscr{N}(H)^{\perp}) \leq \operatorname{In} A,$$

by Lemma 2.

THEOREM V. (Generalization of the Main Inertia Theorem). *Let A be a given matrix and $\mathscr{N}$ a given subspace. There exists a Hermitian H such that*

$$\mathscr{R}(AH) \geq 0 \tag{3.9}$$

and

$$\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}(H) = \mathscr{N} \tag{3.10}$$

if and only if

$$A\mathscr{N}^{\perp} \subseteq \mathscr{N}^{\perp} \tag{3.11}$$

$$\delta(A \mid \mathscr{N}^{\perp}) = 0 \tag{3.12}$$

If (3.9) and (3.10) hold, then

$$\operatorname{In} H = \operatorname{In} (A \mid \mathscr{N}^{\perp}) \leq \operatorname{In} A.$$

PROOF. Suppose that (3.11) and (3.12) hold. We may assume that A is in form (3.1) with respect to $\mathscr{N}$; then $A_{21} = 0$ and $\delta(A_{11}) = 0$. By the Main Theorem, there exists a nonsingular H_{11} such that $\mathscr{R}(A_{11}H_{11}) > 0$. Let $H = H_{11} \oplus 0$. Then $\mathscr{R}(AH) = \mathscr{R}(A_{11}H_{11}) \oplus 0 \geq 0$, and, as both $\mathscr{R}(A_{11}H_{11})$ and H_{11} are nonsingular, (3.9) and (3.10) hold.

Conversely, suppose we are given matrices A and H satisfying (3.9) and (3.10). Setting $H = H_{11} \oplus 0$, where H_{11} is nonsingular, we see that (3.11) is part of Theorem IV. Also $\mathscr{R}(A_{11}H_{11}) = \mathscr{R}(AH)_{11} \geq 0$. By (3.10), $\mathscr{R}(AH)_{11}$ is nonsingular; hence $\delta(A_{11}) = 0$ by the Main Theorem. This is (3.12).

If (3.9) and (3.10) hold, then we have proved (3.12). By Lemma 3,

$$\operatorname{In} H = \operatorname{In} (A \mid \mathscr{N}^{\perp}) \leq \operatorname{In} A.$$

COROLLARY V. 1. *If $\mathscr{R}(AH) \geq 0$ and rank $\mathscr{R}(AH) =$ rank H, then*

$$\operatorname{In} H = \operatorname{In} (A \mid \mathscr{N}(H)^{\perp}) \leq \operatorname{In} A.$$

PROOF. By Theorem IV, $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}(H)$. The result follows from Theorem V.

COROLLARY V. 2. *If* $\mathscr{R}(AH) \geq 0$ *and*

$$\operatorname{rank} \mathscr{R}(AH) = \operatorname{rank} H = \pi(A) + \nu(A),$$

then $\operatorname{In} H = \operatorname{In} A$.

PROOF. Assuming that $H = H_{11} \oplus 0$, we have $A_{21} = 0$ and $\mathscr{R}(A_{11}H_{11}) > 0$. We must have $\delta(A_{11}) = 0$, and by our hypothesis,

$$\operatorname{rank} H = \operatorname{rank} H_{11} = \pi(A_{11}) + \nu(A_{11}) = \pi(A) + \nu(A).$$

Then $\operatorname{In} H = \operatorname{In} H_{11} = \operatorname{In} A_{11} = \operatorname{In} A$, and the proof is completed.

3d. In Theorem V we found necessary and sufficient conditions for the existence of a K for which $\mathscr{R}(AK) \geq 0$ and $\mathscr{N} = \mathscr{N}(K) = \mathscr{N}(\mathscr{R}(AK))$, $\mathscr{N}$ being given. Now we shall attack the following problem. Suppose there is one K satisfying these conditions. By Theorem IV we know that for every H with $\mathscr{R}(AH) \geq 0$ and $\mathscr{N}(\mathscr{R}(AH) = \mathscr{N}$ we have $\mathscr{N}(H) \subseteq \mathscr{N}$. Under what conditions can we conclude that $\mathscr{N}(H) = \mathscr{N}$ for all such H.

If A and B are square matrices of order n and p respectively, we shall denote by $T(A, B)$ the produce $\pi(\alpha_i + \beta_j)$ over all pairs of eigenvalues of A and B.

We shall write[1] $T(A) = T(A, A^*)$. If A is an empty matrix (i.e., operator or a 0-dimensional space) consistency conditions connected with direct sums of matrices force us to put $T(A, B) = 1$, and this is in conformity with the usual convention that the empty product is 1.

