Matrices leaving a cone invariant

Hans Schneider and Bit-Shun Tam

Generalizations of the Perron-Frobenius theory of nonnegative matrices to linear operators leaving a cone invariant were first developed for operators on a Banach space by Krein and Rutman [KR48], Karlin [Kar59] and Schaefer [Sfr66] although there are early examples in finite dimensions, e.g. [Sch65] and [Bir67]. In this article we describe a generalization, sometimes called the geometric spectral theory of nonnegative linear operators in finite dimensions, which emerged in the late 1980s. Motivated by a search for geometric analogs of results in the previously developed combinatorial spectral theory of (reducible) nonnegative matrices (for reviews see [Sch86] and [Her99]), this area is a study of the Perron-Frobenius theory of a nonnegative matrix and its generalizations from the cone-theoretic viewpoint. The treatment is linear-algebraic and cone-theoretic (geometric) with the facial and duality concepts and occasionally certain elementary analytic tools playing the dominant role. The theory is particularly rich when the underlying cone is polyhedral (finitely generated) and it reduces to the nonnegative matrix case when the cone is simplicial.

1 Perron-Frobenius Theorem for cones

We work with cones in a real vector space, as "cone" is a real concept. To deal with cones in $\mathbb{C}^n$, we can identify the latter space with $\mathbb{R}^{2n}$. For a discussion on the connection between the real and complex case of the spectral theory, see [TS94, Section 8].

DEFINITIONS

A **proper cone** K in a finite-dimensional real vector space V is a closed, pointed, full convex cone, viz.

- $K + K \subseteq K$, viz. $\mathbf{x}, \mathbf{y} \in K \Longrightarrow \mathbf{x} + \mathbf{y} \in K$,
- $\mathbb{R}^+ K \subseteq K$, viz. $\mathbf{x} \in K, \alpha \in \mathbb{R}^+ \Longrightarrow \alpha\mathbf{x} \in K$,
- K is closed in the usual topology of V.
- $K \cap (-K) = \{\mathbf{0}\}$, viz. $\mathbf{x}, -\mathbf{x} \in K \Longrightarrow \mathbf{x} = \mathbf{0}$,
- $\mathrm{int}K \neq \emptyset$, where $\mathrm{int}K$ is the interior of K.

Usually, the unqualified term **cone** is defined by the first two items in the above definition. However, in this section we call a proper cone simply a **cone**. We denote by K a cone in $\mathbb{R}^n$, $n \geq 2$.

The vector $\mathbf{x} \in \mathbb{R}^n$ is K**-nonnegative**, written $\mathbf{x} \geq^K \mathbf{0}$, if $\mathbf{x} \in K$.

The vector $\mathbf{x}$ is K**-semipositive**, written $\mathbf{x} \gneq^K \mathbf{0}$, if $\mathbf{x} \geq^K \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$.

The vector $\mathbf{x}$ is K**-positive**, written $\mathbf{x} >^K \mathbf{0}$, if $\mathbf{x} \in \mathrm{int}\ K$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we write $\mathbf{x} \geq^K \mathbf{y}$ ($\mathbf{x} \gneq^K \mathbf{y},\ \mathbf{x} >^K \mathbf{y}$) if $\mathbf{x} - \mathbf{y}$ is K-nonnegative (K-semipositive, K-positive).

The matrix $A \in \mathbb{R}^{n \times n}$ is K**-nonnegative**, written $A \geq^K \mathbf{0}$, if $AK \subseteq K$.

The matrix A is K**-semipositive**, written $A \gneq^K \mathbf{0}$, if $A \geq^K \mathbf{0}$ and $A \neq \mathbf{0}$.

The matrix A is K**-positive**, written $A >^K \mathbf{0}$, if $A(K \setminus \{\mathbf{0}\}) \subseteq \mathrm{int}\ K$.

For $A, B \in \mathbb{R}^{n \times n}$, $A \geq^K B$ ($A \gneq^K B,\ A >^K B$) means $A - B \geq^K \mathbf{0},\ (A - B \gneq^K \mathbf{0},\ A - B >^K \mathbf{0})$.

A **face** F of a cone $K \subseteq \mathbb{R}^n$ is a subset of K which is a cone in the linear span of F such that $\mathbf{x} \in F,\ \mathbf{x} \geq^K \mathbf{y} \geq^K \mathbf{0} \Longrightarrow \mathbf{y} \in F$.

(In this section, F will always denote a face rather than a field, since the only fields involved are $\mathbb{R}$ and $\mathbb{C}$.) Thus F satisfies all definitions of a cone except that its interior may be empty.

A face F of K is a **trivial face** if $F = \{\mathbf{0}\}$ or $F = K$.

For a subset S of a cone K, the intersection of all faces of K including S is called the **face of K generated by** S and is denoted by $\Phi(S)$. If $S = \{\mathbf{x}\}$, then $\Phi(S)$ is written simply as $\Phi(\mathbf{x})$.

For faces F, G of K, their **meet** and **join** are given respectively by $F \wedge G = F \cap G$ and $F \vee G = \Phi(F \cup G)$.

A vector $\mathbf{x} \in K$ is an **extreme vector** if either $\mathbf{x}$ is the zero vector or $\mathbf{x}$ is nonzero and $\Phi(\mathbf{x}) = \{\lambda\mathbf{x} : \lambda \geq 0\}$; in the latter case, the face $\Phi(\mathbf{x})$ is called an **extreme ray**.

If P is K-nonnegative, then a face F of K is a **P-invariant face** if $PF \subseteq F$.

If P is K-nonnegative, then P is **K-irreducible** if the only P-invariant faces are the trivial faces.

If K is a cone in $\mathbb{R}^n$ then a cone, called the **dual cone** of K, is denoted and given by

$$K^* = \{\mathbf{y} \in \mathbb{R}^n \ : \mathbf{y}^T\mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in K\}.$$

If A is an $n \times n$ complex matrix and $\mathbf{x}$ is a vector in $\mathbb{C}^n$, then the **local spectral radius of A at $\mathbf{x}$** is denoted and given by $\rho_{\mathbf{x}}(A) = \lim\sup_{m\to\infty} \|A^m\mathbf{x}\|^{1/m}$, where $\|\cdot\|$ is any norm of $\mathbb{C}^n$. (For $A \in \mathbb{C}^{n\times n}$, its spectral radius is denoted by $\rho(A)$ (or ρ), cf. §1.4.3.)

FACTS

Let K be cone in $\mathbb{R}^n$.

1. The condition $\text{int}K \neq \emptyset$ in the definition of a cone is equivalent to $K - K = V$, viz. for all $\mathbf{z} \in V$ there exist $\mathbf{x}, \mathbf{y} \in K$ such that $\mathbf{z} = \mathbf{x} - \mathbf{y}$.

2. A K-positive matrix is K-irreducible.

3. [Van68], [SV70] Let P be a K-nonnegative matrix. The following are equivalent:

   (a) P is K-irreducible.

(b) $\sum_{i=0}^{n-1} P^i >^K \mathbf{0}$.

(c) $(I + P)^{n-1} >^K \mathbf{0}$,

(d) No eigenvector of P (for any eigenvalue) lies on the boundary of K.

4. (Generalization of Perron-Frobenius Theorem) [KR48], [BS75] Let P be a K-irreducible matrix with spectral radius ρ. Then

   (a) ρ is positive and is a simple eigenvalue of P,

   (b) There exists an (up to a scalar multiple) unique K-positive (right) eigenvector $\mathbf{u}$ of P corresponding to ρ,

   (c) $\mathbf{u}$ is the only K-semipositive eigenvector for P (for any eigenvalue),

   (d) $K \cap (\rho I - P)\mathbb{R}^n = \{\mathbf{0}\}$.

