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COMPLETELY SIMPLE AND INVERSE SEMIGROUPS

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1. The purpose of this paper is to investigate the structure of certain types of semigroups. Rees (6), (7) has determined the structure of a completely simple semigroup, and has shown that such a system may be realized as a type of matrix semigroup. Clifford (2) and Schwarz (8) have found conditions, namely, the existence of minimal left and minimal right ideals, under which a simple semigroup is completely simple, and have made a more detailed study of such semigroups. Preston (4), (5) has studied inverse semigroups, in which each non-zero element has a unique relative inverse, and has also considered inverse semigroups which contain minimal right or left ideals.

In the present paper we obtain a set of conditions on a simple semigroup, each of which is equivalent to the semigroup being both completely simple and inverse. Section 2 defines the terms used and gives a brief résumé of the main results which have already been proved. Section 3 is devoted to our present considerations.

2. A semigroup S is *simple* if it has no non-trivial two-sided ideals and if $S^2 \neq (0)$. This last condition excludes the semigroups (0) and $(0, a)$ with $a^2 = 0$. The semigroup S is *completely simple* if it is simple, and

(i) to every non-zero element a of S there correspond idempotents e and f of S such that $ea = a = af$;

(ii) if e is an idempotent of S, the only non-zero idempotent f of S for which $ef = fe = f$ is $f = e$.

An idempotent e for which (ii) holds is said to be *primitive*.

A semigroup S is *inverse* if idempotents commute and if to every non-zero element a of S there corresponds at least one element e of S for which $ea = a$, and for which the equation $ax = e$ is soluble for x in S.

A minimal left ideal is a non-zero left ideal which does not properly contain any non-zero subset which is also a left ideal. If S is simple and contains a non-zero idempotent, it is known that the existence of a minimal left ideal implies that of a minimal right ideal. Thus we may prove the following results; let S be a simple semigroup containing at least one non-zero idempotent. Then the following are equivalent:

(i) S is completely simple.

(ii) S contains a minimal right (left) ideal.

(iii) S is the union of all its minimal right ideals and of all its minimal left ideals.

If L is a minimal left, and R a minimal right ideal of S then either $LR = S$ and $R \cap L = RL$ is a group with zero, or $LR = (0)$ and RL is a zero semigroup. $\{(RL)^2 = (0)\}$. For each R there exists at least one L, and for each L at least one R such that $LR = S$.

These results were first proved by Clifford (2), Rees (6), (7) and Schwarz (8). A concise summary of most of the results quoted, with simple proofs, may be found in Bruck (1).

3. THEOREM. *Let S be a simple semigroup. Then the following are equivalent:*

(i) *S is a completely simple semigroup in which idempotents commute.*

(ii) *For each non-zero a in S there exists a unique x in S such that $axa = a$.*

(iii) *For each non-zero a in S there exist unique elements e and e' such that $ea = a = ae'$. (It is easily seen that e and e' are idempotent.)*

(iv) *Every non-zero principal right ideal and every non-zero principal left ideal contains just one non-zero idempotent.*

(v) *S is completely simple. For every minimal right ideal R there exists just one minimal left ideal L, and for every minimal left ideal L there exists just one minimal right ideal R, such that $LR = S$. (Or equivalently, RL is a group with zero. Hence the sets of minimal left and minimal right ideals are in* (1-1) *correspondence.)*

(vi) *S is completely simple. Every minimal right ideal R of S is the union of a group with zero and a zero semigroup which annihilates the right ideal on the left, and every minimal left ideal L of S is the union of a group with zero and a zero semigroup which annihilates the left ideal on the right.*

(vii) *S contains at least one non-zero idempotent, and the product of distinct idempotents is zero.*

Proof. (i) *implies* (ii). If a is any non-zero element of S, there exist idempotents e and f such that $ea = af = a$. Since S is completely simple, e is primitive, so eS is minimal (2). But $eS \supset aS$ and $aS \supset af \neq 0$, whence by the minimality of eS, $eS = aS$. It follows that the equation $ax = e$ has a solution x in S. Hence S is inverse. Further, the equation $axa = a$ is soluble. Clearly, x and xa are non-zero, so since S is completely simple, xS and xaS are minimal; as $xaS \subset xS$ we must have $xS = xaS$. Thus the idempotent xa is a left identity for xS, and since $x \in xS$, $xax = x$. But in an inverse semigroup the equations $axa = a$ and $xax = x$ have a unique common solution, (3), so x is the only element for which $axa = a$.