We remark that in the next theorem we have omitted the usual hypothesis that $\mathscr{R}(AH) \geq 0$.

3e. THEOREM VI. *Let* $\mathscr{N}$ *be a subspace of* $\mathscr{V}$ *and* A *a matrix for which* $A\mathscr{N}^\perp \subseteq \mathscr{N}^\perp$. *If*

$$T(A \,|\mathscr{N}^\perp, A^* \,|\mathscr{N})\, T(A^* \,|\mathscr{N}) \neq 0, \tag{3.13}$$

then $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}^\perp$ *implies that* $\mathscr{N}(H) \supseteq N$. *Conversely, if*

$$T(A \,|\mathscr{N}^\perp, A^* \,|\mathscr{N})\, T(A^* \,|\mathscr{N}) = 0, \tag{3.14}$$

then there exists an H *such that* $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}$, *but* $\mathscr{N}(H) \not\supseteq \mathscr{N}$.

We remark that if $\mathscr{N} = \mathscr{V}$ in Theorem VI then $(A \,|\mathscr{N}^\perp)$ is an empty matrix, whence $T(A \,|\mathscr{N}^\perp, A^* \,|\mathscr{N})\, T(A^* \,|\mathscr{N}) = T(A^*) = T(A)$. Thus in this case the theorem reduces to the *known* result that there exists a nonzero H such that $\mathscr{R}(AH) = 0$ if and only if $T(A) = 0$, [1, Vol. II, p. 225; 7]. This result will be used in the proof of theorem.

[1] In [7] $\triangle(A)$ was written instead of $T(A)$.

PROOF OF THEOREM VI. We assume A is (and all H are) in the form (3.1) with respect to $\mathscr{N}$. Our assumptions on A now state that $A_{21} = 0$. Suppose

$$T(A_{11}, A_{22}^*)\, T(A_{22}^*) = T(A\,|\mathscr{N}^\perp, A^*\,|\mathscr{N})\, T(A^*\,|\mathscr{N}) \neq 0.$$

$$\mathscr{R}(AH) = \begin{bmatrix} \mathscr{R}(A_{11}H_{11} + A_{12}H_{21}) & \frac{1}{2}(A_{11}H_{12} + H_{12}A_{22}^* + A_{12}H_{22}) \\ \frac{1}{2}(A_{22}H_{21} + H_{21}A_{11}^* + H_{22}A_{12}^*) & \mathscr{R}(A_{22}H_{22}) \end{bmatrix}$$

Now suppose that

$$\mathscr{R}(A_{22}H_{22}) = 0, \qquad \frac{1}{2}(A_{11}H_{12} + H_{12}A_{22}^* + A_{12}H_{22}) = 0$$

$(\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N})$. Since $T(A_{22}) \neq 0$, we obtain $H_{22} = 0$, whence $\frac{1}{2}(A_{11}H_{12} + H_{12}A_{22}) = 0$, and so $H_{12} = H_{21} = 0$ by $T(A_{11}, A_{22}) \neq 0$. Hence $H = H_{11} \oplus 0$, or $\mathscr{N}(H) \supseteq \mathscr{N}$.

Conversely, suppose that $T(A_{11}, A_{22}^*)\, T(A_{22}^*) = 0$. We consider two cases.

(α) $T(A_{11}, A_{22}^*) \neq 0$. In this case $T(A_{22}^*) = 0$, and there exists nonzero H_{22}, such that $\mathscr{R}(A_{22}H_{22}) = 0$, and just because $T(A_{11}, A_{22}^*) \neq 0$ there exists H_{12} such that $A_{11}H_{12} + H_{12}A_{12}^* = -A_{12}H_{22}$.

(β) $T(A_{11}, A_{22}^*) = 0$. This time we set $H_{22} = 0$, and our condition now guarantees the existence of a nonzero H_{12} for which $A_{11}H_{12} + H_{12}A_{22}^* = 0$.

In both cases, $\mathscr{R}(AH) = \mathscr{R}(AH)_{11} \oplus 0$, while either $H_{22} \neq 0$ or $H_{12} \neq 0$. Thus $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}$ but $\mathscr{N}(H) \supseteq \mathscr{N}$, and the theorem is proved.

COROLLARY VI. 1. *Let $\mathscr{R}(AH) \geq 0$ and set $\mathscr{N} = \mathscr{N}(\mathscr{R}(AH))$. If $A\mathscr{N}^\perp \subseteq \mathscr{N}^\perp$, and (3.13) holds, then $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}(H)$.*

PROOF. By Theorem IV, $A\mathscr{N}(H)^\perp \subseteq \mathscr{N}(H)^\perp$, and $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}(H)$ But as (3.13) holds, $\mathscr{N}(H) \supseteq \mathscr{N} = \mathscr{N}(\mathscr{R}(AH))$.

