A Matrix Problem Concerning Projections

By Hans Schneider

(*Received* 18*th February* 1953).

The following problem is a slight generalisation of one posed and partly solved by H. Nagler.[1] We shall use A^* for the conjugate transpose of a matrix A. A projection is an idempotent matrix, its latent roots consist of units and zeros.

Problem. Let A be an $n \times m$ matrix with complex elements. Find an $m \times n$ matrix B such that $(I - AB)^* (I - AB)$ is a projection of rank k.

Let the rank of A be r. The rank of AB is less than or equal to r. It is easily proved that the rank of $M = I - AB$ cannot be less than $n - r$. But the rank of M^*M equals that of M, and hence

$$k \geqslant n - r \tag{1}$$

is a necessary condition for the existence of a matrix B with the required property.

We shall next show that (1) is also a sufficient condition. It is enough to find a matrix B such that $I - AB$ is a Hermitian projection of rank k.

The $n \times n$ matrix AA^* is Hermitian of rank r. Hence there exist n Hermitian projections of rank 1 satisfying

$$\sum_1^n E_i = I, \tag{2}$$

$$E_i E_j = 0 \text{ when } i \neq j, \tag{3}$$

$$\sum_1^n \rho_i E_i = AA^*, \tag{4}$$

where the ρ_i are the (non-negative) latent roots of AA^*, supposed arranged so that $\rho_i \neq 0$ for $i = 1, \ldots, r$, and $\rho_i = 0$ for $i = r + 1, \ldots, n$.

Let $$C = \sum_1^{n-k} \sigma_i E_i,$$

where $$\sigma_i = 1/\rho_i \text{ for } i = 1, \ldots, n - k \leqslant r.$$

We note that $$AA^*C = \sum_1^{n-k} E_i,$$

---

[1] H. Nagler, "On a certain matrix product with specified latent roots," *Proc. Edinburgh Math. Soc.* (2), 10 (1953), 21-24

whence it follows that
$$I - AA^*C = \sum_{n-k+1}^{n} E_i$$
is a Hermitian projection of rank k. Thus $B = A^*C$ is a matrix having the desired property.

Let $x_1, \ldots, x_n$ form an orthonormal set of latent column vectors of AA^*, where x_i is associated with the latent root ρ_i. It may be remarked that a set of Hermitian projections of rank 1 satisfying (2), (3) and (4) is given by $E_i = x_i x_i^*$ for $i = 1, \ldots, n$.

Suppose now that $k = n - r$. When $m \leqslant n$ and the rank of A is m, then we assert that our method yields $B = (A^*A)^{-1}A^*$, while if $n \leqslant m$ and the rank of A is n, then $B = A^* (AA^*)^{-1}$. The proof of this is left to the reader. In general, it is clear from the method of construction that the solution we have found is not unique.

An $m \times n$ matrix D for which $I - DA$ is a Hermitian projection of rank k can be found in a similar manner, provided that $k \geqq m - r$

THE QUEEN'S UNIVERSITY,
BELFAST.