BOUNDS FOR THE MAXIMAL CHARACTERISTIC ROOT OF A NON-NEGATIVE IRREDUCIBLE MATRIX

By Alexander Ostrowski and Hans Schneider

1. Let A be an $n \times n$ non-negative irreducible matrix with row sums (all summations go from 1 to n.)

$$r_\nu = \sum_\mu a_{\nu\mu}, \qquad R = \max_\nu r_\nu, \qquad r = \min_\nu r_\nu. \tag{1}$$

We shall suppose throughout that

$$r < R. \tag{2}$$

It is well known that because of (2) the maximal characteristic root of A satisfies the inequality

$$r < \omega < R. \tag{3}$$

In §2 we shall use simple arguments to determine bounds L and U for ω satisfying

$$r < L \leq \omega \leq U < R, \tag{4}$$

which may be computed easily in terms of the elements of A; more precisely, L and U will depend only on r and R in (1) and

$$\rho = \frac{1}{n} \sum_\nu r_\nu \tag{5}$$

$$\lambda = \min_\nu a_{\nu\nu} \tag{6}$$

$$\kappa = \min_{\nu\neq\mu} a_{\nu\mu} \; (a_{\nu\mu} > 0), \tag{7}$$

i.e. κ is the minimum of the non-vanishing $a_{\nu\mu}$ with $\nu \neq \mu$.

In §3 a more refined and longer argument will lead to better bounds which still depend only on the r_ν in (1) and κ and λ, but require more computation.

When A is positive, bounds satisfying the inequality (4) have already been found by Ledermann [2] and improved by Ostrowski [3] and Brauer [1], but these bounds may coincide with r and R if A has zero elements. Of course, there are bounds which for many matrices A are better than r or R, but these may again reduce to r and R in some cases. For example, one of us has proved, [4], that $\omega \leq \max_\nu r_\nu^p z_\nu^{1-p}$ where $z_\nu = \sum_\mu a_{\mu\nu}$ and $0 \leq p \leq 1$, but this bound equals R if A is symmetric.

Received April 6, 1960. This work was sponsored by the U.S. Army under Contract No. DA-11-022-ORD-2059.

2. THEOREM 1. *Let ω be the maximal characteristic root of an $n \times n$ non-negative irreducible matrix A. Let r, R, ρ, λ, and κ be given by* (1), (1), (5), (6), *and* (7), *respectively and set*

$$\epsilon = \left(\frac{\kappa}{R - \lambda}\right)^{n-1}. \tag{8}$$

Then

$$\frac{(n-1)(1-\epsilon)r + n\epsilon\rho}{(n-1)(1-\epsilon) + n\epsilon} \leq \omega \leq \frac{(n-1)(1-\epsilon)R + n\epsilon\rho}{(n-1)(1-\epsilon) + n\epsilon}, \tag{9}$$

so that, independently of n,

$$L = r + \epsilon(\rho - r) = (1-\epsilon)r + \epsilon\rho \leq \omega \leq (1-\epsilon)R + \epsilon\rho = R - \epsilon(R - \rho) = U, \tag{10}$$

and these bounds obviously satisfy (4).

Proof. Let y be the positive characteristic (left-sided) row vector belonging to ω;

$$\sum_\mu y_\mu a_{\mu\nu} = \omega y_\nu, \quad \nu = 1, \cdots, n. \tag{11}$$

Summing these equalities we obtain the identity which is at the basis of our results:

$$\sum_\nu y_\nu r_\nu = \omega \sum_\nu y_\nu. \tag{12}$$

Now we may suppose that after a cogredient transformation and a normalization

$$\sum_\nu y_\nu = 1, \quad y_1 \geq y_2 \geq \cdots \geq y_n, \tag{13}$$

and we shall set

$$\delta = \frac{\min_\nu y_\nu}{\max_\nu y_\nu} = y_n/y_1. \tag{14}$$

Then from (12)

$$R - \omega = \sum_\nu y_\nu(R - \omega) = \sum_\nu y_\nu(R - r_\nu) \geq y_n \sum_\nu (R - r_\nu) = ny_n(R - \rho), \tag{15}$$

and similarly

$$\omega - r = \sum_\nu y_\nu(\omega - r) = \sum_\nu y_\nu(r_\nu - r) \geq y_n \sum_\nu (r_\nu - r) = ny_n(\rho - r). \tag{16}$$

