REGIONS OF EXCLUSION FOR THE LATENT ROOTS OF A MATRIX

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The well-known Theorem 0 is due to S. Gersgorin [2] and A. Brauer [1]:

THEOREM 0. *Let* A: $[a_{ij}]$ *be an* $n \times n$ *matrix with complex elements. Then the latent roots of* A *lie in the union of the* n *circular regions* $|z-a_{ii}| \leqq P_i, i=1, \cdots, n$, *in the complex plane, where* $P_i = \sum_j' |a_{ij}|$ *and* $\sum_j'$ *denotes summation from* $j=1$ *to* $j=n$ *with* $j=i$ *omitted.*

It is the purpose of this note to point out that there may also exist bounded regions which exclude the latent roots of A.

Let B: $[b_{ij}]$ be an $n \times n$ matrix with complex elements. Let $(\mu(1), \cdots, \mu(n))$ be a permutation of $(1, \cdots, n)$, and let B': $[b'_{ij}]$ be the matrix $[b'_{ij}] = [b_{i\mu(j)}]$. Thus B' is obtained from B by a permutation of its columns, and therefore B is nonsingular when B' is nonsingular.

The matrix B' is nonsingular if

$$|b'_{ii}| > \sum_j{}' |b'_{ij}|, \qquad \text{for } i = 1, \cdots, n,$$

(cf. e.g. O. Taussky [5]), and so B is nonsingular when

$$(1) \qquad |b_{i\mu(i)}| > \sum_j{}' |b_{i\mu(j)}| = \sum_{j \neq \mu(i)} |b_{ij}|, \qquad \text{for } i = 1, \cdots, n.$$

Now let $B = \lambda I - A$, where λ is a latent root of A. Then $b_{ii} = \lambda - a_{ii}$, and $b_{ij} = -a_{ij}$, when $i \neq j$. Since B is singular not all the inequalities (1) can hold. Hence there is an i, $1 \leqq i \leqq n$, such that either

$$|\lambda - a_{ii}| \leqq P_i \quad \text{and} \quad i = \mu(i),$$

or

$$|a_{i\mu(i)}| \leqq |\lambda - a_{ii}| + \sum_j{}'' |a_{ij}| \quad \text{and} \quad i \neq \mu(i),$$

where $\sum_j''$ denotes summation from $j=1$ to $j=n$, with $j=i$ and $j=\mu(i)$ omitted. We now immediately obtain Theorem 1.

THEOREM 1. *Let* A *be an* $n \times n$ *matrix with complex elements. Let* $(\mu(1), \cdots, \mu(n))$ *be a permutation of* $(1, \cdots, n)$. *Then the latent roots of* A *lie in the union of the* n *regions*

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$$|z - a_{ii}| \leqq P_i \quad \textit{when} \quad i = \mu(i),$$
$$|z - a_{ii}| \geqq Q_i \quad \textit{when} \quad i \neq \mu(i),$$

where $Q_i = |a_{i\mu(i)}| - \sum_j'' |a_{ij}|$.

If $\mu(i) = i$ when $i = 1, \cdots, n$ then Theorem 1 reduces to Theorem 0. In this case we obtain a bounded region in the complex plane within which all latent roots of A lie. If some $i \neq \mu(i)$, Theorem 1 may yield a bounded region of the complex plane within which no latent root of A can lie. Let I be the union of the interiors and boundaries of the circles $|z - a_{ii}| = P_i$ when $i = \mu(i)$. Let CI be the complement of I. Let E be the intersection of the interiors of the circles $|z - a_{ii}| = Q_i$ when $i \neq \mu(i)$, such an interior being empty when $Q_i \leqq 0$. It is easily seen that the region of exclusion for the latent roots of A given by Theorem 1 is the intersection of CI and E. If E is empty, or if E is contained in I, there will be no region of exclusion. In particular, there is no region of exclusion if $Q_i \leqq 0$ for some $i \neq \mu(i)$.

We shall apply Theorem 1 to the matrix

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 3 & -2 \end{bmatrix}.$$

(In the case of nondiagonal elements only the absolute values are relevant.) From the permutation (1, 2, 3, 4) it follows by means of Theorem 1 (or Theorem 0) that the latent roots of A lie within or on the circles $|z-1| = 5$ or $|z+2| = 4$. From the permutation (2, 1, 3, 4) it follows that no latent root of A lies in that part of the interior of the circle $|z-1| = 3$ which is outside $|z+2| = 4$. From the permutation (2, 1, 4, 3) it follows that there is no latent root of A within $|z+1| = 1$. No region of exclusion is obtained from (1, 2, 4, 3) even though Q_3 and Q_4 are positive.

If $n = 2$ the only possible permutations of (1, 2) are (1, 2) and (2, 1). Let λ be a latent root of the 2×2 matrix A. Applying Theorem 1 with the permutation (1, 2) we obtain that λ lies within or on one of the circles

$$(2) \qquad |z - a_{11}| = |a_{12}|, \qquad |z - a_{22}| = |a_{21}|.$$

When Theorem 1 is used with the permutation (2, 1) it follows that λ lies outside or on one of the circles (2). Thus λ lies inside or on one of the circles (2), but not inside both. This is a slightly weakened form of a theorem due to O. Taussky [6].

Some other known results may be generalized in the same way as Theorem 0. P. Stein has proved that if B' is singular and its latent root 0 has m associated linearly independent latent column vectors, then at least m of the inequalities

$$|b'_{ii}| \leqq \sum_j{}' |b'_{ij}|, \qquad i = 1, \cdots, n,$$

are satisfied (P. Stein [4], O. Taussky [7]).

The latent column vectors of B' associated with 0 are just those of B after the permutation $(\mu(1), \cdots, \mu(n))$ has been applied to the indices of the elements of the latter. Hence B has precisely as many linearly independent latent column vectors associated with 0 as B'. If we now apply the argument leading to Theorem 1 to Stein's result we obtain Theorem 2.

THEOREM 2. *Let λ be a latent root of A which has m linearly independent latent column vectors associated with it. Let $(\mu(1), \cdots, \mu(n))$ be a permutation of $(1, \cdots, n)$. Then λ lies in at least m of the regions*

$$|z - a_{ii}| \leqq P_i, \quad \textit{when} \quad i = \mu(i),$$
$$|z - a_{ii}| \geqq Q_i, \quad \textit{when} \quad i \neq \mu(i).$$

Results which apply to irreducible matrices only do not seem to be extendable in this simple manner. The reason is that B' may be reducible when A and $B = \lambda I - A$ are irreducible. It is possible to apply the above method to more general bounds for the latent roots of a matrix, such as those of Ostrowski [3], but the regions of exclusion obtained can apparently not be expressed in a simple form.

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