

CSR EXPANSIONS OF MATRIX POWERS IN MAX ALGEBRA *

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Abstract. We study the behavior of max-algebraic powers of a reducible nonnegative matrix $A \in \mathbb{R}_+^{n \times n}$. We show that for $t \geq 3n^2$, the powers A^t can be expanded in max-algebraic sums of terms of the form CS^tR , where C and R are extracted from columns and rows of certain Kleene stars, and S is diagonally similar to a Boolean matrix. We study the properties of individual terms and show that all terms, for a given $t \geq 3n^2$, can be found in $O(n^4 \log n)$ operations. We show that the powers have a well-defined ultimate behavior, where certain terms are totally or partially suppressed, thus leading to ultimate CS^tR terms and the corresponding ultimate expansion. We apply this expansion to the question whether $\{A^t y, t \geq 0\}$ is ultimately linear periodic for each starting vector y , showing that this question can be also answered in $O(n^4 \log n)$ time. We give examples illustrating our main results.

Key words. Max algebra, matrix factorization, cyclicity, Boolean matrices, diagonal similarity, Kleene star

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1. Introduction. By *max algebra* we understand the analogue of linear algebra developed over the max-times semiring $\mathbb{R}_{\max, \times}$ which is the set of nonnegative numbers \mathbb{R}_+ equipped with the operations of “addition” $a \oplus b := \max(a, b)$ and the ordinary multiplication $a \otimes b := a \times b$. Zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to the nonnegative matrices and vectors in the same way as in conventional linear algebra. That is if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from \mathbb{R}_+ , we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} b_{kj} = \max_k(a_{ik} b_{kj})$ for all i, j .

If A is a square matrix over \mathbb{R}_+ then the iterated product $A \otimes A \otimes \dots \otimes A$ in which the symbol A appears k times will be denoted by A^k . These are the *max-algebraic powers* of nonnegative matrices, the main object of our study.

The *max-plus semiring* $\mathbb{R}_{\max, +} = (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$, developed over the set of real numbers \mathbb{R} with adjoined element $-\infty$ and the ordinary addition playing the role of multiplication, is another isomorphic “realization” of max algebra. In particular, $x \mapsto \exp(x)$ yields an isomorphism between $\mathbb{R}_{\max, +}$ and $\mathbb{R}_{\max, \times}$. In the max-plus setting, the zero element is $-\infty$ and the unity is 0.

The main results of this paper are formulated in the max-times setting, since some important connections with nonnegative and Boolean matrices are more transparent there. However, the max-plus setting is left for the examples in the last section, in order to appeal to the readers who work with max-plus algebra and applications in scheduling problems and discrete event systems [1, 7, 16].

The Cyclicity Theorem is a classical result of max algebra, in the max-times setting it means that the max-algebraic powers of any irreducible nonnegative matrix are ultimately periodic (up to a scalar multiple), with the period equal to cyclicity of the critical graph. This theorem can be found in Heidergott et al. [16, Theorem 3.9], see also Baccelli et al. [1, Theorem 3.109] and Cuninghame-Green [7, Theorem 27-6]

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(all stated in the max-plus setting). However, the length of the pre-periodic part can be arbitrarily large and the result does not have an evident extension to reducible matrices.

The behavior of max-algebraic powers of reducible matrices relies on connections between their strongly connected components and the hierarchy of their max-algebraic eigenvalues. This has been well described in a monograph of Gavalec [15], see also [8, 14, 19].

The relation of the Cyclicity Theorem to the periodicity of Boolean matrices is understood but not commonly and explicitly used in max algebra. For instance the construction of cyclic classes [2, 3] appears in the proof of Lemma 3.3 in [16] (without references to literature on Boolean matrices). This relation becomes particularly apparent after application of a certain $D^{-1}AD$ similarity scaling called visualization in [24, 25], see also [9, 10]. Semancíková [21, 22] realized that the cyclic classes of Boolean matrices are helpful in treating computational complexity of the periodicity problems in max-min algebra, with analogous max-algebraic applications in mind [23].

A result by Nachtigall [20] states that, though the length of the preperiodic part cannot be polynomially bounded, the behavior of matrix powers after $O(n^2)$ can be represented as max-algebraic sums of matrices from certain periodic sequences. Molnárová [18] studies this *Nachtigall expansion* further, showing that for a given matrix power after $O(n^2)$ the representing Nachtigall matrices can be computed in $O(n^5)$ time.

In the general reducible case, the sequences of entries $\{a_{ij}^t, t \geq 0\}$ of A^t are ultimately generalized periodic [8, 19], meaning that for $t \geq T$ where T is sufficiently large, they may consist of several ultimately periodic subsequences which grow with different rates. Gavalec [14, 15] showed that deciding whether $\{a_{ij}^t, t \geq 0\}$ is ultimately periodic is in general NP-hard. In a related study, Butkovič et al. [5] considered *robust matrices*, such that for any given x the sequence $\{A^t x\}$ is ultimately periodic with period 1. The conditions formulated in [5] can be verified in polynomial time, which suggests that such “global” periodicity questions must be tractable.

The main goal of this paper is to find a common ground for the above pieces of knowledge on matrix periodicity in max algebra. We introduce the concept of *CSR expansion*, in which a matrix power A^t is represented as the max-algebraic sum of terms of the form CS^tR , called *CSR products*. Here C and R have been extracted from columns and rows of certain Kleene star (the same for both), and S can be made Boolean by a similarity scaling $D^{-1}SD$. The matrix CR appeared previously as spectral projector [1, 6], and S typically arises as the incidence matrix of a critical graph.

After giving necessary preliminaries in Section 2, we start in Section 3 by studying the *CSR products*. We show that they form a cyclic group and describe the action of this group on the underlying Kleene star. We also emphasize the path sense of these operators, see Theorem 3.3, thus providing connection to the approach of [8, 14, 15, 19, 21, 22]. In Section 4 we establish the algebraic form of the Nachtigall expansion which controls the powers A^t after $t \sim O(n^2)$. See Theorem 4.2. In Section 5 we show that at large t certain Nachtigall terms become totally or partially suppressed by heavier ones, leading to the ultimate expansion of matrix powers. This result, see Theorem 5.6, can be understood as a generalization of the Cyclicity Theorem to reducible case. In Section 6 we treat the computational complexity of computing the terms of *CSR expansions* for a given matrix power, showing that in general this can be done in no more than $O(n^4 \log n)$ operations. In Section 7, which extends the

results of Butkovič et al. [5] on robust matrices, we describe *orbit periodic* matrices A , i.e., such that the orbit $A^t y$ is ultimately linear periodic for all initial vectors y , see Theorem 7.6. We use the ultimate expansion to show that the conditions for orbit periodicity can be verified in no more than $O(n^4 \log n)$ operations, see Theorem 7.8 and its Corollary. We conclude by Section 8 which contains some examples given in the max-plus setting.

2. Preliminaries. In this section we recall some important notions of max algebra. These are the maximum cycle geometric mean, the critical graph and the Kleene star. We close the section with nonnegative similarity scalings and cyclic classes of the critical graph.

Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. The weighted digraph $\mathcal{D}(A) = (N(A), E(A))$, with the set of nodes $N(A) = \{1, \dots, n\}$ and the set of edges $E(A) = \{(i, j) \mid a_{ij} \neq 0\}$ with weights $w(i, j) = a_{ij}$, is called the *digraph associated* with A . Suppose that $P = (i_1, \dots, i_p)$ is a path in \mathcal{D}_A , then the *weight* of P is defined to be $w(P) = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{p-1} i_p}$ if $p > 1$, and 1 if $p = 1$. If $i_1 = i_p$ then P is called a cycle.

When any two nodes in $\mathcal{D}(A)$ can be connected to each other by paths of nonzero weight, the matrix A is called *irreducible*. Otherwise, it is called *reducible*. In the reducible case, there are some (maximal) strongly connected components of $\mathcal{D}(A)$, and a number of nodes that do not belong to any cycle. We will refer to such nodes as to *trivial* components of $\mathcal{D}(A)$.

The *maximum cycle geometric mean* of A , further denoted by $\lambda(A)$, is defined by the formula

$$\lambda(A) = \max_{P_c} (w(P_c))^{1/k}, \tag{2.1}$$

where the maximization is taken over all cycles $P_c = (i_1, \dots, i_k)$, for $k = 1, \dots, n$, in the digraph $\mathcal{D}(A)$.

The Cyclicity Theorem ([16, Theorem 3.9], see also [1, 7]) states that if A is irreducible then after a certain time $T(A)$, there exists a number γ such that $A^{t+\gamma} = \lambda^\gamma(A) A^t$ for all $t \geq T(A)$. Thus λ is the ultimate growth rate of matrix powers in this case. If $\lambda = 1$ then $A^{t+\gamma} = A^t$ for $t \geq T(A)$, in which case we say that $\{A^t, t \geq 0\}$ is *ultimately periodic*.

Remarkably $\lambda(A)$ is also the largest max-algebraic eigenvalue of A , meaning the largest number λ for which there exists a nonzero $x \in \mathbb{R}_+^n$ such that $A \otimes x = \lambda x$, see [1, 4, 7, 16] and references therein.

The operation of taking the maximal cycle geometric mean (m.c.g.m. for short) is homogeneous: $\lambda(\alpha A) = \alpha \lambda(A)$. Hence any matrix, which has $\lambda(A) \neq 0$ meaning that $\mathcal{D}(A)$ is not acyclic, can be scaled so that $\lambda(A/\lambda(A)) = 1$. Following [4], matrix $A \in \mathbb{R}_+^{n \times n}$ with $\lambda(A) = 1$ will be called *definite*.

A cycle $P_c = (i_1, \dots, i_k)$ in $\mathcal{D}(A)$ is called *critical*, if $(w(P_c))^{1/k} = \lambda(A)$. Every node and edge that belongs to a critical cycle is called *critical*. The set of critical nodes is denoted by $N_c(A)$, the set of critical edges is denoted by $E_c(A)$. The *critical digraph* of A , further denoted by $\mathcal{C}(A) = (N_c(A), E_c(A))$, is the digraph which consists of all critical nodes and critical edges of $\mathcal{D}(A)$.

The cyclicity of an irreducible graph is defined as the g.c.d. (greatest common divisor) of the lengths of all its simple cycles. The critical graph defined above may have several strongly connected components, and in this case the cyclicity is the l.c.m. (least common multiple) of their cyclicities. This gives the number γ which appears in the Cyclicity Theorem [1, 16], and it can be shown that the ultimate period cannot be less than γ (in particular, this follows from the approach of the present paper).

There is no obvious subtraction in max algebra, however we have an analogue of $(I - A)^{-1}$ defined by

$$A^* := I \oplus A \oplus A^2 \oplus \dots, \quad (2.2)$$

where I is the identity matrix. This series converges to a finite matrix if and only if $\lambda(A) \leq 1$ [1, 4, 7, 16], and then this matrix $A^* = (a_{ij}^*)$ is called the *Kleene star* of A . This matrix has properties $(A^*)^2 = A^*$ and, clearly, $A^* \geq I$. It is important that the entries of max-algebraic powers $A^k = (a_{ij}^k)$ express the maximal weights of certain paths: a_{ij}^k is equal to the greatest weight of paths P that connect i to j and have length k . The entry a_{ij}^* for $i \neq j$ is equal to the greatest weight of paths that connect i to j with no restriction on their lengths.

