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Hyperplanes and Prime Rings

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1. A ring R is prime if $cad = 0$ for all a in R implies that either $c = 0$ or $d = 0$. Recently one of us has proved ([1], Lemma 2) that in a prime ring $cadaf = 0$ for all a in R implies $c = 0$, or $d = 0$, or $f = 0$. It is still an open question under which conditions on a prime ring R the product $c_1 a c_2 \cdots a c_r = 0$ for all a in R implies that some c_i is 0. However, we settle this question for prime rings with minimal ideals. To prove our theorem we require some lemmas on the covering of a vector space by hyperplanes, which may be of some interest in their own right.

For completeness, we begin by giving a somewhat shorter proof of the result quoted above. Let R be prime, and suppose that $cadaf = 0$ for all a in R. Then for all e', u' in R

$$c(e' + u')\,d(e' + u')\,f = ce'du'f + cu'de'f = 0$$

whence, for all a, e, u,

$$c(eca + afu)\,d(eca + afu)\,f = cecadafuf + cafudecaf = cafudecaf$$

whence for all a, e either $caf = 0$ or $decaf = 0$ and so, for all a, e $decaf = 0$. Thus either $d = 0$, or, for all a, $caf = 0$, whence we conclude that $c = 0$ or $d = 0$, or $f = 0$.

2. We now turn our attention to hyperplanes — or linear functionals — of a vector space. A division ring D we define to be an associative ring with identity in which every non-zero element is invertible. A right (left) vector space F over D is a unitary right (left) D-module of finite or infinite dimension. We shall denote by (x, y) the image of y in F under the linear functional x and shall call the set X of non-zero linear functionals on the vector space F over D a *covering* of F if for each y in F there is an x in X for which $(x, y) = 0$. Clearly vector spaces of dimension 0 and 1 have no coverings.

Lemma 1. *Let F be a vector space over the division ring D, and let E be a space of linear functionals on F, and suppose that* $\dim E \geqq 2$. *Then E contains a covering X of F for which*

(i) $\operatorname{card} X = q + 1$, *if* $\operatorname{card} D = q$ *is finite,*

(ii) $\operatorname{card} X = \operatorname{card} D$, *if* $\operatorname{card} D$ *is infinite.*

Proof. Let x_0 and z be linearly independent elements of E. Let

$$X = \{z\} \cup \{x_\lambda = x_0 + \lambda z : \lambda \in D\}.$$

If $y \in F$, either $(z, y) = 0$, or else $(x_\lambda, y) = 0$ where $\lambda = -(x_0, y)(z, y)^{-1}$, whence X is a covering.

Lemma 2. *Le F be a vector space over D, with* $\dim F \geqq 2$, *and let X be a covering of F by linear functionals. Then*

(i) $\operatorname{card} X \geqq q + 1$, *if* $\operatorname{card} D = q$ *is finite,*

(ii) $\operatorname{card} X \geqq \aleph_0$, *if* $\operatorname{card} D$ *is infinite.*

Proof. It is sufficient to prove that if D has at least r elements, where r is finite, then for all families $\{x_1, \ldots, x_r\}$ of non-zero linear functionals on F there exists a y in F such that $(x_i, y) \neq 0$, $i = 1, \ldots, r$; and this we shall do by induction on r. If $r = 1$, the result is plainly true. Suppose therefore that D has at least r elements and that we can find y' and y'' such that

$$\alpha_i = (x_i, y') \neq 0, \quad i = 1, \ldots, r-1,$$
$$\beta_i = (x_i, y'') \neq 0, \quad i = 2, \ldots, r.$$

If $\alpha_r = (x_r, y') \neq 0$, or $\beta_1 = (x_1, y'') \neq 0$ there is no more to prove. Otherwise, since D has at least r elements, we can choose $0 \neq \lambda \in D$, $\lambda \neq -\beta_i^{-1}\alpha_i$, $i = 2, \ldots, r-1$, and verify immediately that $y = y' + y''\lambda$ satisfies $(x_i, y) \neq 0$, $i = 1, \ldots, r$.

We shall prove Lemma 3 because of its intrinsic interest when compared with Lemma 2.