COROLLARY VI. 2. *Let $\mathscr{R}(AK) \geq 0$ and $\mathscr{N} = \mathscr{N}(K) = \mathscr{N}(\mathscr{R}(AK))$. If (3.13) holds, then $\mathscr{R}(AH) \geq 0$ and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}$ imply $\mathscr{N} = \mathscr{N}(H)$. Conversely, if (3.14) holds, there exists an H such that $\mathscr{R}(AH) \geq 0$ and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}$ yet $\mathscr{N}(H) \subset \mathscr{N}$.*

(Here $\subset$ means "properly contained in.")

PROOF. From the assumptions of the corollary we deduce that $A\mathscr{N}^\perp \subseteq \mathscr{N}^\perp$. Hence Theorem VI applies, and therefore we deduce from (3.13) and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}$ that $\mathscr{N}(H) \supseteq \mathscr{N}$. But, since $\mathscr{R}(AH) \geq 0$, we also have $\mathscr{N}(\mathscr{R}(AH)) \supseteq \mathscr{N}(H)$, whence $\mathscr{N} = \mathscr{N}(H)$.

Now suppose that (3.14) holds. We partition our matrices corresponding to $\mathscr{N}$. As usual we may assume that $K = K_{11} \oplus 0$, where K_{11} is nonsingular, and $\mathscr{R}(AK) = \mathscr{R}(AK)_{11} \oplus 0$, where $\mathscr{R}(AK)_{11} > 0$. By Theorem VI, we can find a Hermitian L such that either $L_{12} \neq 0$ or $L_{22} \neq 0$ ($\mathscr{V}(L) \supseteq \mathscr{N}$) with $\mathscr{R}(AL) = \mathscr{R}(AL)_{11} \oplus 0$ ($\mathscr{N}(\mathscr{R}(AL)) \supseteq \mathscr{N}$). Thus if $\epsilon > 0$ is sufficiently small and $H = K + \epsilon L$ then H_{11} is nonsingular ($\mathscr{N}(H) \subseteq \mathscr{N}$), and either $H_{12} \neq 0$ or $H_{22} \neq 0$ ($\mathscr{N}(H) \supseteq \mathscr{N}$, whence $\mathscr{N}(H) \subset \mathscr{N}$),

$$\mathscr{R}(AH)_{11} = \mathscr{R}(AK)_{11} + \epsilon \mathscr{R}(AL)_{11} > 0,\ \mathscr{R}(AH) = \mathscr{R}(AH)_{11} \oplus 0$$

whence $\mathscr{R}(AH) \geqq 0$ and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}$. The corollary is proved.

As a final corollary to Theorem VI, we shall combine the results of Theorem V and of Corollary VI. 2, into a single statement.

COROLLARY VI. 3. *Let A be a matrix, $\mathscr{N}$ a subspace of $\mathscr{V}$. The following two sets of properties are equivalent.*

$$\mathscr{N}A^{\perp} \subseteq \mathscr{N}^{\perp}, \qquad \delta(A \mid \mathscr{N}^{\perp}) = 0,$$
$$T(A \mid \mathscr{N}^{\dagger}, A^* \mid \mathscr{N})\, T(A^* \mid \mathscr{N}) \neq 0 \tag{3.15}$$

and

There exists a Hermitian H such that $\mathscr{R}(AH) \geq 0$ and $\mathscr{N}(\mathscr{R}(AH)) = \mathscr{N}$, and for every such H, $\mathscr{N} = \mathscr{N}(H)$. (3.16)

IV. AN EFFECTIVE TEST FOR H-STABILITY

4a. By examining Theorem 2 of [7] and the proof of Theorem 4 of [7], it is easy to see that Theorem 4 of [7] may be restated in a somewhat more precise form. We call this the

IMPROVED FORM OF THEOREM 4 OF [7]. *Let A be a matrix for which $\mathscr{R}(A) \geq 0$. If there exists an $H > 0$ such that $\delta(AH) = k$ then there exists a B complex congruent to A which has a skew-Hermitian direct summand of order k.*

Thus $B = B_{11} \oplus iR_{22}$, R_{22} Hermitian of order k. In the special case $k = 0$, R_{22} is empty; i.e., $B = B_{11}$. This will be used in the proof of the next theorem. We also require a lemma.

If B is complex congruent to A ($B = S^*AS$, where S is nonsingular) we shall write $B \sim A$.