As in the nonnegative linear algebra, we have only few invertible matrices, in the sense of the existence of (nonnegative) A^{-1} such that $A^{-1} \otimes A = A \otimes A^{-1} = I$. More precisely, such matrices can be diagonal matrices

$$X = \text{diag}(x) := \begin{pmatrix} x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & x_n \end{pmatrix} \quad (2.3)$$

for a positive $x = (x_1, \dots, x_n)$, or *monomial matrices* obtained from the diagonal matrices by permutations of rows or columns. Nevertheless, such matrices give rise to very convenient *diagonal similarity scalings* $A \mapsto X^{-1}AX$. Such transformations do not change $\lambda(A)$ and $\mathcal{C}(A)$ [11]. They commute with max-algebraic multiplication of matrices and hence with the operation of taking the Kleene star. Geometrically, they correspond to automorphisms of \mathbb{R}_+^n , both in the case of max algebra and in the case of nonnegative linear algebra. The importance of such scalings in max algebra was emphasized already in [7], Ch. 28.

By an observation of Fiedler and Pták [12], for any definite matrix A there is a scaling X such that $X^{-1}AX$ is *visualized* meaning that all critical entries of the matrix equal 1 and all the rest are less than or equal to 1. It is also possible to make $X^{-1}AX$ *strictly visualised* [25], meaning that only critical entries are equal to 1. If a matrix is visualised, or strictly visualised, the same is true for all powers of this matrix, meaning that the critical graph can be seen as a Boolean matrix that “lives by itself”. Thus there is a clear connection to the powers of Boolean matrices.

The periodicity of powers of Boolean matrices is ruled by *cyclic classes* [2] also known as *imprimitivity sets* [3], which we explain below. We note that this notion appeared already in a work of Frobenius [13].

PROPOSITION 2.1 (e.g. Brualdi-Ryser [3]). *Let $G = (N, E)$ be a strongly connected digraph with cyclicity γ_G . Then the lengths of any two paths connecting $i \in N$ to $j \in N$ (with i, j fixed) are congruent modulo γ_G .*

Proposition 2.1 implies that the following equivalence relation can be defined: $i \sim j$ if there exists a path P from i to j such that $l(P) \equiv 0 \pmod{\gamma_G}$. The equivalence classes of G with respect to this relation are called *cyclic classes* [2, 21, 22]. The cyclic class of i will be denoted by $[i]$.

Consider the following *access relations* between cyclic classes: $[i] \rightarrow_t [j]$ if there exists a path P from a node in $[i]$ to a node in $[j]$ such that $l(P) \equiv t \pmod{\gamma_G}$. In this case, a path P with $l(P) \equiv t \pmod{\gamma_G}$ exists between any node in $[i]$ and any node in $[j]$. Further, by Proposition 2.1 the length of any path between a node in $[i]$ and a node in $[j]$ is congruent to t , so the relation $[i] \rightarrow_t [j]$ is well-defined.

Cyclic classes can be computed in $O(|E|)$ time by Balcer-Veinott digraph condensation [2], where $|E|$ denotes the number of edges in G . At each step of this algorithm, we look for all edges which issue from a certain node i , and condense all end nodes of these edges into a single node. Another efficient algorithm is described in [3].

Let $S = (s_{ij})$ be the *incidence matrix* of G , meaning that $s_{ij} = 1$ if $(i, j) \in E$ and $s_{ij} = 0$ otherwise.

The ultimate periodicity of such Boolean matrices has been well studied. If $\gamma_G = 1$ then the periodicity of S^t starts latest after the *Wielandt number* $W(n) := (n-1)^2 + 1$ [3, 17]. This bound is sharp and is due to Wielandt [27].

If $\gamma > 1$ then there are even better sharp bounds due to Schwartz [26]. Assume w.l.o.g. that S is irreducible, and let $n = \alpha\gamma + t$. If $\alpha > 1$ then the periodicity starts at most after $W(\alpha)\gamma + t$ which does not exceed $\frac{n^2}{\gamma} + \gamma$. If $\alpha = 1$ then it starts almost “straightaway”, after at most $\max(1, t)$.

3. CSR products. In this section, given a nonnegative matrix $A \in \mathbb{R}_+^{n \times n}$, we consider max-algebraic products of the form CS^tR , where S is associated with some subdigraph of the critical graph $\mathcal{C}(A)$, and matrices C and R are extracted from a certain Kleene star related to A . We show that these products form a cyclic group and study their periodic properties. We also show that CS^tR are related to a distinguished set of paths which we call \mathcal{C} -heavy.

We start with a remark that the concept of cyclic classes discussed in Section 2 can be generalized to *completely reducible digraphs*, which consist of (possibly several) strongly connected components, not connected with each other. Importantly, the critical digraph of any $A \in \mathbb{R}_+^{n \times n}$ with $\lambda(A) > 0$ is completely reducible.

Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$. Consider any completely reducible subdigraph $\mathcal{C} = (N_c, E_c)$ of $\mathcal{C}(A)$. In particular, \mathcal{C} may consist of several disjoint cycles of $\mathcal{C}(A)$, or we can take a component of $\mathcal{C}(A)$, or we may just have $\mathcal{C} = \mathcal{C}(A)$. Denote by γ the cyclicity of \mathcal{C} and take $B := (A^\gamma)^*$. Define the matrices $C = (c_{ij}) \in \mathbb{R}_+^{n \times n}$, $R = (r_{ij}) \in \mathbb{R}_+^{n \times n}$ and $S = (s_{ij}) \in \mathbb{R}_+^{n \times n}$ by

$$\begin{aligned} c_{ij} &= \begin{cases} b_{ij}, & \text{if } j \in N_c \\ 0, & \text{otherwise,} \end{cases} & r_{ij} &= \begin{cases} b_{ij}, & \text{if } i \in N_c \\ 0, & \text{otherwise,} \end{cases} \\ s_{ij} &= \begin{cases} a_{ij}, & \text{if } (i, j) \in E_c \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

The nonzero entries of C , respectively R , can only be in the submatrix of $B = (A^\gamma)^*$ extracted from columns, respectively rows, in N_c . All nonzero entries of S are in the principal submatrix $S_{N_c N_c}$ extracted from rows and columns in N_c .

We show in the next proposition that S can be assumed to be 0–1. It can be also deduced from the results in [11].

PROPOSITION 3.1. *Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$ and let S be defined as above. There exists a positive $z \in \mathbb{R}_+^n$ such that $D^{-1}SD$ for $D := \text{diag}(z)$ is a 0–1 matrix.*

Proof. As $\lambda(S) = 1$, we can take $z := \bigoplus_{j=1}^n S^*_{.j}$. This vector is positive, and observe that $Sz \leq z$ since $SS^* \leq S^*$. From this and $\mathcal{D}(S) = \mathcal{C}(S) = \mathcal{C}$, it can be deduced by multiplying $z_i^{-1}s_{ij}z_j \leq 1$ along cycles that $z_i^{-1}s_{ij}z_j = 1$ for all $(i, j) \in \mathcal{D}(S)$, while $z_i^{-1}s_{ij}z_j = 0$ for all $(i, j) \notin \mathcal{D}(S)$. \square

As S can be scaled to 0–1 matrix, we conclude that $\{S^t, t \geq 0\}$ becomes periodic at most after the Wielandt number $W(n_c) = (n_c - 1)^2 + 1$, where n_c is the number of nodes in N_c . Matrix A will be called *S-visualized*, if S defined in (3.1) is Boolean. In

the S -visualized case, the asymptotic form of S^t for $t \geq T$ is determined by the cyclic classes of \mathcal{C} [3]:

$$s_{ij}^t = \begin{cases} 1, & \text{if } [i] \rightarrow_t [j], \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

Now we study the *CSR products*

$$\mathcal{P}^{(t)} := CS^tR, \quad (3.3)$$

assuming that $A \in \mathbb{R}_+^{n \times n}$ has $\lambda(A) = 1$.

We start by the observation that if T is the number after which $\{S^t, t \geq 0\}$ becomes periodic, then

$$S^{l\gamma}R = R, \quad CS^{l\gamma} = C, \quad \forall l\gamma \geq T. \quad (3.4)$$

Indeed, (3.2) implies that all diagonal entries of $S^{l\gamma}$ with indices in N_c are 1 if $l\gamma \geq T$, which implies $S^{l\gamma}R \geq R$ and $CS^{l\gamma} \geq C$. On the other hand, $S \leq A$ and so $S^{l\gamma} \leq (A^\gamma)^*$. Further, $(A^\gamma)^*R = R$ and $C(A^\gamma)^* = C$ since $(A^\gamma)^*(A^\gamma)^* = (A^\gamma)^*$, and so $S^{l\gamma}R \leq (A^\gamma)^*R = R$ and $CS^{l\gamma} \leq C(A^\gamma)^* = C$.

As $\{S^t, t \geq T\}$ is periodic, so is $\{\mathcal{P}^{(t)}, t \geq T\}$. Moreover, we conclude from (3.4) that this periodicity starts from the very beginning.

PROPOSITION 3.2 (Periodicity). $\mathcal{P}^{(t+\gamma)} = \mathcal{P}^{(t)}$ for all $t \geq 0$.

Proof. It follows from Eqn. (3.4) that $\mathcal{P}^{(t+l\gamma)} = \mathcal{P}^{(t)}$ and $\mathcal{P}^{(t+(l+1)\gamma)} = \mathcal{P}^{(t+\gamma)}$ for $l\gamma \geq T$. But $\mathcal{P}^{(t+l\gamma)} = \mathcal{P}^{(t+(l+1)\gamma)}$, which implies that also $\mathcal{P}^{(t+\gamma)} = \mathcal{P}^{(t)}$. \square

It is also useful to understand the meaning of $\mathcal{P}^{(t)}$ in terms of paths. Given a set of paths Π , we denote by $w(\Pi)$ the greatest weight of paths in Π , assuming $w(\Pi) = 0$ if Π is empty. A path will be called \mathcal{C} -heavy if it goes through a node in N_c . The set of all \mathcal{C} -heavy paths on $\mathcal{D}(A)$ that connect i to j and have length t will be denoted by $\Pi_{ij,t}^h$. We also denote by τ the maximal cyclicity of the components of $\mathcal{C}(A)$.

THEOREM 3.3 (C-Heavy Paths). Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$ and let $T \geq 0$ be such that $\{S^t, t \geq T\}$ is periodic.

1. For $t \geq 0$,

$$w(\Pi_{ij,t}^h) \leq \mathcal{P}_{ij}^{(t)}. \quad (3.5)$$

2. For $t \geq T + 2\tau(n-1)$,

$$w(\Pi_{ij,t}^h) \geq \mathcal{P}_{ij}^{(t)}. \quad (3.6)$$

Proof. By Proposition 3.1 there exists a diagonal matrix D such that $D^{-1}AD$ is S -visualized. As both sides of (3.5) and (3.6) are stable under similarity scaling of A , we will assume that A is already S -visualized.

1.: Let $P \in \Pi_{ij,t}^h$ and $w(P) \neq 0$. We need to show that

$$w(P) \leq \mathcal{P}_{ij}^{(t)} \quad (3.7)$$

Path P can be decomposed as $P = P_{\text{beg}} \circ P_{\text{end}}$, where P_{beg} connects i to a node $m \in N_c$ and P_{end} connects m to j . Adjoining to P any sufficiently large number of cycles of \mathcal{C} that go through m and whose total length is a multiple of γ , we obtain a path P' that we can decompose as $P' = P'_{\text{beg}} \circ P'_{\text{int}} \circ P'_{\text{end}}$, where P'_{beg} connects i to

$m_1 \in \mathcal{C}$ and $l(P'_{\text{beg}})$ is a multiple of γ , P'_{int} connects m_1 to $m_2 \in N_c$, hss length t and belongs entirely to \mathcal{C} , and P'_{end} connects m_2 to j and $l(P'_{\text{end}})$ is a multiple of γ . We conclude that $w(P'_{\text{beg}}) \leq c_{im_1}$, $w(P'_{\text{end}}) \leq r_{m_2j}$ and $w(P'_{\text{int}}) = s_{m_1m_2}^t = 1$. We obtain $w(P) = w(P') \leq c_{im_1} s_{m_1m_2}^t r_{m_2j}$, which implies (3.7), and hence (3.5).