5. (Generalization of Perron-Frobenius Theorem) Let P be a K-nonnegative matrix with spectral radius ρ. Then

   (a) ρ is an eigenvalue of P.

   (b) There is a K-semipositive eigenvector of P corresponding to ρ.

6. If P, Q are K-nonnegative and $Q \overset{K}{\leq} P$, then $\rho(Q) \leq \rho(P)$. Further, if P is K-irreducible and $Q \overset{K}{\lneq} P$, then $\rho(Q) < \rho(P)$.

7. P is K-nonnegative (K-irreducible) if and only if P^T is K^*-nonnegative (K^*-irreducible).

8. If A is an $n \times n$ complex matrix and $\mathbf{x}$ is a vector in $\mathbf{C}^n$, then the local spectral radius $\rho_{\mathbf{x}}(A)$ of A at $\mathbf{x}$ is equal to the spectral radius of the restriction of A to the A-cyclic subspace generated by $\mathbf{x}$, i.e., span$\{A^i\mathbf{x} : i = 0, 1, \ldots\}$. If $\mathbf{x}$ is nonzero and $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ is the representation of $\mathbf{x}$ as a sum of generalized eigenvectors of A corresponding respectively to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\rho_{\mathbf{x}}(A)$ is also equal to $\max_{1 \leq i \leq k} |\lambda_i|$.

9. Barker and Schneider[BS75] developed Perron-Frobenius theory in the setting of a (possibly infinite-dimensional) vector space over a fully ordered field without topology. They introduced the concepts of irreducibility and strong irreducibility, and show that these two concepts are equivalent if the underlying cone has ascending chain condition on faces. See [ERS95] for the role of real closed ordered fields in this theory.

EXAMPLES

1. The nonnegative orthant $(\mathbb{R}_0^+)^n$ in $\mathbb{R}^n$ is a cone. Then $\mathbf{x} \geq^K \mathbf{0}$ if and only if $\mathbf{x} \geq \mathbf{0}$, viz. the entries of $\mathbf{x}$ are nonnegative, and F is face of $(\mathbb{R}_0^+)^n$ if and only if F is of the form F_J for some $J \subseteq \{1, \ldots, n\}$, where

$$F_J = \{\mathbf{x} \in (\mathbb{R}_0^+)^n \ : \ x_i = 0, \ i \notin J\}.$$

   Further, $P \geq^K \mathbf{0}$ ($P \gneq^K \mathbf{0}, \ P >^K \mathbf{0}, \ P$ is K-irreducible) if and only if $P \geq \mathbf{0}$ ($P \gneq \mathbf{0}, \ P > \mathbf{0}, \ P$ is irreducible) in the sense used for nonnegative matrices, cf. §2.4.

2. The non-trivial faces of the Lorentz (ice-cream) cone K_n in $\mathbb{R}^n$, viz.

$$K_n = \{\mathbf{x} \in \mathbb{R}^n : (x_1^2 + \cdots + x_{n-1}^2)^{1/2} \leq x_n\},$$

   are precisely its extreme rays, each generated by a nonzero boundary vector, that is one for which the equality holds above. The matrix

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   is K_3-irreducible [BP79, p.22].

2 Collatz-Wielandt sets and distinguished eigenvalues

Collatz-Wielandt sets were apparently first defined in [BS75]. However they are so called because they are closely related to Wielandt's proof of the Perron-Frobenius theorem for irreducible nonnegative matrices, [Wie50], which employs an inequality found in Collatz [Col42]. See also [Sch96] for further remarks on Collatz-Wielandt sets and related max-min and min-max characterizations of the spectral radius of nonnegative matrices and their generalizations.

DEFINITIONS

Let P be a K-nonnegative matrix.

The **Collatz-Wielandt sets** associated with P [BS75], [TW89], [TS01], [TS03], [Tam01] are defined by

$$\begin{aligned}
\Omega(P) &= \{\omega \geq 0 : \ \exists \mathbf{x} \in K \backslash \{\mathbf{0}\}, \ P\mathbf{x} \geq^K \omega \mathbf{x}\}. \\
\Omega_1(P) &= \{\omega \geq 0 : \ \exists \mathbf{x} \in \text{int } K, \ P\mathbf{x} \geq^K \omega \mathbf{x}\}. \\
\textstyle\sum(P) &= \{\sigma \geq 0 : \ \exists \mathbf{x} \in K \backslash \{\mathbf{0}\}, \ P\mathbf{x} \ {}^K\!\!\leq \sigma \mathbf{x}\}. \\
\textstyle\sum_1(P) &= \{\sigma \geq 0 : \ \exists \mathbf{x} \in \text{int } K, \ P\mathbf{x} \ {}^K\!\!\leq \sigma \mathbf{x}\}.
\end{aligned}$$

For a K-nonnegative vector $\mathbf{x}$, the **lower and upper Collatz-Wielandt numbers** of $\mathbf{x}$ with respect to P are defined by

$$\begin{aligned}
r_P(x) &= \sup \ \{\omega \geq 0 : \ P\mathbf{x} \geq^K \omega \mathbf{x}\}, \\
R_P(x) &= \inf \ \{\sigma \geq 0 : \ P\mathbf{x} \ {}^K\!\!\leq \sigma \mathbf{x}\},
\end{aligned}$$

where we write $R_P(\mathbf{x}) = \infty$ if no σ exists such that $P\mathbf{x} \ {}^K\!\!\leq \sigma \mathbf{x}$.

A (nonnegative) eigenvalue of P is a **distinguished eigenvalue** for K if it has an associated K-semipositive eigenvector.

The **Perron space** $N_\rho^\nu(P)$ (or N_ρ^ν) is the subspace consisting of all $\mathbf{u} \in \mathbb{R}^n$ such that $(P - \rho I)^k \mathbf{u} = \mathbf{0}$ for some positive integer k. (See §2.1.1 for a more general definition of $N_\lambda^\nu(A)$.)

If F is a P-invariant face of K then the restriction of P to span$F$ is written as $P|F$. The spectral radius of $P|F$ is written as $\rho[F]$ and if λ is an eigenvalue of $P|F$ its index is written as $\nu_\lambda[F]$.

A cone K in $\mathbb{R}^n$ is **polyhedral** if it is the set of linear combinations with nonnegative coefficients of vectors taken from a finite subset of $\mathbb{R}^n$, and is **simplicial** if the finite subset is linearly independent.

FACTS

Let P be a K-nonnegative matrix.

1. [TW89] A real number λ is a distinguished eigenvalue of P for K if and only if $\lambda = \rho_{\mathbf{b}}(P)$ for some K-semipositive vector $\mathbf{b}$.

2. [Tam90] Consider the following conditions:

   (a) ρ is the only distinguished eigenvalue of P for K,

   (b) $\mathbf{x} \geq^K \mathbf{0}$ and $P\mathbf{x} \;{}^K\!\!\leq \rho\mathbf{x}$ imply that $P\mathbf{x} = \rho\mathbf{x}$,

   (c) The Perron space of P^T contains a K^*-positive vector.

   (d) $\rho \in \Omega_1(P^T)$.

   Conditions (a), (b) and (c) are always equivalent and are implied by condition (d). When K is polyhedral, condition (d) is also an equivalent condition.

