GROUP MEMBERSHIP IN RINGS AND SEMIGROUPS

GERALD LOSEY AND HANS SCHNEIDER

1. Introduction. Let R be a semigroup or associative ring. A group G in R is a subset of R which is a group under the multiplication on R. That is, G contains an idempotent e which acts as a multiplicative identity on G and if $\alpha \in G$ then there exists an element $\alpha' \in G$ such that $\alpha\alpha' = \alpha'\alpha = e$. An element α of R is said to be a group element in R if α belongs to some group in R.

The problem of deciding whether a given element of R is a group element has been investigated in various types of rings in [3], [4], [5], [6], [10], [11]. The purpose of the present paper is the generalization and extension of some results of Barnes and Schneider [3], Drazin [4] and Farahat and Mirsky [6].

Section 2 of this paper extends some results of [6] on the imbedding of the groups contained in a ring with identity in the group of units of the ring.

In § 3 use is made of the concept of left π-regularity. McCoy [9] introduced the concept of π-regularity, the consequences of which have developed in [1], [2], and [8]. It imposes a finitness condition satisfied, for example, by rings with minimum condition, by nil rings, by the "divided" rings of [6] and by direct sums of such rings. This condition is found to be sufficient in many of the cases where [6] uses the condition that the ring be a direct sum of divided rings. Moreover, the condition of left π-regularity is applicable to the case of semigroups. Under this condition, it is shown that if S is an extension of a semigroup or ring R, $\alpha \in R$ and α is a group element in S, then α is a group element of R.

Section 4 deals with conditions under which some power of a given element of R is a group element.

Section 5 gives a necessary and sufficient condition for the same property in terms of annihilators.

In order to point up the comparative weakness of the condition of left π-regularity of a ring necessary and sufficient conditions are given in § 6 that a left π-regular ring be a direct sum of divided rings.

2. Groups in rings with identity. Throughout this section R will denote a ring with an identity element 1 and U will denote the group of units of R.

Received August 1, 1960. The research of the second named author was supported by the United States Army under Contract No. DA 11 022 ORD 2059, Mathematics Research Center, United States Army, Madison, Wis.

LEMMA 2.1. (Farahat and Mirsky [6]) *Let G be a group with idempotent e in R. Then the mapping η: $G \to U$ defined by $\eta(g) = g + (1 - e)$ is an isomorphism of G onto a subgroup G_1 of U. The idempotent e commutes with every element of G_1 and $G = eG_1$. Moreover, if $x \in G_1$ then $x = ex + (1 - e)$.*

Proof. Let $g \in G$ and let g' be its inverse in G. Then

$$\{g + (1 - e)\}\{g' + (1 - e)\} = gg' + (1 - e)^2 = e + (1 - e) = 1 ,$$

whence $g + (1 - e) \in U$. The verification that η is an isomorphism is routine.

If $x \in G_1$ then $x = g + (1 - e)$ for some $g \in G$. Hence $ex = xe = g$. It follows that $x = ex + (1 - e)$ and that $G = eG_1$. Thus the lemma is proved.

Now let $C(e)$ denote the set of all elements of U which commute with e. Then, clearly, $eC(e)$ is a group with idempotent e in R. It follows from Lemma 2.1 that every group with idempotent e in R is contained $eC(e)$, whence $eC(e)$ is the unique maximal group with idempotent e in R. We set $M(e) = eC(e)$. If we now apply the isomorphism η of Lemma 2.1 to $M(e)$ we obtain a subgroup $M_1(e)$ of U. It also follows from Lemma 2.1 that $M_1(e) \subsetneqq C(e)$. We shall show that $M_1(e)$ is not only a subgroup of $C(e)$, but is, in fact, a direct factor. This will follow from the more general Theorem 2.2.

If $e_1, e_2, \cdots, e_n$ are idempotents of R, let $C(e_1, e_2, \cdots, e_n)$ denote the set of all elements of U which commute with each of $e_1, e_2, \cdots, e_n$.