2.: There exist indices m_1 and m_2 such that $\mathcal{P}_{ij}^{(t)} = c_{im_1} s_{m_1m_2}^t r_{m_2j}$. This is the the weight of a path P decomposed as $P = P_{\text{beg}} \circ P_{\text{int}} \circ P_{\text{end}}$, where P_{beg} connects i to m_1 , P_{int} connects m_1 to m_2 and P_{end} connects m_2 to j . Here P_{int} has length t and belongs to the component of \mathcal{C} which we denote by \mathcal{T} and whose cyclicity we denote by π . The lengths of P_{beg} and P_{end} are respectively $l(P_{\text{beg}}) = l_1\pi$ and $l(P_{\text{end}}) = l_2\pi$ for some l_1, l_2 , since $l(P_{\text{beg}})$ and $l(P_{\text{end}})$ are multiples of γ and γ is itself a multiple of π . Paths P_{beg} and P_{end} correspond to certain paths on $\mathcal{D}(A^\pi)$ with lengths l_1 and l_2 . If $l_1 \geq n_1$ or $l_2 \geq n_1$, then we can perform cycle deletion (w.r.t. $\mathcal{D}(A^\pi)$) and obtain paths P_{beg}^2 and P_{end}^2 with lengths $k_1\pi$ and $k_2\pi$ where $k_1 < n$ and $k_2 < n$. For the resulting path $P^2 := P_{\text{beg}}^2 \circ P_{\text{int}} \circ P_{\text{end}}^2$ we will have $w(P^2) \geq w(P)$ since $\lambda = 1$. Now we have $l(P_{\text{beg}}^2) + l(P_{\text{end}}^2) \leq 2\tau(n-1)$. If $t \geq T + 2\tau(n-1)$, then the principal submatrix of S^t corresponding to the component \mathcal{T} coincides with that of $S^{t-(k_1+k_2)\pi}$, which implies that P_{int} can be replaced by a path P_{int}^3 with length $t - (k_1 + k_2)\pi$, so that $w(P^3) = w(P^2) \geq \mathcal{P}_{ij}^{(t)}$ where $P^3 := P_{\text{beg}}^2 \circ P_{\text{int}}^3 \circ P_{\text{end}}^2 \in \Pi_{ij,t}^h$. \square

This ‘‘path sense’’ of $\mathcal{P}^{(t)}$ simplifies the proof of the following important law.

THEOREM 3.4 (Group law). $\mathcal{P}^{(t_1+t_2)} = \mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)}$ for all $t_1, t_2 \geq 0$

Proof. As $(A^\gamma)^*(A^\gamma)^* = (A^\gamma)^*$, we have $(RC)_{ii} = (A^\gamma)_{ii}^* = 1$ for all $i \in N_c$, and hence $RCS \geq S$. We use this to obtain that $\mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)} \geq \mathcal{P}^{(t_1+t_2)}$:

$$\mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)} = CS^{t_1}RCS^{t_2}R \geq CS^{t_1+t_2}R = \mathcal{P}^{(t_1+t_2)}. \quad (3.8)$$

But Theorem 3.3 implies that $\mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)} \leq \mathcal{P}^{(t_1+t_2)}$ for all large enough t_1, t_2 , since the concatenation of two \mathcal{C} -heavy paths is again a \mathcal{C} -heavy path. As $\mathcal{P}^{(t)}$ are periodic, it follows that $\mathcal{P}^{(t_1)}\mathcal{P}^{(t_2)} \leq \mathcal{P}^{(t_1+t_2)}$ for all t_1, t_2 , and we obtain the claim combining this with the reverse inequality. \square

Formulas (3.2) and (3.4) imply that if A is S -visualized then all rows of R or columns of C with indices in the same cyclic class of \mathcal{C} coincide. Hence, when working with $\mathcal{P}^{(t)}$ we can assume without loss of generality that all cyclic classes have just 1 element and consequently, that $S_{N_c N_c}$ is a permutation matrix. This captures the structure of $\mathcal{P}^{(t)}$, which form a cyclic group of order γ .

As usual e_i denotes the vector which has all coordinates equal to 0 except for the i th which equals 1. For the rows $\mathcal{P}_{i \cdot}^{(t)}$ and columns $\mathcal{P}_{\cdot j}^{(t)}$ of $\mathcal{P}^{(t)}$ we have:

$$e_i^T \mathcal{P}^{(t)} = \mathcal{P}_{i \cdot}^{(t)}, \quad \mathcal{P}^{(t)} e_j = \mathcal{P}_{\cdot j}^{(t)}. \quad (3.9)$$

Next we study the periodicity of $\mathcal{P}^{(t)}$ in more detail. It turns out that the columns and rows of $\mathcal{P}^{(t+1)}$ with indices in N_c can be obtained from those of $\mathcal{P}^{(t)}$ by means of a permutation on cyclic classes, while the rest of the columns (or rows) are max-linear combinations of the critical ones. We start with the following observation on the spectral projector $\mathcal{P}^{(0)} := CR$, which can be found in [1, 6].

LEMMA 3.5. $\mathcal{P}_{i \cdot}^{(0)} = R_i$. and $\mathcal{P}_{\cdot i}^{(0)} = C_i$ for all $i \in N_c$.

Proof. As $(A^\gamma)_{ii}^* = 1$ for all $i \in N_c$, we obtain $e_i^T C = C_i \cdot \geq e_i^T$ and $Re_i = R_i \geq e_i$. We see that

$$\begin{aligned} \mathcal{P}_{i \cdot}^{(0)} &= e_i^T CR \geq e_i^T R = R_i, \\ \mathcal{P}_{\cdot i}^{(0)} &= CR e_i \geq C e_i = C_i. \end{aligned} \quad (3.10)$$

But $CR \leq ((A^\gamma)^*)^2 = (A^\gamma)^*$, which implies the reverse inequalities $\mathcal{P}_i^{(0)} \leq R_i$ and $\mathcal{P}_i^{(0)} \leq C_i$. \square

THEOREM 3.6. *Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$ and be S -visualized. If $[i] \rightarrow_t [j]$, then*

$$\mathcal{P}_i^{(t+s)} = \mathcal{P}_j^{(s)}, \quad \mathcal{P}_i^{(s)} = \mathcal{P}_j^{(t+s)} \quad (3.11)$$

for all $s, t \geq 0$.

Proof. We prove the first equality of (3.11). Using the group law we assume that $s = 0$. We also assume that $S_{N_c N_c}$ is a permutation matrix, then $e_i^T S^t = e_j$. Using this and Lemma 3.5 we obtain

$$\begin{aligned} \mathcal{P}_j^{(0)} &= e_j^T \mathcal{P}^{(0)} = e_j^T R = e_i^T S^t R \leq \\ &\leq e_i^T C S^t R = e_i^T \mathcal{P}^{(t)} = \mathcal{P}_i^t. \end{aligned} \quad (3.12)$$

Analogously we have $\mathcal{P}_i^{(0)} \leq \mathcal{P}_j^{(\gamma-t)}$. Multiplying this inequality by $\mathcal{P}^{(t)}$ and using the group law and periodicity, we obtain that $\mathcal{P}_i^{(t)} \leq \mathcal{P}_j^{(\gamma)} = \mathcal{P}_j^{(0)}$. Combining this with (3.12) we obtain the desired property. \square

COROLLARY 3.7. *Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$. Then $\mathcal{P}_i^{(t)} = (S^t R)_i$ and $\mathcal{P}_i^{(t)} = (C S^t)_i$ for all $i \in N_c$.*

Proof. We assume that A is S -visualized, and we also assume that $S_{N_c N_c}$ is a permutation matrix. For $[i] \rightarrow_t [j]$, Theorem 3.6 and Lemma 3.5 imply that

$$\begin{aligned} \mathcal{P}_i^{(t)} &= e_i^T \mathcal{P}^{(t)} = e_j^T \mathcal{P}^{(0)} = e_j^T R = e_i^T S^t R = (S^t R)_i, \\ \mathcal{P}_j^{(t)} &= \mathcal{P}^{(t)} e_j = \mathcal{P}^{(0)} e_i = C e_i = C S^t e_j = (C S^t)_j. \end{aligned}$$

The claim is proved. \square

COROLLARY 3.8. *Let $A \in \mathbb{R}_+^{n \times n}$ have $\lambda(A) = 1$. For each $k = 1, \dots, n$ there exist α_{ik} and β_{ki} , where $k \in N_c$, such that*

$$\mathcal{P}_{\cdot k}^{(t)} = \bigoplus_{i \in N_c} \alpha_{ik} \mathcal{P}_i^{(t)}, \quad \mathcal{P}_k^{(t)} = \bigoplus_{i \in N_c} \beta_{ki} \mathcal{P}_i^{(t)}. \quad (3.13)$$

Proof. By Corollary 3.7 we have $\mathcal{P}_i^{(t)} = (C S^t)_i$ and $\mathcal{P}_i^{(t)} = (S^t R)_i$ for all $i \in N_c$. Eqn. (3.13) follows directly from $\mathcal{P}^{(t)} = C S^t R$, the coefficients α_{ik} (resp. β_{ki}) being taken from the k th column of R (resp. the k th row of C). \square

4. Nachtigall expansions. In this section we show that the powers A^t of $A \in \mathbb{R}_+^{n \times n}$ can be expanded for $t \geq 3n^2$ as sum of CSR products. This establishes a more general algebraic form of the Nachtigall expansion studied in [18, 20].

Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. Define $\lambda_1 = \lambda(A)$, and let $\mathcal{C}_1 = (N_1, E_1)$ be a completely reducible subdigraph of $\mathcal{C}(A)$. Set $A_1 := A$ and $K_1 := N$, where $N = \{1, \dots, n\}$.

The elements of a Nachtigall expansion will be now defined inductively for $\mu \geq 2$. Namely, we define $K_\mu := N \setminus \cup_{i=1}^{\mu-1} N_i$ and $A_\mu = (a_{ij}^\mu) \in \mathbb{R}_+^{n \times n}$ by

$$a_{ij}^\mu = \begin{cases} a_{ij}, & \text{if } i, j \in K_\mu, \\ 0, & \text{otherwise.} \end{cases} \quad (4.1)$$

Further define $\lambda_\mu := \lambda(A_\mu)$. If $\lambda_\mu = 0$ then stop, otherwise select a completely reducible subdigraph $\mathcal{C}_\mu = (N_\mu, E_\mu)$ of the critical digraph $\mathcal{C}(A_\mu)$, and proceed as above with $\mu := \mu + 1$.

By the above procedure we define $K_\mu, A_\mu, \lambda_\mu$ and $\mathcal{C}_\mu = (N_\mu, E_\mu)$ for $\mu = 1, \dots, m$, where $m \leq n$ is the last number μ such that $\lambda_\mu > 0$.

Denote $L := \bigcup_{i=1}^m N_\mu$ and $\bar{L} = N \setminus L$.