3. [Tam90] The following conditions are equivalent:

   (a) $\rho(P)$ is the only distinguished eigenvalue of P for K and the index of $\rho(P)$ is one.

   (b) For any vector $\mathbf{x} \in \mathbb{R}^n$, $P\mathbf{x} \;{}^K\!\!\leq \rho(P)\mathbf{x}$ implies that $P\mathbf{x} = \rho(P)\mathbf{x}$.

   (c) $K \cap (\rho I - P)\mathbb{R}^n = \{\mathbf{0}\}$.

   (d) P^T has a K^*-positive eigenvector (corresponding to $\rho(P)$).

4. [TW89] The following statements all hold:

   (a) [BS75] If P is K-irreducible, then

   $\sup \Omega(P) = \sup \Omega_1(P) = \inf \sum(P) = \inf \sum_1(P) = \rho(P)$.

   (b) $\sup \Omega(P) = \inf \sum_1(P) = \rho(P)$.

   (c) $\inf \sum(P)$ is equal to the least distinguished eigenvalue of P for K.

   (d) $\sup \Omega_1(P) = \inf \sum(P^T)$, and hence is equal to the least distinguished eigenvalue of P^T for K^*.

   (e) $\sup \Omega(P) \in \Omega(P)$ and $\inf \sum(P) \in \sum(P)$.

   (f) When K is polyhedral, we have $\sup \Omega_1(P) \in \Omega_1(P)$. For general cones, we may have $\sup \Omega_1(P) \notin \Omega_1(P)$.

   (g) [Tam90] When K is polyhedral, $\rho(P) \in \Omega_1(P)$ if and only if $\rho(A)$ is the only distinguished eigenvalue of P^T for K^*.

   (h) [TS03] $\rho(P) \in \sum_1(P)$ if and only if $\Phi((N^1_\rho(P) \cap K) \cup C) = K$, where C is the set $\{\mathbf{x} \in K : \rho_{\mathbf{x}}(P) < \rho(P)\}$ and $N^1_\rho(P)$ is the Perron eigenspace of P.

5. In the irreducible nonnegative matrix case, statement (b) of the preceding fact reduces to the well-known max-min and min-max characterizations of $\rho(P)$ due to Wielandt. Schaefer [Sfr84] generalized the result to irreducible compact operators in L^p-spaces and more recently Friedland [Fri90], [Fri91] also extended the characterizations in the settings of a Banach space or a C^*-algebra.

6. [TW89, Theorem 2.4(i)] For any $\mathbf{x} \geq^K \mathbf{0}, r_P(\mathbf{x}) \leq \rho_{\mathbf{x}}(P) \leq R_P(\mathbf{x})$. (This fact extends the well-known inequality $r_P(\mathbf{x}) \leq \rho(P) \leq R_P(\mathbf{x})$ in the nonnegative matrix case, due to Collatz[Col42] under the assumption that $\mathbf{x}$ is a positive vector and due to Wielandt[Wie50] under the assumption that P is irreducible and $\mathbf{x}$ is semipositive. For similar results concerning a nonnegative linear continuous operator in a Banach space, see [FN89].)

7. A discussion on estimating $\rho(P)$ or $\rho_x(P)$ by a convergent sequence of (lower or upper) Collatz-Wielandt numbers can be found in [TW89, Section 5], [Tam01, Subsection 3.1.4].

8. [GKT95, Corollary 3.2] If K is strictly convex (i.e. each boundary vector is extreme), then P has at most two distinguished eigenvalues. This fact supports the statement that the spectral theory of nonnegative linear operators depends on the geometry of the underlying cone.

3 The peripheral spectrum, the core, and the Perron-Schaefer condition

In addition to using Collatz-Wielandt sets to study Perron-Frobenius theory we may also approach this theory by considering the core (whose definition will be given below). This geometric approach started with the work of Pullman[Pul71] who succeeded in rederiving the Frobenius theorem for irreducible nonnegative matrices. Naturally, this approach was also taken up in geometric spectral theory. It was found that there are close connections between the core, the peripheral spectrum, the Perron-Schaefer condition, and the distinguished faces of a K-nonnegative linear operator. This led to a revival of interest in the Perron-Schaefer condition and associated conditions for the existence of a cone K such that a preassigned matrix is K-nonnegative. (see [Bir67], [Sfr66], [Van68], [Sch81]). The study has also led to the identification of necessary and equivalent conditions for a collection of Jordan blocks to correspond to the peripheral eigenvalues of a nonnegative matrix (see [TS94] and [McD03]). The local Perron-Schaefer condition was identified in [TS01] and has played a role in the subsequent work. In the course of this investigation methods were found for producing invariant cones for a matrix with the Perron-Schaefer condition, see [TS94], [Tam06]. These constructions may also be useful in the study of allied fields, such as linear dynamical systems. There invariant cones for matrices are often encountered (see, for instance, [BNS89]).

DEFINITIONS

If P is K-nonnegative, then a nonzero P-invariant face F of K is a **distinguished face** (associated with λ) if for every P-invariant face G with $G \subset F$ we have $\rho[G] < \rho[F]$ (and $\rho[F] = \lambda$).

If λ is an eigenvalue of $A \in \mathbf{C}^{n\times n}$, then $\ker(A - \lambda I)^k$ is denoted by $N_\lambda^k(A)$ for $k = 1, 2, \ldots$, the **index** of λ is denoted by $\nu_A(\lambda)$ (or ν_λ when A is clear), and the **generalized eigenspace** at λ is denoted by $N_\lambda^\nu(A)$. See §2.1.1 for more information.

Let $A \in \mathbb{C}^{n\times n}$.

The **order** of a generalized eigenvector $\mathbf{x}$ for λ is the smallest positive integer k such that $(A-\lambda I)^k\mathbf{x} = \mathbf{0}$. The maximal order of all K-semipositive generalized eigenvectors in $N_\lambda^\nu(A)$ is denoted by ord_λ.

The matrix A satisfies the **Perron-Schaefer condition** ([Sfr66], [Sch81]) if

- $\rho = \rho(A)$ is an eigenvalue of A,
- If λ is an eigenvalue of A and $|\lambda| = \rho$, then $\nu_A(\lambda) \le \nu_A(\rho)$.

If K is a cone and P is K-nonnegative, then the set $\bigcap_{i=0}^{\infty} P^i K$, denoted by $\mathrm{core}_K(P)$, is called the **core** of P relative to K.

An eigenvalue λ of A is called **a peripheral eigenvalue** if $|\lambda| = \rho(A)$. The peripheral eigenvalues of A constitute the **peripheral spectrum** of A.

Let $\mathbf{x} \in \mathbb{C}^n$. Then A satisfies the **local Perron-Schaefer condition at x** if there is a generalized eigenvector $\mathbf{y}$ of A corresponding to $\rho_{\mathbf{x}}(A)$ that appears as a term in the representation of $\mathbf{x}$ as a sum of generalized eigenvectors of A. Furthermore, the order of $\mathbf{y}$ is equal to the maximum of the orders of the generalized eigenvectors that appear in the representation and correspond to eigenvalues with modulus $\rho_{\mathbf{x}}(A)$.

FACTS

1. [Sfr66, Chapter V] Let K be a cone in $\mathbb{R}^n$ and let P be a K-nonnegative matrix. Then P satisfies the Perron-Schaefer condition.

2. [Sch81] Let K be a cone in $\mathbb{R}^n$ and let P be a K-nonnegative matrix with spectral radius ρ. Then P has at least m linearly independent K-semipositive eigenvectors corresponding to ρ, where m is the number of Jordan blocks in the Jordan form of P of maximal size that correspond to ρ.