THEOREM 2.2. *If $e_1, e_2, \cdots, e_n$ are mutually orthogonal idempotents in R and $e_1 + e_2 + \cdots + e_n = 1$ then*

$$\begin{aligned} C(e_1, e_2, \cdots, e_n) &= M_1(e_1) \otimes M_1(e_2) \otimes \cdots \otimes M_1(e_n) \\ &\cong M(e_1) \otimes M(e_2) \otimes \cdots \otimes M(e_n) . \end{aligned}$$

Proof. Let $C = C(e_1, e_2, \cdots, e_n)$.

(1) We shall first show that if $i \neq j$ then $M_1(e_i)$ and $M_1(e_j)$ commute elementwise. Let $x \in M_1(e_i)$ and $y \in M_1(e_j)$. Then $x = e_ix + (1 - e_i)$ and $y = e_jy + (1 - e_j)$. Therefore

$$xy = e_ix(1 - e_j) + (1 - e_i)e_jy = e_ix + e_jy$$

which, by symmetry, is also equal to yx.

(2) Next, $C = M_1(e_1) \times M_1(e_2) \times \cdots \times M_1(e_n)$. For suppose $x \in C$. Then $e_ix + (1 - e_i) \in M_1(e_i)$, $i = 1, \cdots, n$. Now

$$\{e_1 x + (1 - e_1)\} \cdots \{e_n x + (1 - e_n)\}$$
$$= e_1 x + \cdots + e_n x = (e_1 + \cdots + e_n)x = x .$$

(3) We now prove that $M_1(e_i) \cap \prod_{j \neq i} M_1(e_j) = 1$. For let x belong to this intersection. Then

$$x = e_i x + (1 - e_i) = \prod_{j \neq i} \{e_j x_j + (1 - e_j)\} ,$$

where $x_j \in U$. Hence, from the commutativity of the $e_j x_j + (1 - e_j)$ and the fact that $e_i\{e_j x_j + (1 - e_j)\} = e_i$, it follows that

$$e_i x = x e_i = e_i \times e_1 \times \cdots \times e_i = e_i ,$$

and so

$$x = e_i x + (1 - e_i) = e_i + (1 - e_i) = 1 .$$

From (1), (2) and (3) it follows that C is the direct product of the $M_1(e_i)$.

COROLLARY 2.3. $C(e) = M_1(e) \otimes M_1(1 - e) \cong M(e) \otimes M(1 - e)$.

Proof. It is merely necessary to notice that $C(e) = C(e, 1 - e)$.

3. Group elements in extensions of π-regular semigroups and rings. In this section R will generally denote a semigroup; results in which R must be assumed to be a ring will be so indicated.

Let R be a semigroup and $\alpha \in R$. We say that α is left π-regular ([8], [2]) if there exists an element x in R and a positive integer n such that $x\alpha^{n+1} = \alpha^n$. The semigroup R is said to be left π-regular if every element of R is left π-regular. Similar definitions are made for right π-regularity. Evidently, if α is both left and right π-regular then there exist x and y in R and a positive integer n for which $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$.

Left π-regularity is a finiteness condition in the following sense: The element α is left π-regular if and only if the descending sequence of left ideals $R\alpha \supseteq R\alpha^2 \supseteq R\alpha^3 \supseteq \cdots$ terminates in a finite number of steps. More precisely, $x\alpha^{n+1} = \alpha^n$ implies that $R\alpha^{n+1} = R\alpha^n$, and conversely, $R\alpha^{n+1} = R\alpha^n$ implies that $x\alpha^{n+2} = \alpha^{n+1}$ for some $x \in R$, and, if R has an identity, implies $x\alpha^{n+1} = \alpha^n$.

