A Matrix Problem Concerning Projections

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The following problem is a slight generalisation of one posed and partly solved by H. Nagler.\(^1\) We shall use \(A^*\) for the conjugate transpose of a matrix \(A\). A projection is an idempotent matrix, its latent roots consist of units and zeros.

**Problem.** Let \(A\) be an \(n \times m\) matrix with complex elements. Find an \(m \times n\) matrix \(B\) such that \((I - AB)^* (I - AB)\) is a projection of rank \(k\).

Let the rank of \(A\) be \(r\). The rank of \(AB\) is less than or equal to \(r\). It is easily proved that the rank of \(M = I - AB\) cannot be less than \(n - r\). But the rank of \(M^*M\) equals that of \(M\), and hence

\[
k \geq n - r \quad (1)
\]

is a necessary condition for the existence of a matrix \(B\) with the required property.

We shall next show that \((1)\) is also a sufficient condition. It is enough to find a matrix \(B\) such that \(I - AB\) is a Hermitian projection of rank \(k\).

The \(n \times n\) matrix \(AA^*\) is Hermitian of rank \(r\). Hence there exist \(n\) Hermitian projections of rank 1 satisfying

\[
\sum_{i=1}^{n} E_i = I, \quad (2)
\]

\[
E_iE_j = 0 \text{ when } i \neq j, \quad (3)
\]

\[
\sum_{i=1}^{n} \rho_i E_i = AA^*, \quad (4)
\]

where the \(\rho_i\) are the (non-negative) latent roots of \(AA^*\), supposed arranged so that \(\rho_i > 0\) for \(i = 1, \ldots, r\), and \(\rho_i = 0\) for \(i = r + 1, \ldots, n\).

Let

\[
C = \sum_{i=1}^{n-k} \sigma_i E_i,
\]

where

\[
\sigma_i = 1/\rho_i \text{ for } i = 1, \ldots, n - k \leq r.
\]

We note that

\[
AA^*C = \sum_{i=1}^{n-k} E_i.
\]

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whence it follows that \[ I - AA^*C = \sum_{n-k+1}^{n} E_i \]

is a Hermitian projection of rank \( k \). Thus \( B = A^*C \) is a matrix having the desired property.

Let \( x_1, \ldots, x_n \) form an orthonormal set of latent column vectors of \( AA^* \), where \( x_i \) is associated with the latent root \( \rho_i \). It may be remarked that a set of Hermitian projections of rank 1 satisfying (2), (3) and (4) is given by \( E_i = x_i x_i^* \) for \( i = 1, \ldots, n \).

Suppose now that \( k = n - r \). When \( m \leq n \) and the rank of \( A \) is \( m \), then we assert that our method yields \( B = (A^*A)^{-1}A^* \), while if \( n \leq m \) and the rank of \( A \) is \( n \), then \( B = A^* (AA^*)^{-1} \). The proof of this is left to the reader. In general, it is clear from the method of construction that the solution we have found is not unique.

An \( m \times n \) matrix \( D \) for which \( I - DA \) is a Hermitian projection of rank \( k \) can be found in a similar manner, provided that \( k \geq m - r \).

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