3. [Van68] Let $A \in \mathbb{R}^{n\times n}$. Then there exists a cone K in $\mathbb{R}^n$ such that A is K-nonnegative if and only if A satisfies the Perron-Schaefer condition.

4. [TS94] Let $A \in \mathbb{R}^{n\times n}$ that satisfies the Perron-Schaefer condition. Let m be the number of Jordan blocks in the Jordan form of A of maximal size that correspond to $\rho(A)$. Then for each positive integer $k, m \leq k \leq$ dim $N_\rho^1(A)$, there exists a cone K in $\mathbb{R}^n$ such that A is K-nonnegative and dim span$(N_\rho^1(A) \cap K) = k$.

5. Let $A \in \mathbb{R}^{n\times n}$. Let k be a nonnegative integer and let $\omega_k(A)$ consist of all linear combinations with nonnegative coefficients of $A^k, A^{k+1}, \ldots$. The closure of $\omega_k(A)$ is a cone in its linear span if and only if A satisfies the Perron-Schaefer condition. (For this fact in the setting of complex matrices see [Sch81].)

6. Necessary and sufficient conditions involving $\omega_k(A)$ so that $A \in \mathbb{C}^{n\times n}$ has a positive (nonnegative) eigenvalue appear in [Sch81]. For the corresponding real versions, see [Tam06].

7. [Pul71],[TS94] If K is a cone and P is K-nonnegative, then $\mathrm{core}_K(P)$ is a cone in its linear span and $P(\mathrm{core}_K(P)) = \mathrm{core}_K(P)$. Furthermore, $\mathrm{core}_K(P)$ is polyhedral (or simplicial) whenever K is. So when $\mathrm{core}_K(P)$ is polyhedral, P permutes the extreme rays of $\mathrm{core}_K(P)$.

8. For a K-nonnegative matrix P, a characterization of K-irreducibility (as well as K-primitivity) of P in terms of $\text{core}_K(P)$, which extends the corresponding result of Pullman for a nonnegative matrix, can be found in [TS94].

9. [Pul71] If P is an irreducible nonnegative matrix, then the permutation induced by P on the extreme rays of $\text{core}_{(\mathbb{R}_0^+)^n}(P)$ is a single cycle of length equal to the number of distinct peripheral eigenvalues of P. (This fact can be regarded as a geometric characterization of the said quantity (cf. the known combinatorial characterization, see Fact 5(c) of §2.4.2), whereas part(b) of the next fact is its extension.)

10. [TS94, Theorem 3.14] For a K-nonnegative matrix P, if $\text{core}_K(P)$ is a nonzero simplicial cone then:

    (a) There is a one-to-one correspondence between the set of distinguished faces associated with nonzero eigenvalues and the set of cycles of the permutation τ_P induced by P on the extreme rays of $\text{core}_K(P)$.

    (b) If σ is a cycle of the induced permutation τ_P, then the peripheral eigenvalues of the restriction of P to the linear span of the distinguished P-invariant face F corresponding to σ are simple and are exactly $\rho[F]$ times all the d_σth roots of unity, where d_σ is the length of the cycle σ.

11. [TS94] If P is K-nonnegative and $\text{core}_K(P)$ is nonzero polyhedral, then:

    (i) $\text{core}_K(P)$ consists of all linear combinations with nonnegative coefficients of the distinguished eigenvectors of positive powers of P corresponding to nonzero distinguished eigenvalues.

    (ii) $\text{core}_K(P)$ does not contain a generalized eigenvector of any positive powers of P other than eigenvectors.

This fact indicates that we cannot expect that the index of the spectral radius of a nonnegative linear operator can be determined from a knowledge of its core.

12. A complete description of the core of a nonnegative matrix (relative to the nonnegative orthant) can be found in [TS94, Theorem 4.2].

13. For $A \in \mathbb{R}^{n\times n}$, in order that there exists a cone K in $\mathbb{R}^n$ such that $AK = K$ and A has a K-positive eigenvector it is necessary and sufficient that A is nonzero, diagonalizable, all eigenvalues of A are of the same modulus and $\rho(A)$ is an eigenvalue of A. For further equivalent conditions, see [TS94, Theorem 5.9].

14. For $A \in \mathbb{R}^{n\times n}$, an equivalent condition given in terms of the peripheral eigenvalues of A so that there exists a cone K in $\mathbb{R}^n$ such that A is K-nonnegative and (i) K is polyhedral, or (ii) $\mathrm{core}_K(A)$ is polyhedral (simplicial or a single ray) can be found in [TS94, Theorems 7.9, 7.8, 7.12, 7.10].

15. [TS94, Theorem 7.12] Let $A \in \mathbb{R}^{n\times n}$ with $\rho(A) > 0$ that satisfies the Perron-Schaefer condition. Let S denote the multi-set of peripheral eigenvalues of A with maximal index (i.e., $\nu_A(\rho)$), the multiplicity of each element being equal to the number of corresponding blocks in the Jordan form of A of order $\nu_A(\rho)$. Let T be the multi-set of peripheral eigenvalues of A for which there are corresponding blocks in the Jordan form of A of order less than $\nu_A(\rho)$, the multiplicity of each element being equal to the number of such corresponding blocks. The following conditions are equivalent:

    (a) There exists a cone K in $\mathbb{R}^n$ such that A is K-nonnegative and $\mathrm{core}_K(A)$ is simplicial.

(b) There exists a multi-subset $\widetilde{T}$ of T such that $S \cup \widetilde{T}$ is the multi-set union of certain complete sets of roots of unity multiplied by $\rho(A)$.

16. McDonald[McD03] refers to the condition(b) that appears in the preceding result as the **Tam-Schneider condition**. She also provides another condition, called the **extended Tam-Schneider condition**, which is necessary and sufficient for a collection of Jordan blocks to correspond to the peripheral spectrum of a nonnegative matrix.

17. [TS01] If P is K-nonnegative and $\mathbf{x}$ is K-semipositive, then P satisfies the local Perron-Schaefer condition at $\mathbf{x}$.

18. [Tam06] Let A be an $n \times n$ real matrix, and let $\mathbf{x}$ be a given nonzero vector of $\mathbb{R}^n$. The following conditions are equivalent:

    (a) A satisfies the local Perron-Schaefer condition at $\mathbf{x}$.

    (b) The restriction of A to $\text{span}\{A^i\mathbf{x} : i = 0, 1, \ldots\}$ satisfies the Perron-Schaefer condition.

    (c) For every (or, for some) nonnegative integer k, the closure of $\omega_k(A, \mathbf{x})$, where $\omega_k(A, \mathbf{x})$ consists of all linear combinations with nonnegative coefficients of $A^k\mathbf{x}, A^{k+1}\mathbf{x}, \ldots$, is a cone in its linear span.

    (d) There is a cone C in a subspace of $\mathbb{R}^n$ containing $\mathbf{x}$ such that $AC \subseteq C$.

19. The local Perron-Schaefer condition has played a role in the work of [TS01], [TS03] and [Tam04]. Further work involving this condition and the cones $\omega_k(A, \mathbf{x})$ (defined in the preceding fact) will appear in [Tam06].

20. One may apply results on the core of a nonnegative matrix to rederive simply many known results on the limiting behavior of Markov chains. An illustration can be found in [Tam01, Section 4.6].