Left π-regularity does not imply right π-regularity and, of course, conversely. In the case of semigroups this is shown by the following example. Let σ and τ be two infinite cardinals with $\tau \leqq \sigma$, and let E be a set of cardinal σ. Let B be the semigroup of all one-to-one mappings of E into itself for which the complement of αE in E is of cardinal τ. It is easy to see that for each $\alpha \in B$ there is an $x \in B$ such that $x\alpha$ is the identity map on αE and so $x\alpha^2 = \alpha$, whence B is left π-regular. But for all integers n and all $y \in B$, $\alpha^{n+1}yE$ is properly contained in $\alpha^{\sigma}E$ and, hence, no element of B is right π-regular. (In the case that $\sigma = \tau = \aleph_0$, the semigoup B is called the semigroup of Baer and Levi.)

THEOREM 3.1. *Suppose R is a semigroup and S is an extension of R. If $\alpha \in R$ is left π-regular and α is a group element in S, then α is also a group element in R.*

Proof. Suppose α is a group element in S. Then there exists α', $e \in S$ such that

$$\alpha e = e\alpha = \alpha\ , \quad \alpha' e = e\alpha' = \alpha'\ , \quad \alpha\alpha' = \alpha'\alpha = e\ .$$

Since $\alpha \in R$ is left π-regular there exists an element $x \in R$ and a positive integer n for which $\alpha^n = x\alpha^{n+1}$. Hence $e = \alpha^n\alpha'^n = x\alpha^{n+1}\alpha'^n = x\alpha e = x\alpha \in R$. Moreover, $\alpha' = e\alpha' = x\alpha\alpha' = xe \in R$. Consequently, α is a group element in R and the theorem is proved.

We note that it follows from this theorem that if S is any extension of the semigroup B of our example and $\alpha \in B$, then α is not a group element in S. For if α were a group element in S it would also be a group element in B and hence right π-regular in B.

An element α of a semigroup R is called *cancellable* (often called regular) if α is both right and left cancellable, viz: $\alpha x = \alpha y$ implies $x = y$ and $x\alpha = y\alpha$ implies $x = y$. In a ring an element is cancellable if and only if it is not a proper divisor of zero.

COROLLARY 3.2. *Let R be a semigroup and let T be an extension of R. Suppose*

(i) *Every element of R is cancellable in T,*

(ii) *For each $\alpha \in R$, $x \in T$ there exist $\alpha' \in R$, $x' \in T$ such that $\alpha x' = x\alpha'$,*

(iii) *Every element of R is left π-regular in R.*

Then T contains an identity and R is a group.

Note that if R is a ring and R^* is the set of non-zero elements of R, then if R^* satisfies (i), (ii) and (iii), the conclusion of the corollary tells us that R is a division ring.

Proof. By a slight modification of an argument of Jacobson [7], p. 118, we may form a semigroup of fractions x/α, $x \in T$, $\alpha \in R$. If we denote this semigroup of fractions by S, we may imbed T, and consequently R, in S by the mapping $x \rightarrow x\alpha/\alpha$. The element α/α is an identity for S and, of course, also for T. Every element of R is invertible in S with respect to 1, namely its inverse is α/α^2. In view of (iii) it follows that $1 \in R$, and thus to T, and that each element of R has an inverse in R, whence R is a group.

We note that if R is assumed to be in the center of T then condition (ii) is automatically satisfied with $\alpha = \alpha'$ and $x = x'$.

COROLLARY 3.3. *Let R be a ring, finitely generated as a module over its center C. Suppose*

(i) *Every non-zero element of R is cancellable, that is, R has no proper zero divisors,*

(ii) *Every element of C is left π-regular in C,*

then R is a division ring.

Proof. By Corollary 3.2, C is a field. Thus R is a finite dimensional algebra over C. But, by (i), this algebra has no zero divisors and a finite dimension algebra without zero divisors is a division ring.

4. Powering elements into group elements. Under certain conditions an element α of a semigroup R may not be a group element itself although some power of it may be. An example of this is the case when R contains a zero element and α is a nilpotent element of R.

