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Comparison theorems for supremum norms

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The problem of characterizing all supremum norms on a space of matrices or linear transformations is still unsolved. The theorems of this note are intended as a step towards solving this problem. Our most general result is Theorem 3, in which the assumption of finite dimensionality is not needed. Theorems 1 and 2 are special cases of Theorem 3. In view of the independent interest of Theorem 1, we have thought it desirable to include a separate proof of this case.

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In this note all vector spaces V considered will be over the real or complex numbers. By a norm ν on V we shall mean an *equilibrated* norm; thus ν maps V into the nonnegative numbers and, if $x, y \in V$ and c is real or complex, then

$$\nu(x) > 0 \quad \text{if} \quad x \neq 0, \tag{1}$$

$$\nu(x+y) \leqq \nu(x) + \nu(y), \tag{2}$$

$$\nu(c\,x) = |c|\,\nu(x). \tag{3}$$

Let U be a vector space normed by μ and V a vector space normed by ν. If T is a space of transformations of U into V, for which $\frac{\nu(A\,x)}{\mu(x)}$ is bounded over x in U for all A in T, then *sup* defined by

$$\sup(A) = \sup_{0 \neq x \in U} \frac{\nu(A\,x)}{\mu(x)}, \qquad A \in T \tag{4}$$

is a norm on T. In this case it is natural to call sup the *supremum norm* belonging to (μ, ν). If $U = V$ and $\mu = \nu$, we call sup the supremum norm belonging to the *single norm* ν. A special case of a supremum norm is the dual norm ν' of ν on the dual space V' of V. Here V' consists of all bounded linear functionals on V. Thus

$$\nu'(y') = \sup_{0 \neq x \in V} \frac{|(y', x)|}{\nu(x)}, \qquad y' \in V'. \tag{5}$$

If V is identified with a subspace of V'' in the canonical way, then it is known (e.g. HOUSEHOLDER [2]) that $\nu''(x) = \nu(x)$, for all $x \in V$. This result will be used

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in the proof of Theorem 1. We shall also use a result which applies to finite dimensional spaces only and which is a consequence of the compactness of the unit ball $\{x: \nu(x) \leqq 1\}$ in these spaces:

If ν_1 and ν_2 are norms on the finite dimensional space V, then $\frac{\nu_1}{\nu_2}$ is bounded, say

$$\sup_V \frac{\nu_1}{\nu_2} = n, \tag{6}$$

and the equality is attained [3]; i.e., for some $v \in V$,

$$\frac{\nu_1(v)}{\nu_2(v)} = n. \tag{7}$$

In equation (6) we used the notational convention of writing $\sup_V \frac{\nu_1}{\nu_2}$ to mean $\sup_{0 \neq x \in V} \frac{\nu_1(x)}{\nu_2(x)}$. Similarly we shall write $\nu_1 \leqq \nu_2$ to mean $\nu_1(x) \leqq \nu_2(x)$ for all $x \in V$.

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Theorem 1. *Let V be a finite dimensional vector space. Let ν_1 and ν_2 be norms on V, and let $\sup_i$, $i = 1, 2$, be the norm on the space L of all linear transformations of V into itself belonging to the single norm ν_i. If*

$$\sup_1 \leqq \sup_2 \tag{8}$$

then actually

$$\sup_1 = \sup_2 \tag{9}$$

and, for some constant n,

$$\nu_1 = n\,\nu_2. \tag{10}$$

Proof. Let n be defined by (6) and suppose that v satisfies (7). It is an immediate consequence of (6) and of the definition (5) of the dual norm that

$$\nu_2' \leqq n\,\nu_1'. \tag{11}$$

Under the hypothesis (8), we shall prove the reverse inequality. For every $y' \in V'$ we may define the projection E on V by

$$E\,x = (y', x)\,v, \tag{12}$$

for all $x \in V$. By (3) and (12) we have

$$\sup_{0 \neq x \in V} \frac{\nu_1(E\,x)}{\nu_1(x)} = \sup_{0 \neq x \in V} \frac{|(y', x)|}{\nu_1(x)}\,\nu_1(v) \tag{13}$$

whence by (5)

$$\sup_1(E) = \nu_1'(y')\,\nu_1(v), \tag{14}$$

and similarly

$$\sup_2(E) = \nu_2'(y')\,\nu_2(v). \tag{15}$$

Hence the hypothesis (8) and equation (7) imply

$$n\,\nu_1' \leqq \nu_2'. \tag{16}$$

Thus $n\nu_1'=\nu_2'$, by (11) and (16), whence $\nu_1''=n\nu_2''$, and therefore (10) follows by the duality theorem $\nu''(x)=\nu(x)$, for $x\in V$. The equality (9) is an immediate consequence of (10).