For each $\mu = 1, \dots, m$, let γ_μ be the cyclicity of \mathcal{C}_μ . Since $\lambda((A_\mu/\lambda_\mu)^{\gamma_\mu}) = \lambda(A_\mu/\lambda_\mu) = 1$, the Kleene star $B_\mu := ((A_\mu/\lambda_\mu)^{\gamma_\mu})^*$ is finite. Define the matrices $C_\mu = (c_{ij}^\mu) \in \mathbb{R}_+^{n \times n}$, $R_\mu = (r_{ij}^\mu) \in \mathbb{R}_+^{n \times n}$ and $S_\mu = (s_{ij}^\mu) \in \mathbb{R}_+^{n \times n}$ by

$$\begin{aligned} c_{ij}^\mu &= \begin{cases} b_{ij}^\mu, & \text{if } j \in N_\mu, \\ 0, & \text{otherwise,} \end{cases} & r_{ij}^\mu &= \begin{cases} b_{ij}^\mu, & \text{if } i \in N_\mu, \\ 0, & \text{otherwise,} \end{cases} \\ s_{ij}^\mu &= \begin{cases} a_{ij}^\mu/\lambda_\mu, & \text{if } (i, j) \in E_\mu \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (4.2)$$

It follows from Proposition 3.1 that each S_μ can be scaled to a 0–1 matrix using a certain vector denoted here by z^μ . Note that the sets N_μ are pairwise disjoint. Defining $z \in \mathbb{R}_+^n$ by

$$z_i = \begin{cases} z_i^\mu, & \text{if } i \in N_\mu, \\ 1, & \text{if } i \in \bar{L}, \end{cases} \quad (4.3)$$

and letting $D := \text{diag}(z)$, we obtain that the matrix $\tilde{A} := D^{-1}AD$ is *totally S -visualized*, meaning that all corresponding matrices \tilde{S}_μ are Boolean.

As S_μ can be scaled to be Boolean, the sequences of their max-algebraic powers $\{S_\mu^t \mid t \geq 0\}$, being powers of Boolean matrices when scaled, are ultimately periodic with periods γ_μ . This periodicity starts at most after the corresponding Wielandt numbers $(k_\mu - 1)^2 + 1$ where k_μ is the number of elements in N_μ .

We proceed with some notation. Denote $\mu(i) = \mu$ if $i \in N_\mu$, and $\mu(i) = +\infty$ if $i \in \bar{L}$. Denote by $\Pi_{ij,t}$ the set of paths on $\mathcal{D}(A)$ which connect i to j and have length t . Denote by $\Pi_{ij,t}^\mu$ the set of paths $P \in \Pi_{ij,t}$ such that $\min_{i \in N_P} \mu(i) = \mu$, where N_P is the set of nodes visited by P . (Note that greater values of μ correspond to smaller λ_μ .) The paths in $\Pi_{ij,t}^\mu$ will be called μ -heavy, since they are \mathcal{C}_μ -heavy (see Sect. 3) in A_μ .

Any path $P \in \Pi_{ij,t}$ with $t \geq n$ has at least one cycle. Hence there are no paths with length $t \geq n$ that visit only the nodes in \bar{L} , for otherwise the subdigraph of $\mathcal{D}(A)$ induced by \bar{L} would contain a cycle and the number of components would be more than m . We can express the entries of $A^t = (a_{ij}^t)$ and $A_\mu^t = (a_{ij}^{\mu,t})$ for $t \geq n$ as follows:

$$\begin{aligned} a_{ij}^t &= w(\Pi_{ij,t}) = \bigoplus_{\mu=1}^m w(\Pi_{ij,t}^\mu), & t \geq n \\ a_{ij}^{\mu,t} &= \bigoplus_{\nu=\mu}^m w(\Pi_{ij,t}^\nu), & t \geq n. \end{aligned} \quad (4.4)$$

Denote

$$\mathcal{N}_\mu^{(t)} := C_\mu S_\mu^t R_\mu. \quad (4.5)$$

These *CSR* products are defined from A_μ and \mathcal{C}_μ in the same way as $\mathcal{P}^{(t)}$, see Sect. 3, were defined from A and \mathcal{C} , and it follows that the sequence $\mathcal{N}_\mu^{(t)}$ is periodic with period γ_μ . Further denote by τ_μ the greatest cyclicity of a component in \mathcal{C}_μ and by n_μ the number of nodes in K_μ . The following is a version of Theorem 3.3 for $\mathcal{N}_\mu^{(t)}$.

THEOREM 4.1 (μ -Heavy Paths). *Let T_μ be such that $\{S_\mu^t, t \geq T_\mu\}$ is periodic.*

1. For $t \geq 0$,

$$w(\Pi_{ij,t}^\mu) \leq \lambda_\mu^t (\mathcal{N}_\mu^{(t)})_{ij} \quad (4.6)$$

2. For $t \geq T_\mu + 2\tau_\mu(n_\mu - 1)$,

$$w(\Pi_{ij,t}^\mu) \geq \lambda_\mu^t (\mathcal{N}_\mu^{(t)})_{ij} \quad (4.7)$$

Proof. We can w.l.o.g. assume that $\lambda_\mu = 1$, since both (4.6) and (4.7) are stable under scalar multiplication of A . After this, the claim follows from Theorem 3.3. \square

In the theorem above, we can choose T_μ equal to each other and of the order $O(n^2)$ for all μ . The main result of this section immediately follows now from Theorem 4.1 and Eqn. (4.4), noting that $T_\mu + 2\tau_\mu(n_\mu - 1) \leq 3n^2$ for all μ .

THEOREM 4.2 (Nachtigall expansion). *Let $A \in \mathbb{R}_+^{n \times n}$. Then for all $t \geq 3n^2$*

$$A_\mu^t = \bigoplus_{\nu=\mu}^m \lambda_\nu^t \mathcal{N}_\nu^{(t)}. \quad (4.8)$$

In particular,

$$A^t = \bigoplus_{\nu=1}^m \lambda_\nu^t \mathcal{N}_\nu^{(t)}. \quad (4.9)$$

5. Ultimate expansion. In this section we construct a different expansion of A^t which we call the ultimate expansion, in order to describe the ultimate behavior of A^t . This expansion is related to the Nachtigall expansion of Section 4 with the selection rule $\mathcal{C}_\mu := \mathcal{C}(A_\mu)$. The latter expansion will be called the *canonical Nachtigall expansion*.

The elements of the ultimate expansion will be labeled by \circ , since we need to distinguish them from those of the canonical Nachtigall expansion. For instance, we will write A_μ° versus A_μ , \mathcal{C}_μ° versus \mathcal{C}_μ and λ_μ° versus λ_μ , etc.

Let $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$. Define $\lambda_1^\circ = \lambda(A)$, let $\mathcal{C}_1^\circ = (N_1^\circ, E_1^\circ)$ be the critical graph of A and denote by M_1° the set of nodes in all components of $\mathcal{D}(A)$ that contain the components of \mathcal{C}_1° . Set $A_1^\circ := A$ and $K_1^\circ := N$.

By induction for $\mu \geq 2$, define $K_\mu^\circ := N \setminus \cup_{i=1}^{\mu-1} M_i^\circ$ (instead of $K_\mu = N \setminus \cup_{i=1}^{\mu-1} N_i$ in the case of a Nachtigall expansion), and define $A_\mu^\circ = (a_{ij}^{\mu^\circ}) \in \mathbb{R}_+^{n \times n}$ by

$$a_{ij}^{\mu^\circ} = \begin{cases} a_{ij}, & \text{if } i, j \in K_\mu^\circ, \\ 0, & \text{otherwise.} \end{cases} \quad (5.1)$$

Define $\lambda_\mu^\circ := \lambda(A_\mu^\circ)$. If $\lambda^\circ = 0$ then stop, otherwise let $\mathcal{C}_\mu^\circ = (N_\mu^\circ, E_\mu^\circ)$ be the critical graph of A_μ° , let M_μ° be the set of all nodes in the components of $\mathcal{D}(A)$ which contain the components of \mathcal{C}_μ° , and proceed with the above definition for $\mu := \mu + 1$.

By the above procedure we define $K_\mu^\circ, A_\mu^\circ, \lambda_\mu^\circ, C_\mu^\circ = (N_\mu^\circ, E_\mu^\circ)$ and M_μ° for $\mu = 1, \dots, m^\circ$, where $m^\circ \leq n$ is the last number μ such that $\lambda_\mu^\circ \neq 0$. Note that $\{\lambda_\mu^\circ, \mu = 1, \dots, m^\circ\}$ is the set of m.c.g.m. of all nontrivial components of $\mathcal{D}(A)$, and each C_μ° consists of the critical digraphs of (possibly several) such components.

Let γ_μ° be the cyclicity of C_μ° , and let $B_\mu^\circ, C_\mu^\circ, S_\mu^\circ$ and R_μ° be defined from $(A_\mu^\circ/\lambda_\mu^\circ)^{\gamma_\mu^\circ}$ in full analogy with B_μ, C_μ, S_μ and R_μ in Sect. 4. The matrices S_μ° can be again simultaneously scaled to 0 – 1 form.

Essentially in the new construction we contract by the components of $\mathcal{D}(A)$ instead of the components of $\mathcal{C}(A_\mu)$ in the case of the canonical Nachtigall expansion.

Denote by $\lambda(i)$ the m.c.g.m. of the component of $\mathcal{D}(A)$ to which i belongs, and let $\lambda(i) = 0$ if i does not belong to any component of $\mathcal{D}(A)$. For a path P define $\lambda(P) := \max_{i \in N_P} \lambda(i)$ where N_P is the set of nodes visited by P . Recall that $\Pi_{ij,t}$ denotes the set of paths on $\mathcal{D}(A)$ which connect i to j and have length t . Denote by $\Pi_{ij,t}^{\mu^\circ}$ the set of paths $P \in \Pi_{ij,t}$ such that P contains a node in N_μ° and $\lambda(P) = \lambda_\mu^\circ$. Such paths will be called μ -hard. Note that they are $\mathcal{C}(A_\mu^\circ)$ -heavy with respect to A_μ° .

Denote

$$\mathcal{U}_\mu^{(t)} := C_\mu^\circ (S_\mu^\circ)^t R_\mu^\circ. \quad (5.2)$$

These CSR products are defined from A_μ° and C_μ° in the same way as $\mathcal{P}^{(t)}$ were defined in Sect. 3 from A and \mathcal{C} , and it follows that the sequence $\{\mathcal{U}_\mu^{(t)}, t \geq 0\}$ has the period γ_μ° . Denote by τ_μ° the greatest cyclicity of a component in C_μ° and by n_μ° the number of nodes in K_μ° . The next result follows from Theorem 3.3.

THEOREM 5.1 (μ -Hard paths). *Let T_μ° be such that $\{(S_\mu^\circ)^t, t \geq T_\mu^\circ\}$ is periodic.*

1. For $t \geq 0$,

$$w(\Pi_{ij,t}^{\mu^\circ}) \leq (\lambda_\mu^\circ)^t (\mathcal{U}_\mu^t)_{ij}. \quad (5.3)$$

2. For $t \geq T_\mu^\circ + 2\tau_\mu^\circ(n_\mu^\circ - 1)$,

$$w(\Pi_{ij,t}^{\mu^\circ}) \geq (\lambda_\mu^\circ)^t (\mathcal{U}_\mu^t)_{ij}. \quad (5.4)$$

Comparing the constructions above with those of the canonical Nachtigall expansion (see Sect. 4 assuming that $C_\mu := \mathcal{C}(A_\mu)$), we see that \mathcal{C}_1 is the same as \mathcal{C}_1° and λ_1 is the same as λ_1° , however, other components and values may be not the same. We next describe relation between them.