THEOREM 4.1. *Let R be a semigroup and S an extension of R such that for each $x \in S$ there is a positive integer $m = m(x)$ for which $x^m \in R$. Suppose $\alpha \in R$ and α is both left and right π-regular in S.*[1] *Then α^n is a group element in R for some n and conversely.*

Proof. Since α is both left and right π-regular, find $x, y \in S$ and a positive integer n such that $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$. If $p \geqq m$ then it follows that $x\alpha^{p+1} = \alpha^p = \alpha^{p+1}y$. Find $m(x)$ and $m(y)$ such that $x^{m(x)}$, $y^{m(y)} \in R$ and set $p = m(x)m(y)n$. Then $x^p, y^p \in R$ and $x\alpha^{p+1} = \alpha^p = \alpha^{p+1}y$. Note that

$$x\alpha^p = x^2\alpha^{p+1} = \alpha^{p+1}y^2 = \alpha^p y$$

and so we may set $\beta = x^{2p}\alpha^p = \alpha^p y^{2p}$ and $e = x^p\alpha^p = \alpha^p y^p$. Then β, $e \in R$ and

$$\beta\alpha^p = x^{2p}\alpha^{2p} = x^{2p-1}x\alpha^{p+1}\alpha^{p-1} = x^{2p-1}\alpha^{2p-1} = \cdots = x^p\alpha^p = e\ .$$

Similarly $\alpha^p\beta = e$. Also

$$e\alpha^p = x^p\alpha^{2p} = x^{p-1}x\alpha^{p+1}\alpha^{p-1} = x^{p-1}\alpha^{2p-1} = \cdots = \alpha^p\ .$$

By another similar argument $\alpha^p e = \alpha^p$. Further,

$$e\beta = x^p\alpha^{2p}y^{2p} = x^p\alpha^p y^p = x^{2p}\alpha^p = \beta$$

[1] Drazin [4] calls an element which is both left and right π-regular a pseudo-invertible element.

and $\beta e = \beta$. Thus, in order to show that α^p is a group element, it is now sufficient to show that e is idempotent. But $e^2 = x^q\alpha^{2p}y^p = \alpha^p y^p = e$.

Conversely, if α^n is a group element in R and β is the group inverse of α^n, then $\beta\alpha^{2n} = \alpha^n = \alpha^{2n}\beta$. Hence, α is both left and right π-regular in R, and therefore in S.

COROLLARY 4.2. *If R is a semigroup, $\alpha \in R$ and α is both left and right π-regular, then α^n is a group element in R for some n and conversely. More precisely, α^n is a group element if and only if*

$$x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$$

for some $x, y \in R$.

Proof. Take $S = R$ is the preceding theorem.

This result appears in a somewhat different guise in a paper by M. P. Drazin, [4].

COROLLARY 4.3. *Let R be a semigroup and α^n a group element with identity e in R. If $e\alpha = \alpha$ then α is a group element in R, and conversely.*

Proof. Let α^n be a group element with identity e and inverse β. Then α^{n+1} is also a group element with the same identity and so $\alpha = e\alpha = \beta\alpha^n \cdot \alpha = \beta\alpha^{n+1}$ a product of group elements with idempotent e and thus a group element itself.

Conversely, if α is a group element with identity f and α^n is a group element with identy e, then $e = f$ since α^n is also a group element with identity f.

5. Annihilator conditions that a given element be a group element.

In a ring R we define the left and right annihilators of an element α in the usual manner:

$$A_l(0, \alpha) = \{z \in R: z\alpha = 0\} \quad \text{and} \quad A_r(0, \alpha) = \{z \in R: \alpha z = 0\} .$$

So that we may state our next results for semigroups as well as rings we shall generalize the concept of an annihilator. In a semigroup R we shall set

$$A_l(x, \alpha) = \{z \in R: \ z\alpha = x\alpha\} ,$$
$$A_r(x, \alpha) = \{z \in R: \ \alpha z = \alpha x\} .$$