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Our first generalization of Theorem 1 is

Theorem 2. *Let V and L be as defined in Theorem 1. If $\sup_i$, $i=1, 2$, is the supremum norm belonging to a single norm ν_i, then*

$$\inf_L \frac{\sup_1}{\sup_2} = \inf_L \frac{\sup_2}{\sup_1}. \tag{17}$$

Theorem 2 is clearly a consequence of Theorem 3 below. For, let U and V be vector spaces normed by μ_i, ν_i, $i=1, 2$, and suppose

$$m_1=\inf_U \frac{\mu_2}{\mu_1}, \quad m_2=\inf_U \frac{\mu_1}{\mu_2}, \quad n_1=\inf_V \frac{\nu_2}{\nu_1}, \quad n_2=\inf_V \frac{\nu_1}{\nu_2}. \tag{18}$$

If we put $U=V$ and $\mu_i=\nu_i$, $i=1, 2$, then $m_1=n_1$, $m_2=n_2$. Hence $m_1n_2=m_2n_1$ and Theorem 2 follows from Theorem 3. Of course, if U and V are finite dimensional, then m_i and n_i are non-zero, and it will be convenient to assume this for the infinite-dimensional case which follows. Thus the norms will be topologically equivalent, and a transformation bounded with respect to μ_1, ν_1 is bounded with respect to μ_2, ν_2, and vice versa.

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Theorem 3. *Let U and V be vector spaces (possibly infinite dimensional), and let μ_i, ν_i, $i=1, 2$, be norms on U and V respectively, and let m_1, n_2 be defined by (18) and be non-zero. Let T be a set of bounded transformations which contains all bounded linear transformations of rank 1. Let $\sup_1$ and $\sup_2$ be the norms on T belonging to μ_1, ν_1, and μ_2, ν_2. Then*

$$\inf_T \frac{\sup_1}{\sup_2} = m_1 n_2. \tag{19}$$

Proof. Let $A\in T$, and $x\in U$. Then, if $Ax\neq 0$,

$$\frac{\nu_1(Ax)}{\mu_1(x)} = \frac{\mu_2(x)}{\mu_1(x)}\,\frac{\nu_1(Ax)}{\nu_2(Ax)}\,\frac{\nu_2(Ax)}{\mu_2(x)} \geqq m_1 n_2 \frac{\nu_2(Ax)}{\mu_2(x)}, \tag{20}$$

and, since this inequality is trivial if $Ax=0$, we have*

$$\frac{\sup_1}{\sup_2} \geqq m_1 n_2. \tag{21}$$

Let m and n be any positive numbers which satisfy

$$m_1 < m, \qquad n_2 < n. \tag{22}$$

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* The argument leading to (21) is equivalent to a remark of HOUSEHOLDER's [2], Chapter 2. We are grateful to Dr HOUSEHOLDER for permitting us to see his book in manuscript form.

To complete the proof of the theorem, we shall demonstrate the existence of a linear transformation E of U into V of rank 1 for which

$$\frac{\sup_1(E)}{\sup_2(E)} \leqq m\,n. \tag{23}$$

In virtue of the definitions (18) there exist $u \in U$, $v \in V$ such that

$$\frac{\mu_2(u)}{\mu_1(u)} \leqq m, \qquad \frac{\nu_1(v)}{\nu_2(v)} \leqq n. \tag{24}$$

On the one dimensional subspace $\{u\}$ we define a linear functional y_0' by

$$|(y_0', x)| = \mu_1(x)\,\mu_2(u), \qquad x \in \{u\}. \tag{25}$$

It follows by the Hahn-Banach Theorem ([*1*], p. 11) that there exists an extension linear functional y' of y_0' for which

$$|(y', x)| \leqq \mu_1(x)\,\mu_2(u), \tag{26}$$

for all $x \in U$. If we now define E by

$$E\,x = (y', x)\,v, \tag{27}$$

we obtain immediately that for all $x \in U$

$$\frac{\nu_1(E\,x)}{\mu_1(x)} = \frac{|(y', x)|\,\nu_1(v)}{\mu_1(x)} \leqq \mu_2(u)\,\nu_1(v), \tag{28}$$

whence

$$\sup_1(E) \leqq \mu_2(u)\,\nu_1(v). \tag{29}$$

But

$$\frac{\nu_2(E\,u)}{\mu_2(u)} = \frac{|(y', u)|\,\nu_2(v)}{\mu_2(u)} = \mu_1(u)\,\nu_2(v), \tag{30}$$

whence

$$\sup_2(E) \geqq \mu_1(u)\,\nu_2(v). \tag{31}$$

The inequality (23) now follows by (24) and the theorem is proved.