PROPOSITION 5.2. *Each $\nu = 1, \dots, m^\circ$ corresponds to a unique $\mu = 1, \dots, m$ such that $\lambda_\nu^\circ = \lambda_\mu$ and all components of \mathcal{C}_ν° are also components of \mathcal{C}_μ .*

Proof. Consider the canonical Nachtigall expansion. Note that λ_μ strictly decrease, as at each step of the definition we remove the whole critical digraph. Now pick arbitrary λ_ν° , which is the m.c.g.m. of some component of $\mathcal{D}(A)$. There is a reduction step when \mathcal{C}_μ for the first time intersects with a component of $\mathcal{D}(A)$ whose m.c.g.m. is λ_ν° . Then $\lambda_\mu = \lambda_\nu^\circ$, and \mathcal{C}_μ has to contain all components of \mathcal{C}_ν° , precisely as they are. This proves the claim. \square

Further we renumber λ_ν° so that $\nu = \mu$ if $\lambda_\nu^\circ = \lambda_\mu$, meaning that the numbering of λ_ν° is adjusted to that of λ_μ . This defines a subset Σ of $\{1, \dots, m\}$, such that $\lambda_\mu = \lambda_\mu^\circ$ if and only if $\mu \in \Sigma$.

COROLLARY 5.3. γ_μ is a multiple of γ_μ° for each $\mu \in \Sigma$.

Proof. With the new numbering, all components of \mathcal{C}_μ° are also components of \mathcal{C}_μ by Proposition 5.2. \square

Unlike μ -heavy paths, μ -hard paths do not cover the whole path sets $\Pi_{ij,t}$ in general. However evidently

$$(A_\mu^\circ)_{ij}^t = w(\Pi_{ij,t}^{\mu^\circ}), \quad \text{if } i \in N_\mu^\circ \text{ or } j \in N_\mu^\circ. \quad (5.5)$$

From this and Theorem 5.1 we deduce the following.

PROPOSITION 5.4. *Let $A \in \mathbb{R}_+^{n \times n}$. For all $t \geq 3n^2$*

$$(A_\mu^\circ)_{ij}^t = \lambda_\mu^t \mathcal{U}_\mu^{(t)}, \quad \text{if } i \in N_\mu^\circ \text{ or } j \in N_\mu^\circ. \quad (5.6)$$

For the sequel we need to establish some relation between connectivity on $\mathcal{D}(A)$ and nonzero entries of $\mathcal{U}_\mu^{(t)}$.

We denote by γ° the l.c.m. of all cyclicities γ_μ° of all components \mathcal{C}_μ° . Recall that we denote by N_P the set of nodes visited by a path P and by $\lambda(P)$ the greatest m.c.g.m. of a component visited by P .

PROPOSITION 5.5. *Let $i, j \in N$, $l \geq 0$ and $\mu \in \Sigma$. The following are equivalent.*

1. $(\mathcal{U}_\mu^{(l)})_{ij} \neq 0$;
2. For all $t \equiv l \pmod{\gamma^\circ}$ such that $t \geq 3n^2$, there is a μ -hard path of length t connecting i to j ;
3. For some $t \equiv l \pmod{\gamma^\circ}$ there exists a μ -hard path of length t connecting i to j ;
4. For some $t \equiv l \pmod{\gamma^\circ}$ there exists a path P of length t such that $\lambda(P) = \lambda_\mu$.

Proof. Implications 1. \Leftrightarrow 2. and 3. \Leftrightarrow 1. follow from Theorem 5.1 and the periodicity of $\mathcal{U}_\mu^{(t)}$. Implications 2. \Rightarrow 3. \Rightarrow 4. are evident. It remains to prove 4. \Rightarrow 3. Let k be a node in N_P which belongs to M_μ° , and let l be a node in N_μ° (that is, a critical node) in the same component of $\mathcal{D}(A)$ as k . There exists a cycle containing both k and l . Adjoining γ° copies of this cycle to P we obtain a μ -hard path, whose length is congruent to $l \pmod{\gamma^\circ}$. \square

Now we establish the ultimate expansion of matrix powers, as an ultimate form of the canonical Nachtigall expansion. We will write $a(t) \stackrel{T}{=} b(t)$ if $a(t) = b(t)$ for all $t \geq t'$ where t' is an unknown integer, and analogously for inequalities.

THEOREM 5.6 (Ultimate expansion). *Let $A \in \mathbb{R}_+^{n \times n}$. For all $\mu \in \Sigma$*

$$(A_\mu^\circ)^t \stackrel{T}{=} \bigoplus_{\nu \in \Sigma: \nu \geq \mu} \lambda_\nu^t \mathcal{U}_\nu^{(t)}. \quad (5.7)$$

In particular,

$$A^t \stackrel{T}{=} \bigoplus_{\nu \in \Sigma} \lambda_\nu^t \mathcal{U}_\nu^{(t)}. \quad (5.8)$$

Proof. It suffices to prove (5.8). First note that $\mathcal{U}_\mu^{(t)} \leq \mathcal{N}_\mu^{(t)}$ for all $\mu \in \Sigma$, since any μ -hard path is a μ -heavy path. As $A^t \stackrel{T}{=} \bigoplus_{\mu} \lambda_\mu^t \mathcal{N}_\mu^{(t)}$ by the canonical Nachtigall expansion, it suffices to prove that

$$\bigoplus_{\mu=1}^m \lambda_\mu^t \mathcal{N}_\mu^{(t)} \leq \bigoplus_{\mu \in \Sigma} \lambda_\mu^t \mathcal{U}_\mu^{(t)} \quad (5.9)$$

For all μ , i and j such that $(\mathcal{N}_\mu^{(t)})_{ij} \neq 0$, we will show that either $\mu \in \Sigma$ and $(\mathcal{U}_\mu^{(t)})_{ij} = (\mathcal{N}_\mu^{(t)})_{ij}$, or there exists $\nu \in \Sigma$ such that $\lambda_\nu > \lambda_\mu$ and $(\mathcal{U}_\nu^{(t)})_{ij} \neq 0$.

Assume that either $\mu \notin \Sigma$, or $\mu \in \Sigma$ but $(\mathcal{N}_\mu^{(t)})_{ij} > (\mathcal{U}_\mu^{(t)})_{ij}$. Theorem 4.1 implies that for all l such that $l \geq 3n^2$ and $l \equiv t \pmod{\gamma_\mu}$ there exist paths $P \in \Pi_{ij,l}^\mu$ such that $w(P) = \lambda_\mu^l (\mathcal{N}_\mu^{(t)})_{ij}$. If $\mu \in \Sigma$ then by Corollary 5.3 γ_μ is a multiple of γ_μ° , and hence $l \equiv t \pmod{\gamma_\mu^\circ}$. Paths P cannot be μ -hard, because then $\mu \in \Sigma$ and $w(P) \leq \lambda_\mu^l (\mathcal{U}_\mu^{(t)})_{ij}$ by Theorem 5.1, which implies $(\mathcal{U}_\mu^{(t)})_{ij} \geq (\mathcal{N}_\mu^{(t)})_{ij}$ contradicting our assumptions. Hence for any such path there exists $\nu \in \Sigma$ such that $\lambda_\nu = \lambda(P) > \lambda_\mu$. Applying Proposition 5.5, we obtain that $(\mathcal{U}_\nu^{(t)})_{ij} \neq 0$ with $\lambda_\nu > \lambda_\mu$. The claim is proved. \square

If A is irreducible, then the ultimate expansion has only one term, which corresponds to its critical graph $\mathcal{C}(A)$. In general, it has several terms (up to n) corresponding to the critical graphs of the components of $\mathcal{D}(A)$. Thus, Theorem 5.6 can be regarded as a generalization of the Cyclicity Theorem, see [16] Theorem 3.9 or [1] Theorem 3.108, which it implies as a special irreducible case.

6. Computational complexity. Given $A \in \mathbb{R}_+^{n \times n}$, we investigate the computational complexity of the following problems.

(P1) For given t , reconstruct all terms $\mathcal{N}_\mu^{(t)}$ of a Nachtigall expansion with a prescribed selection rule for \mathcal{C}_μ .

(P2) For given t : $0 \leq t < \gamma$, reconstruct all terms $\mathcal{U}_\mu^{(t)}$ of the ultimate expansion.

In (P1) we assume that selecting the subdigraph \mathcal{C}_μ of $\mathcal{C}(A_\mu)$ does not take more than $O(n^3)$ operations. This holds in particular if \mathcal{C}_μ is an arbitrary cycle of $\mathcal{C}(A_\mu)$ as in [18, 20].

Problem (P1) is close to the problem considered by Molnárová [18], and Problem (P2) is extension of a problem regarded by Sergeev [24]. An $O(n^4 \log n)$ solution of these problems is given below. It is based on visualisation, square multiplication and permutation of cyclic classes. See Semančíková [21, 22] for closely related studies in max-min algebra.

THEOREM 6.1. *For any $A \in \mathbb{R}_+^{n \times n}$, problems (P1) and (P2) can be solved in $O(n^4 \log n)$ operations.*

Proof. (P1): First we need to compute λ_μ , A_μ and \mathcal{C}_μ for all μ . At each step the computation requires no more than $O(n^3)$ operations, based on Karp and Floyd-Warshall methods applied to each component of $\mathcal{D}(A_\mu)$. The total complexity is no more than $O(n^4)$. After this, we find all cyclic classes in each \mathcal{C}_μ , which has total complexity $O(n^2)$, and hence the cyclicities γ_σ of all components of the graphs \mathcal{C}_μ . At this stage we can also find a scaling which leads to a total S -visualization of A (and hence all A_μ). This relies on Floyd-Warshall method applied to each S_μ and takes no more than $O(n^3)$ operations in total.

By Theorem 4.2, A_μ^t admit Nachtigall expansion for all μ and all $t \geq 3n^2$. The rows and columns of A_μ^t with indices in N_μ are determined at $t \geq 3n^2$ only by $\mathcal{N}_\mu^t = C_\mu S_\mu^t R_\mu$, since by construction these rows and columns are zero in all terms $\mathcal{N}_\nu^{(t)}$ for $\nu > \mu$. This means in particular that these rows and columns become periodic after $3n^2$ time. By repeated squaring $A_\mu, A_\mu^2, A_\mu^4, \dots$, we reach a power A_μ^r with $r \geq 3n^2$, which requires no more than $O(n^3 \log n)$ operations. Now we can use Corollary 3.7 identifying $C_\mu S_\mu^r$ and $S_\mu^r R_\mu$ as submatrices extracted from columns, resp. rows, of A_μ^r with indices in N_μ . By Theorem 3.6 we can obtain $S_\mu^t R_\mu$ from $S_\mu^r R_\mu$ and C_μ from $C_\mu S_\mu^r$ by the permutation on cyclic classes determined by the remainders $r \pmod{\gamma_\sigma}$ and $(t-r) \pmod{\gamma_\sigma}$, for each cyclicity γ_σ of a component of \mathcal{C}_μ . This takes $O(n^2)$ overrides. Finally we compute $\mathcal{N}_\mu^t = C_\mu S_\mu^t R_\mu$ ($O(n^3)$ operations). We conclude that the total complexity for all μ does not exceed $O(n^4 \log n)$ operations.