Several consequences of these definitions are easily proved (though we shall makes no use of these properties). The sets $A(x, \alpha)$ are equivalence classes modulo the equivalence relation $z \equiv_l x$ defined by $z\alpha = x\alpha$, and similarly for the sets $A_r(x, \alpha)$. An element $\alpha \in R$ is cancellable if and only if $A_l(x, \alpha) = A_r(x, \alpha) = \{x\}$ for all $x \in R$. If R is a ring, then $A_l(x, \alpha) = x + A_l(0, \alpha)$ and $A_r(x, \alpha) = x + A_r(0, \alpha)$. Finally, (and this we shall need) it is evident that

$$A_l(x, \alpha) \subseteq A_l(x, \alpha^2) \subseteq A_l(x, \alpha^3) \subseteq \cdots$$

and that $A_l(x, \alpha^n) = A_l(x, \alpha^{n+1})$ implies that $A_l(x, \alpha^n) = A_l(x, \alpha^m)$ for all $m \geqq n$, and similarly for the sets $A_r(x, \alpha)$, $A_r(x, \alpha^2)$, etc.

In analogy with the phrase left π-regular we introduce the following terminology: An element α of a semigroup R is called *left A-regular* in R if there exists a positive integer n for which $x\alpha^{n+1} = z\alpha^{n+1}$ implies that $x\alpha^n = z\alpha^n$ for all x and z in R. Thus α in R is let A-regular if and only if the ascending chains $A_l(x, \alpha) \subseteq A_l(x, \alpha^2) \subseteq \cdots$ terminate in finitely many steps for all $x \in R$.

It is easy to see that a left π-regular element of a semigroup R is right A-regular, and a right π-regular element is left A-regular. In this connection, a slight generalization of a theorem Azumaya [2] proved for rings is of interest.

THEOREM 5.1. *Let R be a semigroup and α a left π-regular element of R. If α is left A-regular then α is right π-regular, and conversely.*

Proof. Suppose α is left A-regular in R. Then we may choose a positive integer n such that $z\alpha^{n+1} = z'\alpha^{n+1}$ implies $z\alpha^n = z'\alpha^n$ for all z, $z' \in R$ and $x\alpha^{n+1} = \alpha^n$ for some $x \in R$. We wish to prove that $\alpha^{n+1}y = \alpha^n$ for some $y \in R$. It is clearly sufficient to prove that $\alpha^m x^m \alpha^n = \alpha^n$ for $m = 0, 1, 2, \cdots$. Since the case $m = 0$ is trivial, we proceed by induction. Thus, assume that $\alpha^m x^m \alpha^n = \alpha^n$. Then

$$\alpha^{m+1}x^{m+1}\alpha^{n+1} = \alpha^{m+1}x^m\alpha^n = \alpha^{n+1} = x\alpha\cdot\alpha^{n+1} ,$$

whence $\alpha^{m+1}x^{m+1}\alpha^n = x\alpha\cdot\alpha^n = \alpha^n$.

That the converse is true has already been remarked before the statement of the theorem.

The second theorem of this section relates the integers n which occur in the definitions of π-regularity and A-regularity. Let n be a positive integer and let z and α be elements of the semigroup R. We shall say that condition $A(z, \alpha, n)$ holds if both $A_l(z, \alpha^{n+1}) = A_l(z, \alpha^n)$ and $A_r(z, \alpha^{n+1}) = A_r(z, \alpha^n)$. Thus an element α is both right and left A-regular if there exists a positive integer n such that $A(z, \alpha, n)$ holds for all z, and conversely. If R is a ring and $A(z, \alpha, n)$ is satisfied for some $z \in R$ then $A(z, \alpha, n)$ is satisfied for all $z \in R$.

THEOREM 5.2. *Let R be a semigroup and let α be an element of R*

which is both left and right π-regular in R. The following results hold:

(i) *If α^n is a group element in R then $A(z, \alpha, n)$ holds for all $z \in R$.*

(ii) *If $A(\alpha, \alpha, n)$ holds then α^{n+1} is a group element in R.*

(iii) *If R has identity 1 and $A(1, \alpha, n)$ holds then α^n is a group element.*

We remark that under the hypotheses of the theorem, α^n is a group element for some n and α is both left and right A-regular.