4 Spectral theory of K-reducible matrices

In this subsection we touch upon the geometric version of the extensive combinatorial spectral theory of reducible nonnegative matrices first found in [Fro12, Section 11] and continued in [Sch56]. Many subsequent developments are reviewed in [Sch86] and [Her99]. Results on the geometric spectral theory of reducible K-nonnegative matrices may be largely found in a series of papers by B.S. Tam, some joint with Wu and H. Schneider, [TW89], [Tam90], [TS94],[TS01], [TS03], [Tam04]. For a review containing considerably more information than this subsection see [Tam01].

In some studies the underlying cone is lattice-ordered (for a definition and much information see [Sfr74]) and in some studies the Frobenius form of a reducible nonnegative matrix is generalized; see the work by Jang and Victory [JV93] on positive eventually compact linear operators on Banach lattices. However in the geometric spectral theory the Frobenius normal form of a nonnegative reducible matrix is not generalized as the underlying cone need not be lattice-ordered. Invariant faces are considered instead of the classes which play an important role in combinatorial spectral theory of nonnegative matrices; in particular, distinguished faces and semi-distinguished faces are used in place of distinguished classes and semi-distinguished classes respectively. (For definitions of the preceding terms, see [TS01].)

It turns out that the various results on a reducible nonnegative matrix are extended to a K-nonnegative matrix in different degrees of generality. In particular, the Frobenius-Victory theorem ([Fro12], [Vic85]) is extended to a K-nonnegative matrix on a general cone. The following are extended to a polyhedral cone: the Rothblum index theorem ([Rot75]), a characterization (in terms of the accessibility relation between basic classes) for the spectral radius to have geometric multiplicity 1, for the spectral radius to have index 1 ([Sch56]), and a majorization relation between the (spectral) height characteristic and the (combinatorial) level characteristic of a nonnegative matrix ([HS91b]). Various conditions are used to generalize the theorem on equivalent conditions for equality of the two characteristics ([RiS78], [HS89], [HS91a]). Even for polyhedral cones there is no complete generalization for the nonnegative-basis-theorem, not to mention the preferred-basis theorem ([Rot75], [RiS78], [Sch86], [HS88]). There is a natural conjecture for the latter case ([Tam04]). The attempts to carry out the extensions have also led to the identification of important new concepts or tools. For instance, the useful concepts of semi-distinguished faces and of spectral pairs of faces associated with a K-nonnegative matrix are introduced in [TS01] in proving the cone version of some of the combinatorial theorems referred to above. To achieve these ends certain elementary analytic tools are also brought in.

DEFINITIONS

Let P be a K-nonnegative matrix.

A nonzero P-invariant face F is a **semi-distinguished face** if F contains in its relative interior a generalized eigenvector of P and if F is not the join of two P-invariant faces that are properly included in F.

A K**-semipositive Jordan chain** for P of length m (corresponding to $\rho(P)$) is a sequence of m K-semipositive vectors $\mathbf{x}, (P - \rho(P)I)\mathbf{x}, \ldots, (P - \rho(P)I)^{m-1}\mathbf{x}$ such that $(P - \rho(P)I)^m\mathbf{x} = \mathbf{0}$.

A basis for $N_\rho^\nu(P)$ is called a K**-semipositive basis** if it consists of K-semipositive vectors.

A basis for $N_\rho^\nu(P)$ is called a K**-semipositive Jordan basis** for P if it is composed of K-semipositive Jordan chains for P.

The set $C(P, K) = \{\mathbf{x} \in K : (P - \rho(P)I)^i\mathbf{x} \in K \text{ for all positive integers } i\}$ is called the **spectral cone** of P (for K corresponding to $\rho(P)$).

Denote ν_ρ by ν.

The **height characteristic of** P is the ν-tuple $\eta(P) = (\eta_1, ..., \eta_\nu)$ given by:

$$\eta_k = \dim(N_\rho^k(P)) - \dim(N_\rho^{k-1}(P)).$$

The **level characteristic of** P is the ν-tuple $\lambda(P) = (\lambda_1, \ldots, \lambda_\nu)$ given by:

$$\lambda_k = \dim\,\mathrm{span}(N_\rho^k(P) \cap K) - \dim\,\mathrm{span}(N_\rho^{k-1}(P) \cap K).$$

The **peak characteristic of** P is the ν-tuple $\xi(P) = (\xi_1, ..., \xi_\nu)$ given by:

$$\xi_k = \dim(P - \rho(P)I)^{k-1}(N_\rho^k \cap K).$$

If $A \in \mathbb{C}^{n\times n}$ and $\mathbf{x}$ is a nonzero vector of $\mathbb{C}^n$ then the **order of $\mathbf{x}$ relative to** A, denoted by $\mathrm{ord}_A(\mathbf{x})$, is defined to be the maximum of the orders of the generalized eigenvectors, each corresponding to an eigenvalue of modulus $\rho_{\mathbf{x}}(A)$, that appear in the representation of $\mathbf{x}$ as a sum of generalized eigenvectors of A.

The ordered pair $(\rho_{\mathbf{x}}(A), \mathrm{ord}_A(\mathbf{x}))$ is called the **spectral pair of $\mathbf{x}$ relative to** A and is denoted by $\mathrm{sp}_A(\mathbf{x})$. We also set $\mathrm{sp}_A(\mathbf{0}) = (0, 0)$ to take care of the zero vector $\mathbf{0}$.

We use $\preceq$ to denote the lexicographic ordering between ordered pairs of real numbers, i.e. $(a, b) \preceq (c, d)$ if either $a < c$, or $a = c$ and $b \leq d$. In case $(a, b) \preceq (c, d)$ but $(a, b) \neq (c, d)$, we write $(a, b) \prec (c, d)$.

FACTS

1. If $A \in \mathbb{C}^{n\times n}$ and $\mathbf{x}$ is a vector of $\mathbb{C}^n$, then $\mathrm{ord}_A(\mathbf{x})$ is equal to the size of the largest Jordan block in the Jordan form of the restriction of A to the A-cyclic subspace generated by $\mathbf{x}$ for a peripheral eigenvalue.

   Let P be a K-nonnegative matrix.

2. In the nonnegative matrix case, the present definition of the level characteristic of P is equivalent to the usual graph-theoretic definition; see [NS94, (3.2)] or [Tam04, Remark 2.2].

3. [TS01] For any $\mathbf{x} \in K$, the smallest P-invariant face containing $\mathbf{x}$ is equal to $\Phi(\hat{\mathbf{x}})$, where $\hat{\mathbf{x}} = (I+P)^{n-1}\mathbf{x}$. Furthermore, $\mathrm{sp}_P(\mathbf{x}) = \mathrm{sp}_P(\hat{\mathbf{x}})$. In the nonnegative matrix case, the said face is also equal to F_J, where F_J is as defined in Example 1 of Subsection 1 and J is the union of all classes of P having access to $\mathrm{supp}(\mathbf{x}) = \{i : \mathbf{x}_i > 0\}$. (For definitions of classes and the accessibility relation, see §2.4.)

4. [TS01] For any face F of K, P-invariant or not, the value of the spectral pair $\mathrm{sp}_P(\mathbf{x})$ is independent of the choice of $\mathbf{x}$ from the relative interior of F. This common value, denoted by $\mathrm{sp}_A(F)$, is referred to as the **spectral pair of F relative to A.**

5. [TS01] For any faces $F,\ G$ of K, we have
   (a) $\mathrm{sp}_P(F) = \mathrm{sp}_P(\hat{F})$, where $\hat{F}$ is the smallest P-invariant face of K including F.
   (b) If $F \subseteq G$, then $\mathrm{sp}_P(F) \preceq \mathrm{sp}_P(G)$. If F, G are P-invariant faces and $F \subset G$ then $\mathrm{sp}_P(F) \preceq \mathrm{sp}_P(G)$; viz. either $\rho[F] < \rho[G]$ or $\rho[F] = \rho[G]$ and $\nu_{\rho[F]}[F] \leq \nu_{\rho[G]}[G]$.