Lemma 3. *Let F be a finite-dimensional vector space over D, with* $\dim F \geqq 2$, *and let X be a covering of F by linear functionals. Then*

(i) $\operatorname{card} X \geqq q + 1$, *if* $\operatorname{card} D = q$ *is finite,*

(ii) $\operatorname{card} X \geqq \operatorname{card} D$, *if* $\operatorname{card} D$ *is infinite.*

Proof. This lemma will be proved by induction on the dimension n of F. If $n = 2$, there is a one-one correspondence between one-dimensional subspaces of F and their one-dimensional annihilators in the dual space E of F. Hence a covering X intersects each one-dimensional subspace of E, and since E has card D, resp. $(q+1)$, one-dimensional subspaces the result is proved. Now suppose the lemma is true for spaces of dimension $n-1$, and let E be n-dimensional. Every covering X contains two linearly independent functionals x_0, z. If for every $x_\lambda = x_0 + \lambda z$ $(\lambda \in D)$ there exist $\lambda' \in D$ such that $\lambda' x_\lambda \in X$, then the set $\{z\} \cup \{\lambda' x_\lambda : \lambda \in D\}$ is contained in X and there is no more to prove. Otherwise, suppose $\lambda' x_\lambda$ does not belong to X for all λ' in D and let F' be the $(n-1)$-dimensional subspace of F annihilated by x_λ. Then the restrictions of the functionals of X to F' are non-zero and hence form a covering of F'. The result now follows from the inductive assumption.

In general, Lemma 2 cannot be improved for infinite dimensional spaces F. For write $F = F' \oplus F''$, where F' is of countable dimension, and let (y_i), $i = 1, 2, \ldots$, be a basis for F'. Define the linear functionals x_i, $i = 1, 2, \ldots$, by $(x_i, y_j) = \delta_{ij}$, the Kronecker delta, and $(x_i, F'') = 0$. Since each element of F is a linear combination of an element of F'' and a finite number of the y_i, the countable set $\{x_i\}$, $i = 1, 2, \ldots$, covers F. However, if the linear functionals are restricted to lie in a fixed dual space of F, then more than a countable number of linear functionals may be required to cover F. For example, let F be a Hilbert space and identify F with one of its duals in the canonical way. The hyperplanes orthogonal to elements of F are closed and nowhere dense in F in the metric topology. It now follows from the Baire Category Theorem ([2], p. 200), applied to the set-theoretic complements of the hyperplanes, that the union of a countable set of hyperplanes is properly contained in F.

In geometrical language, our lemmas refer to the coverings of a vector space by hyperplanes through the origin, but it is easy to see that very similar results may be proved for the coverings by hyperplanes that need not pass through this point. In fact, in the finite part (i) of our three lemmas, we need only replace $(q + 1)$ by q, while the infinite part (ii) goes through without change.

3. We now return to the study of prime rings with minimal left ideals. It is proved in JACOBSON ([3], p. 195 prop. 2), that a ring R is prime if and only if every product of two non-zero ideals in R is non-zero. JACOBSON's proof also establishes the following condition, which we wish to use: A ring R is prime if and only if every product of two non-zero *left* ideals in R is non-zero.

A ring R is called (left) primitive if there exists a faithful irreducible left R-module M. In order to make use of the structure theory of primitive rings we shall show that a ring with minimal left ideal is prime if and only if it is primitive. It is known ([3], p.195, prop. 1) that a primitive ring is prime. For proof, let M be a faithful irreducible left R-module, and let J, K be non-zero left ideals in R. Since M is faithful, $KM \neq 0$, whence $KM = M$ since M is irreducible. Similarly $JM = J(KM) = M$, $(JK)M = M$, whence $JK \neq 0$, and R is prime. It may not be so well known that a prime ring with minimal left ideal is primitive. For let M be a minimal left ideal and therefore (since $RM \neq 0$ by primeness) an irreducible R-module. Since the left annihilator of M is a left ideal I for which $IM = 0$, we conclude, again by primeness, that $I = 0$ and so R is faithful on M. Thus the prime rings with minimal left ideal are precisely the primitive rings with minimal left ideal, or equivalently ([3], Chapter IV, 7—9) the subrings of the ring of linear transformations on a right vector space F over a division ring D which consist solely of transformations continuous with respect to a dual space E and contain all continuous transformations of finite rank. (A transformation on F is continuous with respect to E if it possesses an adjoint on E.) In the proof of our theorem we may therefore assume that R is such a ring of linear transformations and thereby we link our lemmas on coverings of vector spaces to the properties of prime rings.