Proof. (i) If α^n is a group element, then it follows from Corollary 4.2 that $x\alpha^{n+1} = \alpha^n = \alpha^{n+1}y$ for some $x, y \in R$. Suppose that u is any element of $A_l(z, \alpha^{n+1})$ where $z \in R$. Then $z\alpha^n = z\alpha^{n+1}y = u\alpha^{n+1}y = u\alpha^n$, whence $u \in A_l(z, \alpha^n)$ which implies $A_l(z, \alpha^{n+1}) = A_l(z, \alpha^n)$. The proof of $A_r(z, \alpha^{n+1}) = A_r(z, \alpha^n)$ is similar.

(ii) Suppose that $A(\alpha, \alpha, n)$ holds. Since α is let π-regluar there exists a positive integer m and an $x \in R$ such that $x\alpha^{m+1} = \alpha^m$. We shall show that $x\alpha^{n+2} = \alpha^{n+1}$. If $m < n + 1$, we obtain this equality by multiplying the previous equality by α^{n-m+1}. If $m = n + 1$ there is nothing to prove. If $m > n + 1$ then $x\alpha^2\alpha^{m-1} = \alpha\alpha^{m-1}$ and $A(\alpha, \alpha, n)$ implies that

$$x\alpha^{n+2} = x\alpha^2\alpha^n = \alpha\alpha^n = \alpha^{n+1} .$$

The existence of an element $y \in R$ satisfying $\alpha^{n+2}y = \alpha^{n+1}$ is proved similarly. It now follows from Corollary 4.2 that α^{n+1} is a group element.

(iii) The proof is similar to the proof of (ii). This time it follows from $x\alpha^{m+1} = 1\alpha^m$ when m n that $x\alpha^{n+1} = x\alpha\alpha^n = 1\alpha^n = \alpha^n$ by virture of $A(1, \alpha, n)$. Hence α^n is a group element.

6. A criterion that a ring be semi-divided. A ring is said to be *divided* if it has an identity and every element is invertible or nilpotent. A ring is *semi-divided* if it is the direct sum of (possibly infinitely many) divided rings. The terminology is that of [6]. In this section we shall give necessary and sufficient conditions that a left π-regular ring be semi-divided.

LEMMA 6.1. *Let R be a semigroup both left and right π-regular. Then every non-nil (left) ideal of R contains a non-zero idempotent.*

Proof. Let I be a non-nil left ideal of R and α a non-nilpotent element of I. α^n is a group element with respect to the non-zero idempotent e for some n. Let β be the inverse of α^n. Then $e = \beta\alpha^n \in I$.

If e and f are idempotents then we say that e *dominates* f if $ef = fe = f$. An idempotent $e \neq 0$ is *primitive* if it dominates only 0 and itself. For rings this is equivalent to saying that e is primitive if it is not the sum of two non-zero orthogonal idempotents.

THEOREM 6.2. *Let R be a ring satisfying the following conditions:*

(i) *R is left and right π-regular;*

(ii) *Every primitive idempotent of R is in the center of R;*

(iii) *Every non-zero idempotent of R dominates a primitive idempotent;*

(iv) *If $x \in R$ then $xe = 0$ for all but finitely many primitive idempotents e.*

Then R is the direct sum of a semi-divided ring and a nil ring, and conversely. If, in addition, R satisfies the condition:

(v) *Every element of R has a left or right identity,*

then R is semi-divided, and conversely.

Proof. Let e be a primitive idempotent. Then $Re = eR$, since e is in the center of R, and e is the identity of Re. Since e is primitive e is unique non-zero idempotent of Re. If $\alpha \in Re$ is not nilpotent then α^n is invertible in Re. But $e\alpha = \alpha$ and so, by 4.3, α is invertible in Re. Hence Re is a divided ring.