6. [TS01] If K is a cone with the property that the dual cone of each of its faces is a facially exposed cone, for instance, when K is a polyhedral cone, a perfect cone or equals $P(n)$ (see [TS01] for definitions), then for any nonzero P-invariant face G, G is semi-distinguished if and only if $\mathrm{sp}_P(F) \prec \mathrm{sp}_P(G)$ for all P-invariant faces F properly included in G.

7. [Tam04] (Cone version of the Frobenius-Victory theorem, [Fro12], [Vic85], [Sch86])

   (i) For any real number λ, λ is a distinguished eigenvalue of P if and only if $\lambda = \rho[F]$ for some distinguished face F of K.

   (ii) If F is a distinguished face, then there is a (up to multiples) unique eigenvector $\mathbf{x}$ of P corresponding to $\rho[F]$ that lies in F. Furthermore, $\mathbf{x}$

belongs to the relative interior of F.

(iii) For each distinguished eigenvalue λ of P, the extreme vectors of the cone $N_\lambda^1(P) \cap K$ are precisely all the distinguished eigenvectors of P that lie in the relative interior of certain distinguished faces of K associated with λ.

8. Let P be a nonnegative matrix. The Jordan form of P contains only one Jordan block corresponding to $\rho(P)$ if and only if any two basic classes of P are comparable (with respect to the accessibility relation); all Jordan blocks corresponding to $\rho(P)$ are of size 1 if and only if no two basic classes are comparable, [Sch56]. An extension of these results to a K-nonnegative matrix on a class of cones which contains all polyhedral cones can be found in [TS01, Theorems 7.2 and 7.1].

9. [Tam90, Theorem 7.5] If K is polyhedral, then:

   (i) There is a K-semipositive Jordan chain for P of length ν_ρ; thus, there is a K-semipositive vector in $N_\rho^\nu(P)$ of order ν_ρ, viz. $\mathrm{ord}_\rho = \nu_\rho$.

   (ii) The Perron space $N_\rho^\nu(P)$ has a basis consisting of K-semipositive vectors.

   However, when K is nonpolyhedral, there need not exist a K-semipositive vector in $N_\rho^\nu(P)$ of order ν_ρ, viz. $\mathrm{ord}_\rho < \nu_\rho$. For a general distinguished eigenvalue λ, we always have $\mathrm{ord}_\lambda \leq \nu_\lambda$, no matter whether K is polyhedral or not.

10. Part(ii) of the preceding fact is not yet a complete cone version of the nonnegative-basis theorem, as the latter theorem guarantees the existence of a basis for the Perron space which consists of semipositive vectors that satisfy certain combinatorial properties. For a conjecture on a cone version of the nonnegative-basis theorem, see [Tam04, Conjecture 9.1].

11. [TS01, Theorem 5.1] (Cone version of the (combinatorial) generalization of the Rothblum index theorem, [Rot75], [HS88])
Let K be a polyhedral cone. Let λ be a distinguished eigenvalue of P for K. Then there is a chain $F_1 \subset F_2 \subset \ldots \subset F_k$ of $k = \mathrm{ord}_\lambda$ distinct semi-distinguished faces of K associated with λ, but there is no such chain with more than ord_λ members. When K is a general cone, the maximum cardinality of a chain of semi-distinguished faces associated with a distinguished eigenvalue λ may be less than, equal to, or greater than ord_λ, see [TS01, Examples 5.3, 5.4, 5.5].

12. For $K = (\mathbb{R}_0^+)^n$, viz. P is a nonnegative matrix, characterizations of different types of P-invariant faces (in particular, the distinguished and semi-distinguished faces) are given in [TS01] (in terms of the concept of an initial subset for P; see [HS88] or [TS01] for definition of an initial subset).

13. [Tam04] The spectral cone $C(P, K)$ is always invariant under $P - \rho(P)I$ and satisfies:
$$N_\rho^1(P) \cap K \subseteq C(A, K) \subseteq N_\rho^\nu(P) \cap K.$$
If K is polyhedral, then $C(A, K)$ is a polyhedral cone in $N_\rho^\nu(P)$.

14. (Generalization of corresponding results on nonnegative matrices, [NS94]) We always have $\xi_k(P) \leq \eta_k(P)$ and $\xi_k(P) \leq \lambda_k(P)$ for $k = 1, \ldots, \nu_\rho$.

15. [Tam04, Theorem 5.9] Consider the following conditions:
    (a) $\eta(P) = \lambda(P)$.
    (b) $\eta(P) = \xi(P)$.
    (c) For each k, $k = 1, \ldots, \nu_\rho$, $N_\rho^k(P)$ contains a K-semipositive basis.
    (d) There exists a K-semipositive Jordan basis for P.
    (e) For each k, $k = 1, \ldots, \nu_\rho$, $N_\rho^k(P)$ has a basis consisting of vectors taken from $N_\rho^k(P) \cap C(P, K)$.

(f) For each k, $k = 1, \ldots, \nu_\rho$, we have

$$\eta_k(P) = \dim(P - \rho(P)I)^{k-1}[N_\rho^k(P) \cap C(P, K)].$$

Conditions (a)–(c) are equivalent and so are conditions (d)–(f). Moreover, we always have (a)$\Longrightarrow$(d), and when K is polyhedral, conditions (a)–(f) are all equivalent.

16. As shown in [Tam04], the level of a nonzero vector $\mathbf{x} \in N_\rho^\nu(P)$ can be defined to be the smallest positive integer k such that $\mathbf{x} \in$span$(N_\rho^k(P) \cap K)$; when there is no such k the level is taken to be ∞. Then the concepts of K-semipositive level basis, height-level basis, peak vector, etc., can be introduced and further conditions can be added to the list given in the preceding result.

17. [Tam04, Theorem 7.2] If K is polyhedral, then $\lambda(P) \preceq \eta(P)$.

18. Cone-theoretic proofs for the preferred-basis theorem for a nonnegative matrix and for a result about the nonnegativity structure of the principal components of a nonnegative matrix can be found in [Tam04].

5 Linear equations over cones

Given a K-nonnegative matrix P and a vector $\mathbf{b} \in K$, in this subsection we consider the solvability of following two linear equations over cones and some consequences:

$$(\lambda I - P)\mathbf{x} = \mathbf{b}, \;\; \mathbf{x} \in K. \tag{1}$$

and

$$(P - \lambda I)\mathbf{x} = \mathbf{b}, \;\; \mathbf{x} \in K. \tag{2}$$

Equation (1) has been treated by several authors in finite-dimensional as well as infinite-dimensional settings, and several equivalent conditions for its solvability have been found. (See [TS03] for a detailed historical account.) The study of equation (2) is relatively new. A treatment of the equation by graph-theoretic arguments for the special case when $\lambda = \rho(P)$ and $K = (\mathbb{R}_0^+)^n$ can be found in [TW89]. The general case is considered in [TS03]. It turns out that the solvability of equation (2) is a more delicate problem. It depends on whether λ is greater than, equal to, or less than $\rho_{\mathbf{b}}(P)$.

