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PAIRS OF MATRICES WITH A NON-ZERO COMMUTATOR

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1. This note takes its origin in a remark by Brauer (1) and Perfect (5): Let A be a square complex matrix of order n whose characteristic roots are $\alpha_1, \ldots, \alpha_n$. If $\mathbf{x}_1$ is a characteristic column vector with associated root α_1 and $\mathbf{k}$ is any row vector, then the characteristic roots of $A + \mathbf{x}_1\mathbf{k}$ are $\alpha_1 + \mathbf{k}\mathbf{x}_1, \alpha_2, \ldots, \alpha_n$. Recently, Goddard (2) extended this result as follows: If $\mathbf{x}_1, \ldots, \mathbf{x}_r$ are linearly independent characteristic column vectors associated with the characteristic roots $\alpha_1, \ldots, \alpha_r$ of the matrix A, whose elements lie in any algebraically closed field, then any characteristic root of $\Lambda + KX$ is also a characteristic root of $A + XK$, where K is an arbitrary $r \times n$ matrix, $X = (\mathbf{x}_1, \ldots, \mathbf{x}_r)$ and $\Lambda = \operatorname{diag}(\alpha_1, \ldots, \alpha_r)$. We shall prove some theorems of which these and other well-known results are special cases.

2. Let F be an arbitrary field. By $f(x, y)$ we shall denote a polynomial, with coefficients in F, in two non-commutative indeterminates x and y, and we shall set $f_0(x) = f(x, 0)$. By F_n we shall mean the ring of square matrices of order n with elements in F. In order to avoid exceptional cases, we adopt the convention that the characteristic polynomial of the empty matrix (the 0×0 matrix) is unity.

Let A and B be matrices in F_n and F_m respectively. By a *commutator from A to B* we shall mean an $n \times m$ matrix X, with elements in F, satisfying $AX = XB$. If X is a commutator from A to B then X also satisfies $f(A, XK)X = Xf(B, KX)$, where K is any $m \times n$ matrix with elements in F. We shall prove the following theorem.

THEOREM 1. *Let A and B be matrices in F_n and F_m respectively and let X be a commutator from A to B of rank r. Then, for any polynomial $f(x, y)$ and any $m \times n$ matrix K, with elements in F,*

$$|\lambda I_n - f(A, XK)| = \theta(\lambda)p(\lambda),$$

$$|\lambda I_m - f(B, KX)| = \theta(\lambda)q(\lambda),$$

where (i) *$\theta(\lambda), p(\lambda), q(\lambda)$ are polynomials in $F[\lambda]$ of degrees $r, n-r, m-r$, respectively,* (ii) *$p(\lambda)$, $q(\lambda)$, are independent of K and are therefore factors of the characteristic polynomials of $f_0(A)$ and $f_0(B)$ respectively.*

Proof. If $r = 0$ the result is trivial. If $r = \min(n, m)$ then either $p(\lambda)$ or $q(\lambda)$ is unity, and it is easily seen that the proof given below for the case $0 < r < \min(n, m)$ is still valid, provided that some of the submatrices of the partitioned matrices which occur are omitted.

Let P and Q be non-singular matrices in F_n and F_m respectively for which

$$P\overset{X}{\cancel{X}}Q^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = Y, \text{ say.}$$

Property P_n is then property P (cf. McCoy (3)). Theorem 1 implies that the pairs of matrices A, XK and B, KX have properties P_{n-r} and P_{m-r} respectively. In addition we have this theorem.

THEOREM 3. *If, under the conditions of Theorem* 1, *the pair of matrices* A, XK *has property* P (*or property* L, *cf. Motzkin and Taussky* (4)), *then so has the pair* B, KX, *and conversely.*

Proof. We need only remark that with the hypothesis of Theorem 3 we have $\theta(\lambda) = \prod_{i=1}^{r} (\lambda - f(\epsilon_i, \gamma_i))$ for any polynomial $f(x, y)$ (or, in the case of property L, for $f(x, y) = \mu x + \nu y$), where $\lambda^{n-r} \prod_{i=1}^{r} (\lambda - \gamma_i)$ is the characteristic polynomial of XK.

4. In this section we shall show how various known results are obtained from our theorems. Let $f(x, y) = x$. Then Theorem 1 reduces to the result that $AX = XB$ implies that the characteristic polynomials of A and B have a common factor of degree r, where r is the rank of X. Next, set $f(x, y) = y$. Then $f_0(A)$ and $f_0(B)$ are zero, so that our theorem furnishes yet another proof that the characteristic polynomials of XK and KX differ by the factor λ^{n-m}. It is easy to see that the same is true for any polynomial $f(x, y)$, each of whose terms contain y. Finally, let $r = m$ and let X consist of characteristic column vectors of A. Then B is diagonal, in Theorem 2, $(\beta_{r+1}, \ldots, \beta_m)$ is empty, and the result of Goddard (2) quoted in § 1 follows in a slightly stronger form on putting $f(x, y) = x + y$ in Theorem 2. If $m = n$ and $B = 0$, so that $AX = 0$, and if $f(x, y)$ is a polynomial without constant term, then the corollary to Theorem 1 becomes, with $K = I_n$,

$$|\lambda I_n - f(A, X)|\, \lambda^n = |\lambda I_n - f(A, 0)|\; |\lambda I_n - f(0, X)|,$$

which is a result proved elsewhere by Schneider (6).

Note added in proof. We have noticed that Theorem 1 of a paper just published by Miss Hazel Perfect (*Duke math. J.* 22 (1955), 305–11) is a special case of the corollary to Theorem 1 above, namely, where $\lambda = 0$, $f(x, y) = x - y$ and X is of full rank.

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