Let $\{e_i\}$ be the set of all primitive idempotents of R. $e_ie_j = 0$ if $e_i \neq e_j$. The sum ΣRe_i is direct; for if $x \in Re_j \cap \sum_{i\neq j} Re_i$ then $x = xe_j = \sum_{i\neq j} x_ie_ie_j = 0$. Thus $R_1 = \Sigma Re_i$ is semi-divided.

Let R_2 be the set of all $x \in R$ for which $xe_i = 0$ for all primitive idempotents e_i. R_2 is an ideal of R. If R_2 contains a non-zero idempotent then, by condition (iii) R_2 contains a primitive idempotent e. But then we would have $e = e^2 = 0$. Hence, by 6.1, R_2 must be nil.

The sum $R_1 + R_2$ is direct; for if $x \neq 0$ is an element of R_1 then $xe_i \neq 0$ for at least one e_i. Hence, $R_1 \cap R_2 = 0$. We now wish to show that $R = R_1 + R_2$. If $x \in R$ then $xe_i \neq 0$ for only finitely many primitive e_i. Hence, $x' = x - \Sigma xe_i$ is well defined. Moreover, $x'e_i = xe_i - xe_i^2 = 0$ and so $x' \in R_2$. Therefore, $x = \Sigma xe_i + x' \in R_1 + R_2$. Hence, R is the direct sum of a semi-divided ring and a nil ring.

The converse is directly verified.

Now suppose in addition to (i)-(iv) R also satisfies (v). Let $x \in R_2$. Then x has a (say left) identity $e = e_1 + e_2$, $e_1 \in R_1, e_2 \in R_2$, and

$$x = ex = e_1x + e_2x = e_2x ,$$

since $e_1x = 0$. But then $x = e_2^mx$ for all $m \geq 0$. Since R_2 is nil, $e_2^m = 0$ for some m and so $x = 0$. Thus $R_2 = 0$ and $R = R_1$, a semi-divided ring.

Again, the converse is easily verified.

That condition (iv) is actually necessary may be seen from the following example. Let S be the strong direct sum of countably many copies of Z_4, the ring of integers mod 4. Let R be the subring of S generated by the weak direct sum and the element $(2, 2, \cdots, 2, \cdots)$. Then R satisfies (i)–(iii) and is not the direct sum of a semi-divided ring and nil ring.

In conclusion, the authors wish to express their gratitude to M. P. Drazin for several pertinent criticisms.

BIBLIOGRAPHY

1. R. Arens and I. Kaplansky, *Topological representations of algebras*, Trans. Amer. Math. Soc., **63** (1948), 457-481.
2. G. Azumaya, *Strongly π-regular rings*, J. Fac. Sci. Hokkaido University, **13** (1954), 34-39.
3. W. E. Barnes and H. Schneider, *The group membership of a polynomial in an element algebraic over a field*, Arch. Math., **8** (1957), 166-168.
4. M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly, **65** (1958), 506-514.
5. H. K. Farahat and L. Mirsky, *A condition for diagonability of matrices*, Amer. Math. Monthly, **63** (1956), 410-412.
6. H. K. Farahat and L. Mirsky, *Group membership in rings of various types*, Math. Zeitschr., **70** (1958), 231-244.
7. N. Jacobson, *The theory of rings*, Math. Survey II, (1943), American Math. Soc.
8. I. Kaplansky, *Topological representations of rings II*, Trans. Amer. Math. Soc., **68** (1950), 62-75.
9. N. H. McCoy, *Generalized regular rings*, Bull. Amer. Math. Soc., **45** (1939), 175-178.
10. A. Ranum, *The group membership of singular matrices*, Amer. J. Math., **31** (1909), 18-41.
11. A. Ranum, *The groups belonging to a linear associative algebra*, Amer. J. Math., **49** (1927), 285-308.

UNIVERSITY OF WISCONSIN AND
MATHEMATICS RESEARCH CENTER,
U. S. ARMY, MADISON, WISCONSIN