FACTS

Let P be a K-nonnegative matrix, let $\mathbf{0} \neq \mathbf{b} \in \mathbf{K}$, and let λ be a given positive real number.

1. [TS03, Theorem 3.1] The following conditions are equivalent:

   (a) Equation (1) is solvable.

   (b) $\rho_{\mathbf{b}}(P) < \lambda$.

   (c) $\lim\limits_{m\to\infty} \sum\limits_{j=0}^{m} \lambda^{-j} P^j \mathbf{b}$ exists.

   (d) $\lim\limits_{m\to\infty} (\lambda^{-1}P)^m \mathbf{b} = \mathbf{0}$.

   (e) $\langle \mathbf{z}, \mathbf{b} \rangle = 0$ for each generalized eigenvector $\mathbf{z}$ of P^T corresponding to an eigenvalue with modulus greater than or equal to λ.

   (f) $\langle \mathbf{z}, \mathbf{b} \rangle = 0$ for each generalized eigenvector $\mathbf{z}$ of P^T corresponding to a distinguished eigenvalue of P for K which is greater than or equal to λ.

2. For a fixed λ, the set $(\lambda I - P)K \cap K$, which consists of precisely all vectors $\mathbf{b} \in K$ for which equation (1) has a solution, is equal to $\{\mathbf{b} \in K : \rho_{\mathbf{b}}(P) < \lambda\}$ and is a face of K.

3. For a fixed λ, the set $(P - \lambda I)K \cap K$, which consists of precisely all vectors $\mathbf{b} \in K$ for which equation (2) has a solution, is in general not a face of K.

4. [TS03, Theorem 4.1] When $\lambda > \rho_{\mathbf{b}}(P)$, equation (2) is solvable if and only if λ is a distinguished eigenvalue of P for K and $\mathbf{b} \in \Phi(N_\lambda^1(P) \cap K)$.

5. [TS03, Theorem 4.5] When $\lambda = \rho_{\mathbf{b}}(P)$, if equation (2) is solvable then $\mathbf{b} \in (P - \rho_{\mathbf{b}}(P)I)\Phi(N^{\nu}_{\rho_{\mathbf{b}}(P)}(P) \cap K)$.

6. [TS03, Theorem 4.19] Let r denote the largest real eigenvalue of P less than $\rho(P)$. (If no such eigenvalues exist, take $r = -\infty$.) Then for any λ, $r < \lambda < \rho(P)$, we have

$$\Phi((P - \lambda I)K \cap K) = \Phi(N_\rho^\nu(P) \cap K).$$

Thus, a necessary condition for equation (2) to have a solution is that $\mathbf{b} \; {}^K\!\!\leq \mathbf{u}$ for some $\mathbf{u} \in N_\rho^\nu(P) \cap K$.

7. [TS03, Theorem 5.11] Consider the following conditions:

   (a) $\rho(P) \in \sum_1(P^T)$.

   (b) $N_\rho^\nu(P) \cap K = N_\rho^1(P) \cap K$, and P has no eigenvectors in $\Phi(N_\rho^1(P) \cap K)$ corresponding to an eigenvalue other than $\rho(P)$.

   (c) $K \cap (P - \rho(P)I)K = \{\mathbf{0}\}$ (equivalently, $\mathbf{x} \geq^K \mathbf{0}, P\mathbf{x} \geq^K \rho(P)\mathbf{x}$ imply that $P\mathbf{x} = \rho(P)\mathbf{x}$).

   We always have (a)$\Longrightarrow$ (b)$\Longrightarrow$(c). When K is polyhedral, conditions (a), (b) and (c) are equivalent. When K is nonpolyhedral, the missing implications all do not hold.

6 Elementary analytic results

In geometric spectral theory, besides the linear-algebraic method and the cone-theoretic method, certain elementary analytic methods have also been called into play, for example the use of Jordan form or the components of a matrix. This approach may have begun with the work of Birkhoff[Bir67] and it was followed by Vandergraft[Van68] and Schneider[Sch81]. Friedland and Schneider[FS80], and Rothblum[Rot81] have also studied the asymptotic behavior of the powers of a nonnegative matrix, or their variants, by elementary analytic methods. The papers [TS94] and [TS01] in the series also need a certain kind of analytic arguments in their proofs; more specifically, they each make use of the

K-nonnegativity of a certain matrix, either itself a component or a matrix defined in terms of the components of a given K-nonnegative matrix (see Facts 3 and 4 in this subsection). In [HNR90] Hartwig, Neumann and Rose offer a (linear)algebraic-analytic approach to the Perron-Frobenius theory of a nonnegative matrix, one which utilizes the resolvent expansion but does not involve the Frobenius normal form. Their approach is further developed by Neumann and Schneider, [NS92], [NS93], [NS94]. By employing the concept of spectral cone and combining the cone-theoretic methods developed in the earlier papers of the series with this algebraic-analytic method, Tam, [Tam04], offers a unified treatment to reprove or extend (or partly extend) several well-known results in the combinatorial spectral theory of nonnegative matrices. The proofs given in [Tam04] rely on the fact that if K is a cone in $\mathbb{R}^n$, then the set $\pi(K)$ that consists of all K-nonnegative matrices is a cone in the matrix space $\mathbb{R}^{n\times n}$ and if, in addition, K is polyhedral then so is $\pi(K)$, [Fen53, p. 22], [SV70], [Tam77] . See [Tam01, Section 6.5] and [Tam04, Section 9] for further remarks on the use of the cone $\pi(K)$ in the study of the spectral properties of K-nonnegative matrices.

In this subsection we collect a few elementary analytic results (whose proofs rely on the Jordan form) which have proved to be useful in the study of the geometric spectral theory. In particular, Facts 3, 4 and 5 identify members of $\pi(K)$. As such, they can be regarded as nice results, which are difficult to come by for the following reason: If K is non-simplicial, then $\pi(K)$ must contain matrices which are not nonnegative linear combinations of its rank-one members, [Tam77]. However, not much is known about such matrices([Tam92]).

DEFINITIONS

Let P be a K-nonnegative matrix. Denote ν_ρ by ν.

The **principal eigenprojection** of P, denoted by $Z_P^{(0)}$, is the projection of $\mathbb{C}^n$ onto the Perron space N_ρ^ν along the direct sum of other generalized eigenspaces of P.

For $k = 0, \ldots, \nu$, the **kth principal component of P** is given by

$$Z_P^{(k)} = (P - \rho(P))^k Z_P^{(0)}.$$

The kth component of P corresponding to an eigenvalue λ is defined in a similar way.

For $k = 0, \ldots, \rho$, the **kth transform principal component of P** is given by:

$$J_P^{(k)}(\varepsilon) = Z_P^{(k)} + Z_P^{(k+1)}/\varepsilon + \cdots + Z_P^{(\nu-1)}/\varepsilon^{\nu-k-1} \quad \text{for all} \quad \varepsilon \in \mathbb{C}\backslash\{0\}.$$

FACTS

Let P be a K-nonnegative matrix. Denote ν_ρ by ν.

1. [Kar59], [Sch81] $Z_P^{(\nu-1)}$ is K-nonnegative.

2. [TS94, Theorem 4.19(i)] The sum of the νth components of P corresponding to its peripheral eigenvalues is K-nonnegative; it is the limit of a convergent subsequence of $((\nu-1)!P^k/[\rho^{k-\nu+1}k^{\nu-1}])$.

3. [Tam04, Theorem 3.6(i)] If K is a polyhedral cone, then for $k = 0, \ldots, \nu-1$, $J_P^{(k)}(\varepsilon)$ is K-nonnegative for all sufficiently small positive ε.