Theorem. *Let R be a prime ring with minimal left ideal and let D be the divison ring of R-endomorphisms of a faithful irreducible left R-module F. If D contains at least r elements, where r is a positive integer, then for each family $\{P_1, P_2, \ldots, P_{r+1}\}$ of non-zero elements of R, there exists an A in R with $P_1AP_2A \cdots P_rAP_{r+1} \neq 0$. Conversely, if D contains fewer than r elements and R is not a division ring, there exists a family $\{P_1, P_2, \ldots, P_{r+1}\}$ of non-zero elements of R such that $P_1AP_2A \cdots P_rAP_{r+1} = 0$ for all A in R.*

Proof. Let E be such that E, F is a pair of dual vector spaces over the division ring D. Let R contain all continuous transformations of finite rank and consist solely of transformations on F, continuous with respect to the given pairing. Given the family $\{P_1, P_2, \ldots, P_{r+1}\}$ of non-zero elements of R, we will construct an A of rank 1 such that the product $P_1 A P_2 \cdots A P_{r+1} \neq 0$. Since each $P_i \neq 0$, there is a y_i in F for which $P_i y_i \neq 0$. By Lemma 2, applied to coverings of E, the r elements $P_i y_i$, $i = 2, \ldots, r+1$, do not cover E; hence there exists an x_0 in E such that

$$(x_0 P_i^*, y_i) = (x_0, P_i y_i) \neq 0, \quad i = 2, \ldots, r+1.$$

In particular, note that for $i = 2, \ldots, r$, $x_0 P_i^* \neq 0$, and set

$$\alpha_{r+1} = (x_0, P_{r+1} y_{r+1}) \neq 0.$$

Next choose x_1 so that $x_1 P_1^* \neq 0$; by applying Lemma 2 to F this time, choose y_0 in F so that

$$\alpha_1 = (x_1, P_1 y_0) = (x_1 P_1^*, y_0) \neq 0,$$
$$\alpha_1 = (x_0, P_i y_0) = (x_0 P_1^*, y_0) \neq 0, \quad i = 2, \ldots, r.$$

Now define the transformation A on F by

$$Ay = y_0 (x_0, y).$$

Clearly A belongs to R since A is of finite rank and its adjoint A^* is given by $xA^* = (x, y_0)\, x$. Observe that $A P_{r+1} y_{r+1} = y_0 \alpha_{r+1}$, $A P_i y_0 = y_0 \alpha_i$, $i = 2, \ldots, r$, and $(x_1, P_1 y_0) = \alpha_1$. It follows that

$$(x_1, P_1 A P_2 \cdots A P_{r+1} y_{r+1}) = \alpha_1 \alpha_2 \cdots \alpha_{r+1} \neq 0$$

whence, for this A, $P_1 A P_2 \cdots A P_{r+1} \neq 0$.

To prove the converse, we first note that if F has dimension one over D, then E is also one-dimensional and R is a division ring. So suppose that D has a finite number q of elements and R is not a division ring. We need only consider the case $r = q + 1$. By Lemma 1, there exist non-zero $x_1, \ldots, x_{q+1}$ in E which cover F, since F and E have dimension at least 2 over D. Let y_0 be a non-zero element of F, and define the linear transformations P_i by

$$P_i y = y_0 (x_i, y), \quad i = 1, \ldots, q+1,$$
$$P_{q+2} y = y_0 (x_0, y),$$

where x_0 is a non-zero element of E. If A is an arbitrary continuous linear transformation on F, and if we write $(x_i, A y_0) = \beta_i$, $i = 1, \ldots, q+1$, then for all y in E

$$(x_0, P_1 A P_0 \cdots A P_{q+2} y) = \beta_1 \beta_2 \cdots \beta_{q+1} (x_0, y) = 0$$

since some $\beta_i = 0$ by the covering property of $x_1, \ldots, x_{q+1}$. Hence

$$P_1 A P_2 \cdots A P_{q+2} = 0,$$

for all A in R. This proves the theorem.

By inspecting the covering family of Lemma 1, we note that the transformations P_i in the second part of the theorem may be taken as having a common one-dimensional range and a null-space of codimension at most two. Hence, the transformations of our counterexample are direct sums of zero transformations and transformations which can be realized by the family of matrices of order 2:

$$P_i = \begin{bmatrix} 1 & \lambda_i \\ 0 & 0 \end{bmatrix}, \quad i = 1, \dots, q; \; P_{q+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; P_{q+2} = \begin{bmatrix} \mu_1 & \mu_2 \\ 0 & 0 \end{bmatrix},$$

where $\lambda_1 = 0$, $\lambda_2, \dots, \lambda_q$ are the q elements of D, and μ_1, μ_2 are any two elements of D not both zero.

References

[1] E. C. Posner, Derivations in prime rings. Proc. Amer. Math. Soc. 8, 1093–1100 (1957).
[2] J. L. Kelley, General Topology. New York 1955.
[3] N. Jacobson, Structure of Rings. Amer. Math. Soc. Coll. Publ., Vol. 37, 1956.

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