LEMMA 4. *Let $A = P + iQ$, $A' = P' + iQ'$, where P, P', Q, Q' are all Hermitian, and $P = P_{11} \oplus 0$, $P_{11} > 0$, $P' = P_{11} \oplus 0$ (partitioned conformably). If $A \sim A'$, then $Q_{22} \sim Q'_{22}$.*

PROOF. Suppose $A' = S^*AS$, where S is nonsingular. We partition P', Q, Q' conformably with P; then $0 = P'_{22} = S^*_{12}P_{11}S_{12}$. If x is any column of S_{12}, $0 = x^*P_{11}x$, whence $x = 0$ since $P_{11} > 0$. Hence $S_{12} = 0$ and S_{22} must be nonsingular. We deduce that $Q'_{22} = S^*_{22}Q_{22}S_{22}$.

THEOREM VII. *Let A be a nonsingular matrix with $\mathscr{R}(A) \geq 0$, and suppose that*

$$\max_{H>0} \delta(AH) = k. \tag{4.1}$$

Let $A \sim A' = P + iQ$; P, Q Hermitian, with $P = P_{11} \oplus 0$ and $P_{11} > 0$. If Q is partitioned conformably with P, then rank $Q_{22} = k$.

In particular, A is H-stable if and only if $Q_{22} = 0$.

PROOF. (a) We set $s = \operatorname{rank} Q_{22}$, and shall first show that $s \leq k$. Since Q_{22} is Hermitian, Q_{22} has a nonsingular principal minor L_{22} of order s, which by a cogradient permutation of rows and columns of Q_{22} may be brought into bottom right position. Repartitioning, we have $A \sim A' = K + iL$, where order K_{22} = order $L_{22} = s$, $K = K_{11} \oplus 0$, and L_{22} is non-singular. Define S by

$$S = \begin{bmatrix} I & 0 \\ -L_{22}^{-1}L_{21} & I \end{bmatrix}.$$

An easy computation shows that

$$S^*KS = K_{11} \oplus 0, \qquad S^*LS = (L_{11} - L_{12}L_{22}^{-1}L_{21}) \oplus L_{22}.$$

It is clear that $\delta = \delta(S^*A'S) \geq s$, and since $S^{*-1}(S^*A'S)\,S^* = A'(SS^*)$ and $SS^* > 0$, we must have $\delta = \delta(A'(SS^*)) = \delta(AH)$ for some $H > 0$. Thus $s \leq \delta \leq k$, by (4.1).

(b) We next show that $s \geq k$. By the improved version of Theorem 4 of [7], there exists a $B \sim A \sim A'$ with $B = C \oplus iR_{33}$, with R_{33} Hermitian, of order k. (The reason for our choice of subscripts will be clear later.) As B is nonsingular, rank $R_{33} = k$. We apply a congruence transformation to C and we obtain $B \sim B' = C' \oplus iR_{22}$, and

$$\mathscr{R}(C') = S_{11} \oplus 0, \qquad S_{11} > 0.$$

Thus if $R' = \mathscr{I}(B')$ is partitioned conformably, then

$$R' = \begin{bmatrix} R'_{11} & R'_{12} & 0 \\ R_{21} & R'_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix}$$

so that the minor complementary to R_{11} is $R_{22} \oplus R_{33}$. Note that order S_{11} = order P_{11} since $A' \sim B'$. Hence we can apply Lemma 4, and obtain $(R_{22} \oplus R_{33}) \sim Q_{22}$, whence

$$s = \operatorname{rank} Q_{22} = \operatorname{rank}\,(R_{22} \oplus R_{33}) \geq \operatorname{rank}\ R_{33} = k.$$

As $s \leq k$ by (a), $s = k$.

We have proved Theorem VII (and incidentally also that $R'_{22} = 0$.)
We conclude with a consequence of H-stability.

COROLLARY VII. 1. *If A has order n and is H-stable, then* rank $\mathscr{R}(A) \geq n/2$.

PROOF. As in Theorem 4.2, $A \sim A' = P + iQ$, where $P = P_{11} \oplus 0$ and $P_{11} > 0$. By Theorem 4.2, $Q_{22} = 0$. Thus

$$A' = \begin{bmatrix} Q_{11} + P_{11} & Q_{12} \\ Q_{21} & 0 \end{bmatrix}.$$

As rank $\mathscr{R}(A) =$ order P_{11}, obviously rank $A' \leq 2$ rank P_{11}. But A' ($\sim A$) must be nonsingular; hence rank $P_{11} \geq n/2$.

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