Since

$$y_n = \frac{y_n}{\sum_\nu y_\nu} \geq \frac{y_n}{(n-1)y_1 + y_n} = \frac{\delta}{(n-1) + \delta}, \tag{17}$$

it now follows from (15) that

$$\frac{R-\omega}{R-\rho} \geq \frac{n\,\delta}{(n-1)+\delta} \tag{18}$$

and from (16) that

$$\frac{\omega-r}{\rho-r} \geq \frac{n\,\delta}{(n-1)+\delta}. \tag{19}$$

We note that the right-hand side of (18) and (19) increases monotonically with δ for δ non-negative and hence in both (18) and (19) we may replace δ by any positive lower bound. Such a bound is obtained by applying the argument of the lemma in [6] to the y_ν :

$$\delta \geq \left(\frac{\kappa}{\omega-\lambda}\right)^{n-1}, \tag{20}$$

and hence by (3), ϵ in (8) is also a lower bound. Thus

$$\frac{R-\omega}{R-\delta} \geq \frac{n\epsilon}{(n-1)+\epsilon}, \tag{21}$$

and

$$\frac{\omega-r}{\rho-r} \geq \frac{n\epsilon}{(n-1)+\epsilon}, \tag{22}$$

and (21) and (22) are equivalent to (9).

For (10) we simply note that

$$\begin{aligned}&\frac{(n-1)(1-\epsilon)R+n\epsilon\rho}{(n-1)(1-\epsilon)+n\epsilon}\\ &\qquad < \frac{(n-1)(1-\epsilon)R+n\epsilon\rho+(1-\epsilon)R}{(n-1)(1-\epsilon)+n\epsilon+(1-\epsilon)} = (1-\epsilon)R+\epsilon\rho,\end{aligned} \tag{23}$$

and similarly for the lower bound. Thus our theorem is proved.

We remark that (10) could have been obtained directly by using

$$y_n = y_n/\sum_\nu y_\nu \geq \frac{y_n}{ny_1} = \frac{1}{n}\,\delta \tag{24}$$

in place of (17).

3. In this section we shall no longer assume (13) but suppose instead that after a convenient cogredient transformation

$$R = r_1 \geq r_2 \geq \cdots \geq r_n = r. \tag{25}$$

For any vector $x = (x_1, \cdots, x_n) \neq 0$ with $x_\nu \geq 0$, $\nu = 1, \cdots, n$ we shall define

$$\psi(x) = \frac{\sum_\nu r_\nu x_\nu}{\sum_\nu x_\nu} \tag{26}$$

and prove:

LEMMA 1. *Let $1 \geq \epsilon > 0$, and let X_ϵ be the set of vectors $(x_1, \cdots, x_n)$ with*

$$1 \geq x_\nu \geq \epsilon, \qquad \nu = 1, \cdots, n. \tag{27}$$

The function ψ is continuous on the closed bounded set X_ϵ and thus attains its least upper bound there, say

$$\psi^* = \max \psi(x), \qquad x \in X_\epsilon . \tag{28}$$

If x belongs to X_ϵ, then $\psi(x) = \psi^*$ if and only if

$$\begin{cases} x_\nu = 1 & \text{if} \quad r_\nu - \psi^* > 0, \\ x_\nu = \epsilon & \text{if} \quad r_\nu - \psi^* < 0. \end{cases} \tag{29}$$

Proof. We note that

$$\frac{\partial \psi}{\partial x_\nu}(x) = \frac{1}{\sum_\nu x_\rho}(r_\nu - \psi(x)), \tag{30}$$

whence by the mean value theorem

$$\begin{aligned} \psi(x + z) &= \psi(x) + \sum_\nu z_\nu \frac{\partial \psi}{\partial z_\nu}(x + \theta z) \\ &= \psi(x) + \frac{1}{\sum_\nu (x_\nu + \theta z_\nu)} \left(\sum_\nu z_\nu (r_\nu - \psi(x + \theta z))\right) \end{aligned} \tag{31}$$

for some θ, $0 < \theta < 1$.
Now suppose that $\psi(x) = \psi^*$ and that $z \neq 0$ and

$$\left\{ z_\nu \begin{matrix} \geq \\ = \\ \leq \end{matrix} 0 \quad \text{according as} \quad r_\nu - \psi^* \begin{matrix} > \\ = \\ < \end{matrix} 0. \right. \tag{32}$$

Let z be any vector whose modulus is so small that for all ν and all θ, with $0 < \theta < 1$, the signs of $r_\nu - \psi(x + \theta z)$ and $r_\nu - \psi^* = r_\nu - \psi(x)$ are the same whenever $r_\nu - \psi^* \neq 0$. For all such z it follows from (31) and (32) that

$$\psi(x + z) > \psi(x) = \psi^*. \tag{33}$$

But this implies that for these z the vector $x + z$ does not belong to X_ϵ, and (29) follows immediately.