(P2): It is clear that the computation of all prerequisites for the ultimate expansion is done like in the first para of the proof of (P1), and takes no more than $O(n^4)$ operations. After that we use Proposition 5.4 which means that the critical rows and columns in each $(A_\mu^\circ)^r$ for $r \geq 3n^2$ are determined only by $\mathcal{U}_\mu^{(r)}$. Hence the factors of each $\mathcal{U}_\mu^{(t)}$ can be computed by matrix squaring of A_μ° , followed by a permutation on cyclic classes and matrix multiplication, which overall takes no more than $O(n^3 \log n)$ operations. We conclude that the total complexity for all μ does not exceed $O(n^4 \log n)$ operations. \square

7. Orbit periodic matrices. Being motivated by the results of Butkovič et al. [5] on robust matrices, we are going to derive necessary and sufficient conditions for orbit periodicity and to show that they can be verified in polynomial time.

A matrix $A \in \mathbb{R}_+^{n \times n}$ is called *orbit periodic* if for each $y \in \mathbb{R}_+^n$ there exists $\lambda(y) \in \mathbb{R}_+^n$ such that $A^{t+\gamma^\circ} y = (\lambda(y))^{\gamma^\circ} A^t y$ for all sufficiently large t , where (as above) γ° is the joint cyclicity (l.c.m.) of the critical graphs of all components of $\mathcal{D}(A)$.

A sequence $\{A^t y, t \geq 0\}$ with the above property will be called *ultimately linear periodic* and $\lambda(y)$ will be called its *ultimate growth rate*. The same wording will be used for the sequences $\{a_{ij}^t, t \geq 0\}$. We say that a subsequence $\{a_{ij}^{l+\gamma^\circ s}, s \geq 0\}$ has ultimate growth rate λ , if there exists $\alpha_{ij} \neq 0$ such that $a_{ij}^{l+\gamma^\circ s} = \alpha_{ij} \lambda^{l+\gamma^\circ s}$ for all s starting from a sufficiently large number.

It may seem more general if in the above definition of linear periodicity we replace γ° by $\gamma(y)$. But using the ultimate expansion (5.8) we conclude that $\{A^t y, t \geq 0\}$ is ultimately linear periodic if and only if there exists $\mu \in \Sigma$ such that $A^t y \stackrel{T}{=} \lambda_\mu^t \mathcal{U}_\mu^{(t)} y$. As $\mathcal{U}_\mu^{(t+\gamma^\circ)} = \mathcal{U}_\mu^{(t)}$ for all μ and t , we conclude that the exact period of $\{A^t y, t \geq 0\}$ has to divide γ° .

The ultimate expansion leads to the following properties of the sequences $\{a_{ij}^t, t \geq 0\}$, already known in max algebra [8, 15, 19]. As above, $\lambda(P)$ is the largest m.c.g.m. of the components of $\mathcal{D}(A)$ visited by P , and $\Pi_{ij,t}$ denotes the set of paths of length t connecting i to j .

LEMMA 7.1. *For each $l: 0 \leq l < \gamma^\circ$, the subsequence $\{a_{ij}^{l+\gamma^\circ s}, s \geq 0\}$ is ultimately zero or has an ultimate growth rate λ_{ij} . In the latter case, for each such l and each sufficiently large $t \equiv l \pmod{\gamma^\circ}$ there exists $P \in \Pi_{ij,t}$ such that $\lambda(P) = \lambda_{ij}$.*

Proof. The ultimate expansion (5.8) implies that for each l there exists $\mu \in \Sigma$ such that $a_{ij}^{l+\gamma^\circ s} = \lambda_\mu^{l+\gamma^\circ s} (\mathcal{U}_\mu^{(l)})_{ij}$ at sufficiently large s , so the subsequence has growth rate $\lambda_\mu = \lambda_{ij}$. The second part of the statement follows from Proposition 5.5. \square

LEMMA 7.2. *If $P \in \Pi_{ij,l}$ then $\{a_{ij}^{l+\gamma^\circ s}, s \geq 0\}$ has ultimate growth rate at least $\lambda(P)$.*

Proof. Using Proposition 5.5 part 4, we obtain that $(\mathcal{U}_\mu^{(l)})_{ij} \neq 0$ for μ such that $\lambda_\mu = \lambda(P)$, and then $a_{ij}^{l+\gamma^\circ s} \geq (\lambda(P))^{l+\gamma^\circ s} (\mathcal{U}_\mu^{(l)})_{ij}$ at sufficiently large s . \square

Denote by L° the set of nodes in the nontrivial components of $\mathcal{D}(A)$. The next statement follows from the Cyclicity Theorem [1, 16]. For the sake of completeness we deduce it from the ultimate expansion.

LEMMA 7.3. *If $i, j \in L^\circ$ belong to the same component of $\mathcal{D}(A)$, then $\{a_{ij}^t, t \geq 0\}$ is ultimately linear periodic and its growth rate is the m.c.g.m. of that component. Further, for each $i \in L^\circ$ and all t there exist k and l in the same component of $\mathcal{D}(A)$ such that $a_{ik}^t \neq 0$ and $a_{li}^t \neq 0$.*

Proof. All paths which connect i to j are μ -hard, with $\lambda_\mu = \lambda(i) = \lambda(j)$. Using Theorem 5.1 we obtain that only $(\mathcal{U}_\mu^{(t)})_{ij}$ is nonzero and hence the ultimate expansion reads $a_{ij}^t \stackrel{T}{=} \lambda_\mu^t (\mathcal{U}_\mu^{(t)})_{ij}$. For the second part note that i belongs to a cycle with nonzero weight. \square

If i is connected to j by a path, we denote this by $i \rightarrow j$. Observe that if $i \rightarrow j$ then also $k \rightarrow l$ for each k in the same component of $\mathcal{D}(A)$ with i and for each l in the same component of $\mathcal{D}(A)$ with j . We also denote $i \leftrightarrow j$ if both $i \rightarrow j$ and $j \rightarrow i$ (i.e., if i and j are in the same component of $\mathcal{D}(A)$). In the next theorem we describe, in terms of such relations, when the sequences of columns $\{A^t e_i, t \geq 0\}$ are ultimately linear periodic.

PROPOSITION 7.4. *Let $A \in \mathbb{R}_+^{n \times n}$ and $j \in L^\circ$. The sequence $\{A^t e_j, t \geq 0\}$ is ultimately linear periodic if and only if for all $i \in L^\circ$, $i \rightarrow j$ implies $\lambda(i) \leq \lambda(j)$.*

Proof. The “only if” part: Lemma 7.3 implies that for each t there exists k such that $a_{kj}^t \neq 0$, and the sequence $\{a_{kj}^{t+s\gamma^\circ}, s \geq 0\}$ has ultimate growth rate $\lambda(j)$. If the condition does not hold, there exists a path P leading from i to j such that $\lambda(P) = \lambda(i) > \lambda(j)$, and by Lemma 7.2 there is a subsequence of $\{a_{ij}^l, l \geq 0\}$ with ultimate growth rate at least $\lambda(i)$.

The “if” part: If the sequence $\{A^t e_j, t \geq 0\}$ is not ultimately linear periodic, then some of its entries have ultimate growth rate $\lambda(j)$ and there is a subsequence of $\{a_{kj}^t, t \geq 0\}$, for some $k \in N$, which has a different ultimate growth rate. Lemma 7.1 implies that this growth rate has to be greater than $\lambda(j)$, and must be the m.c.g.m. of a component which has access to j . \square

We denote $i \rightrightarrows j$ and say that i *strongly accesses* j , if i can be connected to j by a path of any length starting from a certain number T_{ij} . For example, i strongly accesses j if $i \rightarrow j$ and the cyclicities of the components of $\mathcal{D}(A)$ containing i and j are coprime. Observe that if $i \rightrightarrows j$ then also $k \rightrightarrows l$ for each $k \leftrightarrow i$ and $l \leftrightarrow j$. Now we show that strong access relations are essential for the ultimate linear periodicity of all sequences $\{A^t(e_i \oplus e_j), t \geq 0\}$.

PROPOSITION 7.5. *Let $A \in \mathbb{R}_+^{n \times n}$ and $i, j \in L^\circ$. Suppose that $\{A^t e_i, t \geq 0\}$ and $\{A^t e_j, t \geq 0\}$ are ultimately linear periodic. Then $\{A^t(e_k \oplus e_l), t \geq 0\}$ is also ultimately linear periodic for all $k \leftrightarrow i$ and $l \leftrightarrow j$, if and only if either $i \rightrightarrows j$, or $j \rightrightarrows i$, or both are false but $\lambda(i) = \lambda(j)$.*

Proof. If $\lambda(i) = \lambda(j)$ then both $\{A^t e_k, t \geq 0\}$ and $\{A^t e_l, t \geq 0\}$ for all $k \leftrightarrow i$ and $l \leftrightarrow j$ have this growth rate and $\{A^t(e_k \oplus e_l), t \geq 0\}$ is ultimately linear periodic with this growth rate. So it remains to consider the case $\lambda(i) < \lambda(j)$. In this case Proposition 7.4 implies $j \not\rightrightarrows i$, and therefore we have to show that $\{A^t(e_k \oplus e_l), t \geq 0\}$ are ultimately linear periodic for all $k \leftrightarrow i$ and $l \leftrightarrow j$ if and only if $i \rightrightarrows j$.

The “if” part: For each $t: 0 \leq t < \gamma^\circ$, if there exists $m \in N$ and $s_1 \geq 0$ such that $a_{mk}^{t+s_1\gamma^\circ} \neq 0$, then there exists a path $P \in \Pi_{mk, t+s_1\gamma^\circ}$. As $k \rightrightarrows l$, this path can be joined with a path from k to l of length $s_2\gamma^\circ$, and we get that $a_{ml}^{t+(s_1+s_2)\gamma^\circ} \neq 0$. Using Lemma 7.2 we obtain $a_{ml}^{t+s\gamma^\circ} \neq 0$ for all sufficiently large s and it dominates over $a_{mk}^{t+s\gamma^\circ}$ since it has larger growth rate.

The “only if” part: The ultimate linear periodicity of $\{A^t(e_k \oplus e_j), t \geq 0\}$, for any $k \leftrightarrow i$, implies that $\text{supp}(A^t e_k) \subseteq \text{supp}(A^t e_j)$ for all large enough t . By Lemma 7.3 there is $k \leftrightarrow i$ such that $(A^t)_{ik} \neq 0$, hence also $(A^t)_{ij} \neq 0$. As we reasoned for any t , it follows that $i \rightrightarrows j$. \square

THEOREM 7.6 (Orbit periodicity). *$A \in \mathbb{R}_+^{n \times n}$ is orbit periodic if and only if the following conditions hold for all $i, j \in L^\circ$:*

1. $i \rightarrow j$ implies $\lambda(i) \leq \lambda(j)$,
2. if neither $j \rightrightarrows i$ nor $i \rightrightarrows j$ then $\lambda(i) = \lambda(j)$,

or equivalently if $\{A^t(e_i \oplus e_j), t \geq 0\}$ are ultimately linear periodic for all $i, j \in L^\circ$.

Proof. We need only prove that 1. and 2. are sufficient for orbit periodicity, the rest relies on Propositions 7.4 and 7.5.