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The following corollary includes assertion (10) of Theorem 1.

Corollary. *If* $\frac{\sup_1}{\sup_2}$ *is a constant on* T, *then* $\frac{\mu_2}{\mu_1}$ *and* $\frac{\nu_1}{\nu_2}$ *are constants on* U *and* V *(and, trivially, conversely).*

Proof. By the definitions (18)

$$m_1\,m_2 \leqq 1, \qquad n_1\,n_2 \leqq 1, \tag{32}$$

and $\frac{\mu_1}{\mu_2}$, $\frac{\nu_1}{\nu_2}$ are constant if and only if

$$m_1\,m_2 = 1, \qquad n_1\,n_2 = 1. \tag{33}$$

But, in virtue of (19), $\frac{\sup_1}{\sup_2}$ is a constant if and only if

$$\frac{\sup_1}{\sup_2} = m_1 n_2, \quad m_2 n_1 = \frac{\sup_2}{\sup_1} = \frac{1}{m_1 n_2}. \tag{34}$$

Hence $\frac{\sup_1}{\sup_2}$ is a constant if and only if

$$(m_1 m_2)(n_1 n_2) = 1 \tag{35}$$

and so, by (32), if and only if (33) holds. The corollary is proved.

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We give now a geometric version of the above proof, assuming for the sake of clarity that $\frac{\mu_2}{\mu_1}$ and $\frac{\nu_1}{\nu_2}$ attain their infima, say

$$\frac{\mu_2(u)}{\mu_1(u)} = m_1, \quad \frac{\nu_1(v)}{\nu_2(v)} = n_2. \tag{36}$$

Suppose M_i, N_i are the convex bodies belonging to the norms μ_i, ν_i: $M_i = \{x \in U: \mu_i(x) \leqq 1\}$. We may choose u and v satisfying (36) on the boundaries M_1^0, N_2^0 of M_1, N_2 respectively. Let $u + H$ be the support plane of M_1 at u, where H is a maximal subspace of U. Then the linear transformation E defined by $Eu = v$, $EH = 0$, has $\frac{\sup_1(E)}{\sup_2(E)} = m_1 n_2$. This construction is in fact analogous to the construction of E in the proof of Theorem 3. We shall only remark here that the Hahn-Banach Theorem implies the existence of a support plane for M_1 at every point of M_1^0: for,* if $u \in M_1^0$ and y' satisfies (25) and (26), then $u + H = \{x \in U: (y', x) = \mu_2(u)\}$ is a support plane for M_1 at u.

We shall also relate the proof of Theorem 1 to the proof of the more general Theorem 3. As already indicated, the proof of Theorem 1 depends on the theorem that $\nu''(x) = \nu(x)$, for all $x \in V$, under the natural identification of V with a subspace of V''. Now, if $x \in V$, then

$$\nu''(x) = \sup_{\nu'(y')=1} |(y', x)|$$

and if $\nu'(y') = 1$ then $|(y', x)| \leqq \nu(x)$ by (5). Hence $\nu''(x) = \nu(x)$ if there exists $y' \in V'$ such that $|(y', x)| = \nu(x)$ and $|(y', z)| \leqq \nu(z)$, for all $z \in V$. Thus the duality theorem is implied by the Hahn-Banach Theorem for one-dimensional subspaces.

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Theorem 1, and therefore, of course, Theorems 2 and 3, are false for non-equilibrated norms μ_i, ν_i, viz. those for which (3) is replaced by

$$\nu(c\,x) = c\,\nu(x), \qquad \text{if} \quad c > 0. \tag{37}$$

* We are grateful to Professor W. RUDIN for this remark.

For let V be the real line, and let

$$\nu_1(x) = \nu_2(x) = x \qquad \text{if} \quad x \geqq 0, \tag{38}$$

but let

$$\nu_1(x) = 2\nu_2(x) = -x \quad \text{if} \quad x < 0. \tag{39}$$

If r is identified with the linear transformation $x \to rx$ then

$$\sup_1(r) = \sup_2(r) = r \qquad \text{if} \quad r > 0, \tag{40}$$

but

$$\sup_2(r) = 2\sup_1(r) = -2r \quad \text{if} \quad r < 0. \tag{41}$$

Hence $\sup_2 \geqq \sup_1$, but $\sup_2 \neq \sup_1$.

Finally we remark that a result slightly more general than theorem 3 may be proved. If the set T contains all transformations of rank 1 then the assumption that $m_i, n_i, i = 1, 2$, are non-zero is not needed, and theorem 3 can be formulated as a theorem on quasi-norms.

References

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