Conversely, suppose the vector x satisfies (33) and belongs to X_ϵ. Since

ψ attains its maximum on X_ϵ, there exists a vector x^* in X_ϵ with $\psi(x^*) = \psi^*$. Put $x = x^* + w$. By the first part of the proof x^* also satisfies (29), whence

$$w_\nu = 0 \quad \text{if} \quad r_\nu - \psi^* \neq 0 \tag{34}$$

and so

$$\sum_\nu r_\nu w_\nu = \psi^* \sum_\nu w_\nu . \tag{35}$$

Thus

$$\psi(x) = \frac{\sum_\nu r_\nu x_\nu}{\sum_\nu x_\nu} = \frac{\sum_\nu r_\nu x_\nu^* + \sum_\nu r_\nu w_\nu}{\sum_\nu x_\nu^* + \sum_\nu w_\nu} = \frac{\psi^* \sum_\nu x_\nu^* + \psi^* \sum_\nu w_\nu}{\sum_\nu x_\nu^* + \sum_\nu w_\nu} = \psi^*, \tag{36}$$

and the lemma is proved.

The analogous result for the greatest lower bound is

LEMMA 2. *Let X_ϵ be defined by (27). The function ψ attains its greatest lower bound on X_ϵ, say*

$$\psi_* = \min \psi(x), \qquad x \in X_\epsilon . \tag{37}$$

If x belongs to X_ϵ then $\psi(x) = \psi_*$ if and only if

$$\begin{cases} x_\nu = \epsilon & \text{if} \quad r_\nu - \psi_* > 0, \\ x_\nu = 1 & \text{if} \quad r_\nu - \psi_* < 0. \end{cases} \tag{38}$$

We shall now state and prove

THEOREM 2. *Let ω be the maximal characteristic root of an $n \times n$ non-negative irreducible matrix A. Set*

$$\rho_\nu = (1/\nu)(r_1 + \cdots + r_\nu), \tag{39}$$

$$\sigma_\nu = (1/\nu)(r_{n-\nu+1} + \cdots + r_n), \tag{40}$$

where the r_ν are ordered by (25), and let ϵ be given by (8). Then

$$\frac{q(1-\epsilon)\sigma_q + n\epsilon\rho}{q(1-\epsilon) + n\epsilon} \leq \omega \leq \frac{p(1-\epsilon)\rho_p + n\epsilon\rho}{p(1-\epsilon) + n\epsilon} \tag{41}$$

where p is the smallest integer ν for which

$$(r_1 - r_{\nu+1}) + \cdots + (r_\nu - r_{\nu+1}) \geq \epsilon((r_{\nu+1} - r_{\nu+2}) + \cdots + (r_{\nu+1} - r_n)) \tag{42}$$

and q is the smallest integer ν for which

$$(r_{n-\nu} - r_{n-\nu+1}) + \cdots + (r_{n-\nu} - r_n) \geq \epsilon((r_1 - r_{n-\nu}) + \cdots + (r_{n-\nu-1} - r_{n-\nu})). \tag{43}$$

Proof. We introduce a new notation for certain vectors: Write

$$x = (x_1, \cdots, x_n) = (\alpha^\nu, \beta^{n-\nu}) \tag{44}$$

if

$$x_1 = \cdots = x_\nu = \alpha, \qquad x_{\nu+1} = \cdots = x_n = \beta, \tag{45}$$

and from now on we shall suppose ϵ is given by (8). Let y again be the characteristic row vector belonging to ω and, since ϵ is a lower bound for δ in (14), we may suppose that y is normalized to belong to X_ϵ .