Let $\mathcal{D}(A, y)$ be the subgraph induced by the set of nodes that have access to $\text{supp}(y) := \{i : y_i \neq 0\}$. Nontrivial components \mathcal{D}_σ of $\mathcal{D}(A, y)$ are ordered by relation $\mathcal{D}_{\sigma_1} \preceq \mathcal{D}_{\sigma_2}$ if $i \rightrightarrows j$ for some (and hence all) $i \in \mathcal{D}_{\sigma_1}$ and $j \in \mathcal{D}_{\sigma_2}$. Consider the maximal components with respect to this relation, by conditions 1. and 2. they must have the same m.c.g.m. and it must be the greatest one. We denote this m.c.g.m. by λ and show that it is the ultimate growth rate of $A^t y$.

By Lemma 7.1 for each $l: 0 \leq l < \gamma$, the subsequence $\{a_{ij}^{l+s\gamma^\circ}, s \geq 0\}$ has a certain ultimate growth rate if it is not ultimately zero. We have to show that λ is the maximal growth rate of $\{a_{ik}^{l+s\gamma^\circ}, s \geq 0\}$ over $k \in \text{supp}(y)$, for every fixed $i \in N$ and $l: 0 \leq l < \gamma^\circ$. Then it follows that $A^{t+\gamma^\circ} y \stackrel{T}{=} \lambda^{\gamma^\circ} A^t y$.

To avoid trivialities we assume that there exists $k \in \text{supp}(y)$ and such that $\{a_{ik}^{l+s\gamma^\circ}, s \geq 0\}$ is not ultimately zero. Then for some $k \in \text{supp}(y)$ there exists a path $P \in \Pi_{ik,t}$ where $t \equiv l \pmod{\gamma^\circ}$ which visits a nontrivial component \mathcal{D}_{σ_1} of $\mathcal{D}(A, y)$. If the m.c.g.m. of \mathcal{D}_{σ_1} is λ , then by Lemma 7.2 the growth rate of $\{a_{ik}^{l+s\gamma^\circ}, s \geq 0\}$ is not less than λ , hence it must be λ and we are done. If the m.c.g.m. of \mathcal{D}_{σ_1} is less than λ , then \mathcal{D}_{σ_1} strongly accesses some component \mathcal{D}_{σ_2} with m.c.g.m. λ , and \mathcal{D}_{σ_2} accesses a node $k' \in \text{supp}(y)$. Due to the strong access we can adjust the length of the path from \mathcal{D}_{σ_1} to \mathcal{D}_{σ_2} if necessary, and we obtain a path $P' \in \Pi_{ik',t'}$ where $t' \equiv l \pmod{\gamma^\circ}$. By Lemma 7.2 we obtain that the growth rate of $\{a_{ik'}^{l+s\gamma^\circ}, s \geq 0\}$ is not less than λ , hence it must be λ . \square

To assess the computational feasibility of condition 2. in Theorem 7.6 we need the following observation which uses the ultimate expansion, see Theorem 5.6 and Section 5.

THEOREM 7.7. *Let $A \in \mathbb{R}_+^{n \times n}$ be such that the sequences $\{A^t e_i, t \geq 0\}$ are ultimately linear periodic for all $i \in L^\circ$. Then A is orbit periodic if and only if the following holds for all $\mu, \nu \in \Sigma$:*

$$\lambda_\mu < \lambda_\nu \Rightarrow \bigcup_{i \in N_\mu^\circ} \text{supp}(\mathcal{U}_\mu^{(1)} e_i) \subseteq \bigcap_{j \in N_\nu^\circ} \text{supp}(\mathcal{U}_\nu^{(1)} e_j). \quad (7.1)$$

Proof. If $\{A^t e_i, t \geq 0\}$ is ultimately linear periodic and $i \in L^\circ$, then $A^t e_i \stackrel{T}{=} \lambda_\mu^t \mathcal{U}_\mu^{(t)} e_i$ for $\lambda_\mu = \lambda(i)$, since λ_μ must be the growth rate of $A^t e_i$ and all other terms of the ultimate expansion have different growth rates. Then also $A^t(e_i \oplus e_j) \stackrel{T}{=} \lambda_\mu^t \mathcal{U}_\mu^{(t)} e_i \oplus \lambda_\nu^t \mathcal{U}_\nu^{(t)} e_j$ where $\lambda_\mu = \lambda(i)$ and $\lambda_\nu = \lambda(j)$ (equivalently, $i \in M_\mu^\circ$ and $j \in M_\nu^\circ$). If $\lambda_\mu < \lambda_\nu$ then $\{A^t(e_i \oplus e_j), t \geq 0\}$ is ultimately linear periodic if and only if $A^t(e_i \oplus e_j) \stackrel{T}{=} \lambda_j \mathcal{U}_\nu^{(t)} e_j$. This happens if and only if

$$\lambda_\mu < \lambda_\nu \Rightarrow \text{supp}(\mathcal{U}_\mu^{(t)} e_i) \subseteq \text{supp}(\mathcal{U}_\nu^{(t)} e_j) \quad \forall i \in M_\mu^\circ, j \in M_\nu^\circ, \quad (7.2)$$

holds for all $\mu, \nu \in \Sigma$.

Corollary 3.8 implies that any column of $\mathcal{U}_\mu^{(t)}$, resp. $\mathcal{U}_\nu^{(t)}$ is a max-linear combination of columns with indices in N_μ° , resp. N_ν° , so that the support of that column is the union of supports of certain columns with indices in N_μ° , resp. N_ν° . Hence we need

to check the support inclusions only for $i \in N_\mu^\circ$ and $j \in N_\nu^\circ$. Further, Corollary 3.7 implies that the columns of $\mathcal{U}_\mu^{(t)}$ (or resp. $\mathcal{U}_\nu^{(t)}$) with indices in N_μ° (or resp. N_ν°) just permute as t changes, so the inclusions need be verified only for $t = 1$. This shows that (7.2) is equivalent to

$$\lambda_\mu < \lambda_\nu \Rightarrow \text{supp}(\mathcal{U}_\mu^{(t)} e_i) \subseteq \text{supp}(\mathcal{U}_\nu^{(t)} e_j) \quad \forall i \in N_\mu^\circ, j \in N_\nu^\circ, \quad (7.3)$$

which is equivalent to (7.1). The claim is proved. \square

Now we give a polynomial bound on the computational complexity of verifying the orbit periodicity of a reducible matrix.

THEOREM 7.8. *Let $A \in \mathbb{R}_+^{n \times n}$. Suppose that all components of $\mathcal{D}(A)$ and access relations between them are known, and λ_μ and $\mathcal{U}_\mu^{(1)}$ are computed for all $\mu \in \Sigma$. Then the orbit periodicity of A can be verified in no more than $O(n^3)$ operations.*

Proof. For all pairs μ, ν , we must verify condition 1. of Theorem 7.6 and condition (7.1). The first of these conditions is verified for the pairs of components of $\mathcal{D}(A)$ and it takes no more than $O(n^2)$, if all access relations between them are known. To verify condition (7.1), we need to compute all unions $\bigcup_{i \in N_\mu^\circ} \text{supp}(\mathcal{U}_\mu^{(1)} e_i)$ and intersections $\bigcap_{i \in N_\mu^\circ} \text{supp}(\mathcal{U}_\mu^{(1)} e_i)$, which requires $O(n^2)$ operations, and then make $O(n^2)$ comparisons of Boolean vectors, which requires no more than $O(n^3)$ operations. \square

We combine the results of Theorems 6.1 and 7.8.

COROLLARY 7.9. *Given $A \in \mathbb{R}_+^{n \times n}$, it takes no more than $O(n^4 \log n)$ operations to verify whether it is orbit periodic or not.*

8. Examples. All examples in this section will be in the max-plus setting $\mathbb{R}_{\max,+} := (\mathbb{R} \cup \{-\infty\}, \oplus = \max, \otimes = +)$.

Example 1. We construct the canonical Nachtigall expansion of A^t for

$$A = \begin{pmatrix} -1 & 0 & -7 & -6 \\ 0 & -1 & -5 & -4 \\ -7 & -5 & -1 & -3 \\ -6 & -4 & -3 & -2 \end{pmatrix}. \quad (8.1)$$

We start with $A_1 = A$. The maximal cycle mean is $\lambda_1 = 0$, and the component \mathcal{C}_1 has two nodes 1, 2 and two edges (1, 2) and (2, 1). The cyclicity is $\gamma_1 := 2$.

We proceed by setting the entries in first two rows and columns to $-\infty$, thus obtaining

$$A_2 = \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -1 & -3 \\ -\infty & -\infty & -3 & -2 \end{pmatrix}. \quad (8.2)$$

The maximum cycle mean is $\lambda_2 = -1$, and the component \mathcal{C}_2 has one node 3 and one edge (3, 3). The cyclicity is $\gamma_2 = 1$.

Now we set everything to $-\infty$ except for the entry (4, 4):

$$A_3 = \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -2 \end{pmatrix}. \quad (8.3)$$

The maximum cycle mean is $\lambda_3 = -2$, and the component \mathcal{C}_3 has one node 4 and one edge (4,4). The cyclicity is $\gamma_3 = 1$.

We obtain matrices S_1 , S_2 and S_3 which correspond, respectively, to \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 :

$$\begin{aligned} S_1 &= \begin{pmatrix} -\infty & 0 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, & S_2 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, \\ S_3 &= \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}. \end{aligned} \tag{8.4}$$

These are Boolean matrices in the max-plus setting, with entries $\infty, 0$ instead of $0, 1$.

Further we need to compute Kleene stars $(A^{\gamma_1})^*$, $((A_2 - \lambda_2)^{\gamma_2})^* = (1 + A_2)^*$ and $((A_3 - \lambda_3)^{\gamma_3})^* = (2 + A_3)^*$, and construct matrices C_1, R_1, C_2, R_2, C_3 and R_3 . The critical parts of the Kleene stars are shown below, the rest of the elements being denoted by \cdot as we do not need them:

$$\begin{aligned} (A^2)^* &= \begin{pmatrix} 0 & -1 & -5 & -4 \\ -1 & 0 & -6 & -5 \\ -5 & -6 & \cdot & \cdot \\ -4 & -5 & \cdot & \cdot \end{pmatrix}, \\ (1 \otimes A_2)^* &= \begin{pmatrix} \cdot & \cdot & -\infty & \cdot \\ \cdot & \cdot & -\infty & \cdot \\ -\infty & -\infty & 0 & -2 \\ \cdot & \cdot & -2 & \cdot \end{pmatrix}, & (2 \otimes A_3)^* &= \begin{pmatrix} \cdot & \cdot & \cdot & -\infty \\ \cdot & \cdot & \cdot & -\infty \\ \cdot & \cdot & \cdot & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}. \end{aligned} \tag{8.5}$$

Further we compute the Nachtigall matrices

$$\begin{aligned} \mathcal{N}_1^{(0)} &= C_1 \otimes R_1 = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -1 & 0 & -6 & -5 \\ -5 & -6 & -10 & -9 \\ -4 & -5 & -9 & -8 \end{pmatrix}, \\ \mathcal{N}_1^{(1)} &= C_1 \otimes S_1 \otimes R_1 = \begin{pmatrix} -1 & 0 & -6 & -5 \\ 0 & -1 & -5 & -4 \\ -6 & -5 & -11 & -10 \\ -5 & -4 & -10 & -9 \end{pmatrix}, \\ \mathcal{N}_2^{(0)} &= C_2 \otimes R_2 = \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -2 \\ -\infty & -\infty & -2 & -4 \end{pmatrix}, \\ \mathcal{N}_2^{(0)} &= C_3 \otimes R_3 = \begin{pmatrix} -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}. \end{aligned} \tag{8.6}$$