7 Splitting theorems and stability

Splitting theorems for matrices have played a large role in the study of convergence of iterations in numerical linear algebra, see [Var62]. Here we present a cone version of a splitting theorem which is proved in [Sch65] and applied to stability (inertia) theorems for matrices. A closely related result is generalized to operators on a partially ordered Banach space in [DH03] and [Dam04]. There it is used to describe stability properties of (stochastic) control systems and to derive non-local convergence results for Newton's method applied to nonlinear operator equations of Riccati type. We also discuss several kinds of positivity for operators involving a cone that are relevant to the applications mentioned.

For recent splitting theorems involving cones see [SSA05]. For applications of theorems of the alternative for cones to the stability of matrices see [CHS97]. Cones occur in many parts of stability theory, see for instance [Her98].

DEFINITIONS Let K be a cone in $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n\times n}$.

A is **positive stable** if $\mathrm{spec}(A) \subseteq \mathbb{C}^+$ viz. the spectrum of A is contained in the open right half plane.

A is **K-inverse nonnegative** if A is nonsingular and A^{-1} is K-nonnegative.

A is **K-resolvent nonnegative** if there exists an $\alpha_0 \in \mathbb{R}$ such that, for all $\alpha > \alpha_0,\ \alpha I - A$ is K-inverse nonnegative.

A is **cross-positive** on K if for all $\mathbf{x} \in K, \mathbf{y} \in K^*, \mathbf{y}^T\mathbf{x} = 0$ implies $\mathbf{y}^T A\mathbf{x} \geq 0$.

A is a **Z**-matrix if all of its off-diagonal entries are nonpositive.

FACTS Let K be a cone in $\mathbb{R}^n$.

1. A is K-resolvent nonnegative if and only if A is cross-positive on K. Other equivalent conditions and also Perron-Frobenius type theorems for the class of cross-positive matrices can be found in [Els74], [SV70] or [BNS89].

2. When K is $(\mathbb{R}_0^+)^n$, A is cross-positive on K if and only if $-A$ is a **Z**-matrix.

3. [Sch65], [Sch97]. Let $T = R - P$ where $R, P \in \mathbb{R}^{n\times n}$ and suppose that P is K-nonnegative. If R satisfies $R(\mathrm{int}\ K) \supseteq \mathrm{int}\ K$ or $R(\mathrm{int}\ K) \cap \mathrm{int}\ K = \emptyset$, then the following are equivalent:

   (a) T is K-inverse nonnegative.

   (b) For all $\mathbf{y} >^K \mathbf{0}$ there exists (unique) $\mathbf{x} >^K \mathbf{0}$ such that $\mathbf{y} = T\mathbf{x}$.

   (c) There exists $\mathbf{x} >^K \mathbf{0}$ such that $T\mathbf{x} >^K \mathbf{0}$.

   (d) There exists $\mathbf{x} \geq^K \mathbf{0}$ such that $T\mathbf{x} >^K \mathbf{0}$.

   (e) R is K-inverse nonnegative and $\rho(R^{-1}P) < 1$.

4. Let $T \in \mathbb{R}^{n \times n}$. If $-T$ is K-resolvent nonnegative, then T satisfies $T(\text{int } K) \supseteq \text{int } K$ or $T(\text{int } K) \cap \text{int } K = \emptyset$. But the converse is false, see Example 1 below.

5. [DH03, Theorem 2.11], [Dam04, Theorem 3.2.10]. Let T, R, P be as given in Fact 3. If $-R$ is K-resolvent nonnegative, then conditions (a)–(e) of Fact 3 are equivalent. Moreover, the following are additional equivalent conditions:

   (f) T is positive stable.

   (g) R is positive stable and $\rho(R^{-1}P) < 1$.

6. If K is $(\mathbb{R}_0^+)^n$, $R = \alpha I$ and P is a nonnegative matrix, then $T = R - P$ is a **Z**-matrix. It satisfies the equivalent conditions (a)–(g) of Facts 3 and 5 if and only if it is a nonsingular M-matrix [BP79, Chapter 6].

7. (Special case of Fact 5 with $P = 0$). Let $T \in \mathbb{R}^{n \times n}$. If $-T$ is K-resolvent nonnegative, then conditions (a)–(d) of Fact 3 and condition (f) of Fact 5 are equivalent.

8. In [GT06] a matrix T is called a **Z**-transformation on K if $-T$ is cross-positive on K. Many properties on **Z**-matrices, such as being a P-matrix, a Q-matrix (which has connection with the linear complementarity problem), an inverse-nonnegative matrix, a positive stable matrix, a diagonally stable matrix, etc., are extended to **Z**-transformations. For a **Z**-transformation, the equivalence of these properties is examined for various kinds of cones, particularly for symmetric cones in Euclidean Jordan algebras.

9. [Sch65], [Sch97]. (Special case of Fact 3 with K equal to the cone of positive semi-definite matrices in the real space of $n \times n$ Hermitian matrices, and $R(H) = AHA^*$, $P(H) = \Sigma_{k=1}^{s} C_k H C_k^*$). Let $A, C_k, k = 1, \ldots, s$

be complex $n \times n$ matrices which can be simultaneously upper triangularized by similarity. Then there exists a natural correspondence $\alpha_i, \gamma_i^{(k)}$ of the eigenvalues of $A, C_k, k = 1, \ldots, s$. For Hermitian H, let $T(H) = AHA^* - \Sigma_{k=1}^{s} C_k H C_k^*$. Then the following are equivalent:

(a) $|\alpha_i|^2 - \Sigma_{k=1}^{s} |\gamma_i^{(k)}|^2 > 0, \ i = 1, \ldots, n.$

(b) For all positive definite G there exists a (unique) positive definite H such that $T(H) = G$.

(c) There exists a positive definite H such that $T(H)$ is positive definite.

10. Gantmacher-Lyapunov [Gan59, Chapter XV] (Special case of Fact 9 with A replaced by $A+I, s = 2, C_1 = A, C_2 = I$, and special case of Fact 7 with K equal to the cone of positive semi-definite matrices in the real space of $n \times n$ Hermitian matrices and $T(H) = AH + HA^*$).

    Let $A \in \mathbf{C}^{n \times n}$. The following are equivalent:

    (a) For all positive definite G there exists a (unique) positive definite H such that $AH + HA^* = G$.

    (b) There exists a positive definite H such that $AH + HA^*$ is positive definite.

    (c) A is positive stable.

11. Stein [Ste52](Special case of Fact 9 with $A = I, s = 1, C_1 = C$, and special case of Fact 7 with $T(H) = H - CHC^*$).

    Let $C \in \mathbf{C}^{n \times n}$. The following are equivalent:

    (a) There exists a positive definite H such that $H - CHC^*$ is positive definite.

    (b) The spectrum of C is contained in the open unit disk.

EXAMPLES

1. Let $K = (\mathbb{R}_0^+)^2$ and take $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $TK = K$ and so $T(\text{int } K) \supseteq \text{int } K$. Note that $\langle T\mathbf{e_1}, \mathbf{e_2}\rangle = 1 > 0$ whereas $\langle \mathbf{e_1}, \mathbf{e_2}\rangle = 0$; so $-T$ is not cross-positive on K and hence not K-resolvent nonnegative. Since the eigenvalues of T are -1 and 1, T is not positive stable. This example tells us that the converse of Fact 4 is false. It also shows that to the list of equivalent conditions of Fact 3 we cannot add condition (f) of Fact 5.

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