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A PAIR OF MATRICES WITH PROPERTY P

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A set $A_1, \cdots, A_s$ of $n\times n$ matrices with coefficients in an algebraically closed field is said to have property P if there exists an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}$, $i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$ for which the characteristic roots of any polynomial $p(A_1, \cdots, A_s)$ are $p(\alpha_i^{(1)}, \cdots, \alpha_i^{(s)})$, $i=1, \cdots, n$. In 1936 McCoy [3] proved that the set $A_1, \cdots, A_s$ has property P if and only if every matrix of the form $(A_iA_j-A_jA_i)R$, where R is a polynomial in the A_i, is nilpotent (for an elementary proof see Drazin, Dungey and Gruenberg [2]). More recently two very special cases of this theorem have been proved separately. Thus in 1950 Parker [5] showed that if $AB=B^2=0$, then the characteristic roots of $A+B$ are the same as those of A. (The characteristic roots of B are all zero.) In 1953 Perfect [6], completing a theorem of A. Brauer [1], showed that if $(C-\lambda I)v=0$, and B is a matrix of rank 1 all of whose columns are multiples of the column vector v, then the characteristic roots of $C+B$ are obtained from those of C by replacing one λ by $\lambda+$trace B. (One characteristic root of B is trace B, the rest are zero, and $AB=0$ if $A=(C-\lambda I)$.) We shall give a very simple proof of a special case of McCoy's theorem which includes the two results quoted above.

In order to make our theorem applicable to $n\times n$ matrices with coefficients in a field which is not necessarily algebraically closed we shall state the result in terms of the characteristic polynomial $|xI-A|$ of a square matrix A.

LEMMA. *Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k. If $(\sum_{i=1}^{j} A_i)A_{j+1}=0$, for $j=1, \cdots, s-1$, then the characteristic polynomial of $\sum_{j=1}^{s} A_j$ is $(\prod_{j=1}^{s} |xI-A_j|)/x^{(s-1)n}$.*

Proof: Since $A_1A_2=0$,

$$\begin{aligned} |xI-A_1|\,|xI-A_2| &= |x^2-x(A_1+A_2)+A_1A_2| \\ &= |x(xI-(A_1+A_2))| = x^n|xI-(A_1+A_2)|. \end{aligned}$$

The result follows if $s=2$. The lemma is now obtained by induction on s.

THEOREM. *Let A and B be $n\times n$ matrices with coefficients in k. If $AB=0$, then the characteristic polynomial of the polynomial $p(A, B)$ without constant term is*

$$|xI-p(A, 0)|\cdot|xI-p(0, B)|/x^n.$$

Proof. Since $AB=0$, we have $p(A, B)=A_1+A_2+A_3$, where $A_1=p(A, 0)$,

$A_3=p(0, B)$ and A_2 is of form $Bq(A, B)A$. Clearly $A_2^2=0$, and so the characteristic polynomial of A_2 is x^n. As $A_1A_2=(A_1+A_2)A_3=0$, it follows by the lemma that

$$\begin{aligned} |xI - p(A, B)| &= |xI - A_1|\,|xI - A_2|\,|xI - A_3|/x^{2n} \\ &= |xI - p(A, 0)|\,|xI - p(0, B)|/x^n. \end{aligned}$$

COROLLARY. *If k is algebraically closed, then the pair of matrices A, B has property P.*

Proof. If $\alpha_i, \beta_i, i=1, \cdots, n$, are the characteristic roots of A and B respectively then $x^n|xI-(A+B)| = |xI-A|\,|xI-B| = \prod_{i=1}^n (x-\alpha_i)(x-\beta_i)$. We may easily show from this that there is an ordering $\alpha_i, \beta_i, i=1, \cdots, n$, for which $\alpha_i\beta_i=0, i=1, \cdots, n$. It is enough to prove that with this ordering the characteristic roots of $p(A, B)$ are $p(\alpha_i, \beta_i), i=1, \cdots, n$, for any polynomial $p(A, B)$ without constant term.

For such a polynomial, $p(\alpha_i, \beta_i)=p(\alpha_i, 0)+p(0, \beta_i)$ and $p(\alpha_i, 0)p(0, \beta_i)=0$, $i=1, \cdots, n$.

By the theorem,

$$\begin{aligned} x^n|xI - p(A, B)| &= |xI - p(A, 0)|\,|xI - p(0, B)| \\ &= \prod_{i=1}^n (x - p(\alpha_i, 0))(x - p(0, \beta_i)) \\ &= x^n \prod_{i=1}^n (x - p(\alpha_i, \beta_i)). \end{aligned}$$

The corollary is proved.

We remark that if $AB=0$ then $((AB-BA)R)^2=0$, where R is any polynomial in A and B. Thus we have indeed proved a special case of McCoy's theorem. A slight extension of the arguments used in the theorem and corollary leads to the following results: Let $A_1, \cdots, A_s$ be a set of $n\times n$ matrices with coefficients in k such that $A_iA_j=0$ if $i<j$. If $p(A_1, \cdots, A_s)$ is a polynomial without constant term, then the characteristic polynomial of $p(A_1, \cdots, A_s)$ is

$$\left(\prod_{j=1}^{s} |xI - p_j(A_j)|\right)/x^{(s-1)n},$$

where $p_1(A_1)=p(A_1, 0, \cdots, 0)$, *etc.* If in addition, k is algebraically closed, then the set $A_1, \cdots, A_s$ has property P. It may also be shown that if $A_1, \cdots, A_s$ satisfy the conditions of the lemma and k is algebraically closed, then the characteristic roots of $\sum_{j=1}^s A_j$ are $\sum_{j=1}^s \alpha_i^{(j)}$, $i=1, \cdots, n$, for an ordering $\alpha_i^{(1)}, \cdots, \alpha_i^{(s)}, i=1, \cdots, n$, of the characteristic roots of $A_1, \cdots, A_s$. This is a property weaker than property L (*cf.* Motzkin and Taussky [4]).

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