The Nachtigall expansion starts to work already at $t = 2$. Indeed,

$$\begin{aligned} A^2 &= \begin{pmatrix} 0 & -1 & -5 & -4 \\ -1 & 0 & -6 & -5 \\ -5 & -6 & -2 & -4 \\ -4 & -5 & -4 & -4 \end{pmatrix} = \mathcal{N}_1^{(0)} \oplus (-2) \otimes \mathcal{N}_2^{(0)} \oplus (-4) \otimes \mathcal{N}_3^{(0)}, \\ A^3 &= \begin{pmatrix} -1 & 0 & -6 & -5 \\ 0 & -1 & -5 & -4 \\ -6 & -5 & -3 & -5 \\ -5 & -4 & -5 & -6 \end{pmatrix} = \mathcal{N}_1^{(1)} \oplus (-3) \otimes \mathcal{N}_2^{(0)} \oplus (-6) \otimes \mathcal{N}_3^{(0)}. \end{aligned} \quad (8.7)$$

Starting from $t = 4$ the third term can be forgotten:

$$A^4 = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -1 & 0 & -6 & -5 \\ -5 & -6 & -4 & -6 \\ -4 & -5 & -6 & -8 \end{pmatrix} = \mathcal{N}_1^{(0)} \oplus (-4) \otimes \mathcal{N}_2^{(0)}. \quad (8.8)$$

The ultimate periodic behavior starts after $T(A) = 10$:

$$A^{10} = \begin{pmatrix} 0 & -1 & -5 & -4 \\ -1 & 0 & -6 & -5 \\ -5 & -6 & -10 & -9 \\ -4 & -5 & -9 & -8 \end{pmatrix} = \mathcal{N}_1^{(0)} = \mathcal{U}_1^{(0)}. \quad (8.9)$$

Example 2. The following example will illustrate the ultimate expansion:

$$A = \begin{pmatrix} -2 & 0 & -3 & -7 & -\infty & -\infty & -\infty \\ 0 & -2 & -5 & -7 & -\infty & -\infty & -\infty \\ -9 & -7 & -9 & -8 & -\infty & -\infty & -\infty \\ -9 & -6 & -4 & -4 & -\infty & -\infty & -\infty \\ -8 & -5 & -5 & -4 & -1 & -7 & -5 \\ -7 & -8 & -5 & -6 & -3 & -6 & -8 \\ -6 & -4 & -9 & -3 & -5 & -5 & -5 \end{pmatrix}. \quad (8.10)$$

We compute the elements of the ultimate expansion.

Firstly, $A_1^\circ = A$, $\lambda_1 = 0$, and the component $\mathcal{C}_1^\circ = \mathcal{C}(A)$ consists of two nodes 1, 2 and two edges (1, 2) and (2, 1).

On the next step we set all entries in the first four rows and columns of A to $-\infty$, thus obtaining matrix A_2° . Its *essential* submatrix with finite entries is extracted from the remaining rows and columns 5 to 7:

$$A_2^\circ{}_{ess} = \begin{pmatrix} -1 & -7 & -5 \\ -3 & -6 & -8 \\ -5 & -5 & -5 \end{pmatrix}, \quad 5 \text{ to } 7 \times 5 \text{ to } 7. \quad (8.11)$$

We compute $\lambda_2 = -1$, and the component $\mathcal{C}_2 = \mathcal{C}(A_2)$ consists of the loop (5, 5).

The components \mathcal{C}_1 and \mathcal{C}_2 determine Boolean (i.e., $0, -\infty$) matrices S_1 and S_2 .

We compute $(A^2)^*$ and $(1 \otimes A_2^\circ)^*$:

$$(A^2)^* = \begin{pmatrix} 0 & -2 & -5 & -7 & -\infty & -\infty & -\infty \\ -2 & 0 & -3 & -7 & -\infty & -\infty & -\infty \\ -7 & -9 & 0 & -12 & -\infty & -\infty & -\infty \\ -6 & -8 & -8 & 0 & -\infty & -\infty & -\infty \\ -5 & -6 & -6 & -5 & 0 & -8 & -6 \\ -8 & -7 & -8 & -7 & -4 & 0 & -8 \\ -4 & -6 & -7 & -7 & -6 & -10 & 0 \end{pmatrix}, \quad (8.12)$$

$$(1 \otimes A_2^\circ)_{ess}^* = \begin{pmatrix} 0 & -6 & -4 \\ -2 & 0 & -6 \\ -4 & -4 & 0 \end{pmatrix}, \quad 5 \text{ to } 7 \times 5 \text{ to } 7. \quad (8.13)$$

Next we build matrices C_1°, R_1° and C_2°, R_2° whose essential parts are shown below:

$$C_1^\circ_{ess} = \begin{pmatrix} 0 & -2 & -7 & -6 & -5 & -8 & -4 \\ -2 & 0 & -9 & -8 & -6 & -7 & -6 \end{pmatrix}^T, \quad 1 \text{ to } 7 \times 1 \text{ to } 2, \quad (8.14)$$

$$R_1^\circ_{ess} = \begin{pmatrix} 0 & -2 & -5 & -7 \\ -2 & 0 & -3 & -7 \end{pmatrix}, \quad 1 \text{ to } 2 \times 1 \text{ to } 4.$$

$$C_2^\circ_{ess} = (0 \quad -2 \quad -4)^T, \quad 5 \text{ to } 7 \times 5 \quad (8.15)$$

$$R_2^\circ_{ess} = (0 \quad -6 \quad -4), \quad 5 \times 5 \text{ to } 7.$$

Using (8.14) and (8.15) we compute the ultimate terms $\mathcal{U}_1^{(0)} = C_1^\circ \otimes R_1^\circ$, $\mathcal{U}_1^{(1)} = C_1^\circ \otimes S_1^\circ \otimes R_1^\circ$ and $\mathcal{U}_2^{(0)} = C_2^\circ \otimes R_2^\circ$. The ultimate expansion starts to work at $t = 9$, meaning

$$A^9 = \mathcal{U}_1^{(1)} \oplus (-9) \otimes \mathcal{U}_2^{(0)}, \quad A^{10} = \mathcal{U}_1^{(0)} \oplus (-10) \otimes \mathcal{U}_2^{(0)}, \quad (8.16)$$

$$A^{11} = \mathcal{U}_1^{(1)} \oplus (-11) \otimes \mathcal{U}_2^{(0)}, \quad A^{12} = \mathcal{U}_1^{(0)} \oplus (-12) \otimes \mathcal{U}_2^{(0)}, \dots$$

In this case, the canonical Nachtigall expansion starts to work already at $t = 3$, and after $t = 4$ only the first two terms are essential. In this expansion, matrix A_2 results from setting only the first *two* (instead of four) columns and rows to $-\infty$. The second Nachtigall term \mathcal{N}_2 is equal to $C_2 \otimes R_2$, where $C_2 = C_2^\circ$ and $R_2 \neq R_2^\circ$ has essential part

$$R_2_{ess} = (-4 \quad -3 \quad 0 \quad -6 \quad -4), \quad 5 \times 3 \text{ to } 7. \quad (8.17)$$

Example 3. We illustrate the total periodicity. Let

$$A = \begin{pmatrix} -\infty & 1 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 1 & -\infty & -\infty \\ 1 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -2 & -\infty & -\infty & -\infty & 0 \\ -\infty & -2 & -\infty & -\infty & 0 & -\infty \end{pmatrix} \quad (8.18)$$

$$B = \begin{pmatrix} -\infty & 1 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 1 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 1 & -\infty & -\infty \\ 1 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -2 & -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & -2 & -\infty & 0 & -\infty \end{pmatrix} \quad (8.19)$$

Note that A and B are almost the same, except for the entries $(6, 2)$ and $(6, 3)$. In both cases the ultimate expansion coincides with the canonical Nachtigall expansion, and we have two critical components $\mathcal{C}_1 = (N_1, E_1)$ with $N_1 = \{1, 2, 3, 4\}$, $E_1 = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ and $\mathcal{C}_2 = (N_2, E_2)$ with $N_2 = \{5, 6\}$ and $E_2 = \{(5, 6), (6, 5)\}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0$, and the cyclicities are $\gamma_1 = 4$ and $\gamma_2 = 2$, their l.c.m. is $\gamma = 4$. Note that $M_1^\circ = N_1$ and $M_2^\circ = N_2$.

In both cases the component with λ_1 does not have access to the component with λ_2 which is smaller, hence condition 1. of Theorem 7.6 is true meaning that all columns of A^t and B^t are ultimately periodic. Condition 2. of Theorem 7.6 holds for A but it does not hold for B . In particular, there are only paths of odd length connecting node 6 to node 3. Hence A is orbit periodic and B is not.

Consider also condition (7.1). We need the terms of the ultimate expansion for $t \pmod{\gamma^\circ} = 1$. In each case there are two terms, which we denote by \mathcal{U}_1^A and \mathcal{U}_2^A , resp. \mathcal{U}_1^B and \mathcal{U}_2^B , for the case of A , resp. B . We have

$$\mathcal{U}_1^A = \begin{pmatrix} -\infty & 0 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -4 & -3 & -6 & -5 & -\infty & -\infty \\ -4 & -3 & -6 & -5 & -\infty & -\infty \end{pmatrix} \quad (8.20)$$

$$\mathcal{U}_2^B = \begin{pmatrix} -\infty & 0 & -\infty & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty & -\infty \\ -\infty & -3 & -\infty & -5 & -\infty & -\infty \\ -4 & -\infty & -3 & -\infty & -\infty & -\infty \end{pmatrix} \quad (8.21)$$

$$\mathcal{U}_{2\text{ ess}}^A = \mathcal{U}_{2\text{ ess}}^B = \begin{pmatrix} -\infty & 0 \\ 0 & -\infty \end{pmatrix}, \quad 5 \text{ to } 6 \times 5 \text{ to } 6. \quad (8.22)$$

Observe that $\text{supp}(\mathcal{U}_2^A e_i) \subseteq \text{supp}(\mathcal{U}_1^A e_j)$ for all $i \in M_2^\circ = \{5, 6\}$ and $j \in M_1^\circ = \{1, 2, 3, 4\}$, but this condition does not hold for B .

To see the difference between $A^t x$ and $B^t x$, take $x = [0 \ -\infty \ -\infty \ -\infty \ -\infty \ 0]^T$. The sequence $\{A^t x\}$ is ultimately periodic starting from $t = 4$ with period 4 and growth rate 1. In particular the last component of $\{A^t x\}$ yields the following number sequence for $t \geq 4$:

$$(A^t x)_6 = \{1 \ 1 \ 1 \ 1 \ 5 \ 5 \ 5 \ 5 \ 9 \ 9 \ 9 \ 9 \dots\}, \quad t \geq 4. \quad (8.23)$$

The sequence $\{B^t x\}$ is not ultimately periodic. In particular, the last component of $\{B^t x\}$ yields the following number sequence for $t \geq 2$:

$$(B^t x)_6 = \{0 \ 0 \ 0 \ 1 \ 0 \ 4 \ 0 \ 5 \ 0 \ 8 \ 0 \ 9 \dots\}, \quad t \geq 2, \quad (8.24)$$

which can be expressed as

$$(B^t x)_6 = \begin{cases} 0, & \text{if } t \text{ is even and } t \geq 2, \\ t - 3, & \text{if } t = 4k + 3 \text{ and } k \geq 0, \\ t - 5, & \text{if } t = 4k + 5 \text{ and } k \geq 0. \end{cases} \quad (8.25)$$

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