The equation (12) asserts that $\omega = \psi(y)$, whence by (28) and Lemma 1

$$\omega \leq \psi^* = \psi(x), \tag{46}$$

where x is any vector satisfying (33). Since

$$r_1 > \psi(x) > r_n \tag{47}$$

for all x in X_ϵ , there is a unique integer p, $1 \geq p > n$ such that

$$r_p > \psi^* \geq r_{p+1} , \tag{48}$$

and as $(1^p, \epsilon^{n-p})$ satisfies (33), we obtain

$$\begin{aligned} \omega \leq \psi^* &= \psi(1^p, \epsilon^{n-p}) \\ &= \frac{(r_1 + \cdots + r_p) + \epsilon(r_{p+1} + \cdots + r_n)}{p + \epsilon(n - p)} = \frac{p(1 - \epsilon)\rho_p + n\epsilon\rho}{p(1 - \epsilon) + n\epsilon} , \end{aligned} \tag{49}$$

which is one of the inequalities of (41).

Now by (48) p obviously satisfies

$$r_{p+1} - \psi(1^p, \epsilon^{n-p}) = r_{p+1} - \psi^* \leq 0, \tag{50}$$

while for $\nu < p$, in virtue of $r_{\nu+1} \geq r_p$ and (48)

$$r_{\nu+1} - \psi(1^\nu, \epsilon^{n-\nu}) > r_p - \psi^* > 0. \tag{51}$$

Hence by (50) and (51) the integer p is the smallest ν with

$$r_{\nu+1} - \psi(1^\nu, \epsilon^{n-\nu}) \leq 0. \tag{52}$$

But

$$\begin{aligned} r_{\nu+1} - \psi(1^\nu, \epsilon^{n-\nu}) = -\frac{1}{\nu + (n - \nu)\epsilon} &((r_1 - r_{\nu+1}) + \cdots + (r_\nu - r_{\nu+1}) \\ &- \epsilon((r_{\nu+1} - r_{\nu+2}) + \cdots + (r_{\nu+1} - r_n))) \end{aligned} \tag{53}$$

whence it follows that p is the smallest integer satisfying (42).

Using Lemma 2 we may prove similarly that $\psi_* = \psi(\epsilon^{n-q}, 1^q)$, where q is the smallest integer ν satisfying (43); or alternatively this result may be obtained by applying Lemma 1 to $r_1 - r_n , \cdots , r_1 - r_1$ in place of $r_1 , \cdots , r_n$. We have completed the proof of Theorem 2.

It is clear that the bounds of Theorem 2 are better than those of Theorem 1, since for all ν

$$\frac{\nu(1-\epsilon)\rho\nu + n\epsilon\rho}{\nu(1-\epsilon)+n\epsilon} \leq \frac{\nu(1-\epsilon)R + n\epsilon\rho}{\nu(1-\epsilon)+n\epsilon} \leq \frac{(n-1)(1-\epsilon)R + n\epsilon\rho}{(n-1)(1-\epsilon)+n\epsilon}, \tag{54}$$

and these upper bounds are equal if and only if $r_1 = \cdots = r_{n-1} > r_n$.

A similar argument holds for the lower bounds, with equality here if and only if $r_1 > r_2 = \cdots = r_n$.

4. Recently one of us [5] has obtained some other bounds for δ, and thus at the expense of introducing more data depending on the elements of the matrix A our results may be improved. When A is a *positive* matrix, a particularly simple bound for δ is m/M where

$$m = \min_{\nu,\mu} a_{\nu\mu} \quad \text{and} \quad M = \max_{\nu,\mu} a_{\nu\mu}. \tag{55}$$

REFERENCES

1. A. BRAUER, *The theorems of Ledermann and Ostrowski on positive matrices*, this Journal, vol. 24(1957), pp. 265–274.
2. W. LEDERMANN, *Bounds for the greatest latent roots of a positive matrix*, Journal of the London Mathematical Society, vol. 25(1950), pp. 265–278.
3. A. OSTROWSKI, *Bounds for the greatest latent root of a positive matrix*, Journal of the London Mathematical Society, vol. 27(1952), pp. 253–256.
4. A. OSTROWSKI, *Ueber das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen*, Compositio Mathematica, vol. 9(1951), pp. 209–226.
5. A. OSTROWSKI, *On the eigenvector belonging to the maximal root of a non-negative matrix*, Report, Mathematics Research Center, U. S. Army, No. 145.
6. H. SCHNEIDER, *Note on the fundamental theorem on irreducible non-negative matrices*, Proceedings of the Edinburgh Mathematical Society, vol. 11(1958), pp. 127–130.

MATHEMATICS RESEARCH CENTER
UNITED STATES ARMY
UNIVERSITY OF WISCONSIN