(ii) *implies* (iii). Solve $axa = a$ for x and set $e = ax$. Then $e^2 = e \neq 0$; suppose that in addition $fa = a$. Then $e = ax = fax = fe$, and so $f = e$ by (ii). The proof of the uniqueness of the right identity is similar.

(iii) *implies* (iv). The principal right (left) ideal generated by any element a in S is defined to be the intersection of all right (left) ideals of S containing a, and under our present assumptions is aS. (Sa). Let e be the unique idempotent such that $ea = a$. If f is any non-zero idempotent such that $ef = fe = f$, (iii) shows that $f = e$, so that e is primitive. As above, $aS = eS$, so the principal right ideal aS contains e. If g is any non-zero idempotent contained in aS, $g \in eS$ so $eg = g$, whence again by (iii), $g = e$. The proof for left ideals is analogous.

(iv) *implies* (v). Let e be a non-zero idempotent of S, and let f be any non-zero idempotent such that $ef = fe = f$. Then $f \in Se$ so $f = e$ and e is primitive. Since S is simple it is therefore completely simple (7). Let R be a minimal right ideal of S; since S is completely simple, R contains an idempotent $e \neq 0$, and $R = eS$. By assumption, e is unique. Let L and L' be minimal left ideals of S such that RL and RL' are groups with zero. Then RL and RL' both contain e, so $RL \cap RL' \neq 0$, whence $L \cap L' \neq 0$. It follows that $L = L'$, since L and L' are minimal. The proof for left ideals is again analogous.

(v) *implies* (vi). Let L be the unique minimal left ideal such that RL is a group with zero. Let $G = RL$ and let Z be the complement of the non-zero part of G in R. Then $R = G \cup Z$ and $ZR = (0)$, since each element of Z belongs to a left ideal L' for which $L'R = (0)$. That Z is a zero semigroup is obvious. The proof for left ideals is completely analogous.

(vi) *implies* (vii). Clearly S contains at least one non-zero idempotent e. Let f be any idempotent such that $ef \neq 0$. Since S is the union of its minimal left ideals, e is in some minimal left ideal L, and dually, f is in some minimal right ideal R. Then $LR \supset ef \neq 0$, so RL is a group with zero, say G. Since $RL \subset L$ and $RL \subset R$ it follows that $R = G \cup Z$, $L = G \cup Z'$ where Z and Z' are the left and right annihilators of R and L, respectively. We conclude that e and f must coincide with the identity element of G, and so the product of distinct idempotents is zero.

(vii) *implies* (i). Let e and f be non-zero idempotents of S; then $ef = fe = f$ implies that $e = f$, so all non-zero idempotents of S are primitive. It follows that S is completely simple(7). That idempotents commute is obvious.

We note that in the proof of the first part of the theorem we have shown that a completely simple semigroup in which idempotents commute is inverse.

To show that the condition of simplicity is necessary for the equivalence of all our conditions, consider the semigroup S with multiplication defined as follows:

	0	a	a_1	a_2	b	b_1
0	0	0	0	0	0	0
a	0	a	a_1	a_2	0	0
a_1	0	a_1	a_2	a	0	0
a_2	0	a_2	a	a_1	0	0
b	0	0	0	0	b	b_1
b_1	0	0	0	0	b_1	b

Then S is commutative and satisfies condition (vii), having a and b as its only non-zero idempotents, but does not satisfy conditions (i), (v) or (vi) as it is not simple. ($A = (0, a, a_1, a_2)$ and $B = (0, b, b_1)$ are ideals of S.)

By considering the semigroup T in which multiplication is defined by $ab = a$ for all non-zero a and b in T, we see that the left and right conditions of the theorem are necessary. It is easily seen that T is completely simple, but is not inverse, for idempotents do not commute; clearly T satisfies the first part of conditions (iii), (iv), (v) and (vi), but not the second.

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