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Comparison theorems using general cones for norms of iteration matrices

Thomas I. Seidman [a], Hans Schneider [b,*], Marina Arav [c]

[a] *Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, MD 21250, USA*
[b] *Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA*
[c] *Department of Mathematics and Statistics, Georgia State University, Atlanta, GA 30303, USA*

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Abstract

We prove comparison theorems for norms of iteration matrices in splittings of matrices in the setting of proper cones in a finite dimensional real space by considering cone linear absolute norms and cone max norms. Subject to mild additional hypotheses, we show that these comparison theorems can hold only for such norms within the class of cone absolute norms. Finally, in a Banach algebra setting, we prove a comparison theorem for spectral radii without appealing to Perron–Frobenius theory.

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* Corresponding author. Tel.: +1 608 271 7252; fax: +1 608 263 8891.
E-mail addresses: seidman@math.umbc.edu (T.I. Seidman), hans@math.wisc.edu (H. Schneider), marav@mathstat.gsu.edu (M. Arav).

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1. Introduction

Extensive literature on the splitting of matrices satisfying various conditions goes back about 50 years; see the books by Varga [22, pp. 94–103] and Axelsson [1, pp. 213–219]. Generally, splittings require a nonnegativity condition, classically with respect to the nonnegative orthant (i.e. elementwise for the matrices involved), but, more recently, with respect to a proper cone in finite dimensional real space; see Marek [15], Marek–Szyld [16] or Climent–Perea [5], where some infinite dimensional generalizations may also be found.

One topic of considerable interest concerns monotonicity properties for the spectral radius of the iteration matrix of the splitting. Such results are usually called *comparison theorems*. These appear in numerous places; see for example the books cited above or the review paper by Woźnicki [23]. However for (right) weak regular splittings, we have found only one result that proves a comparison theorem for a *norm* of splittings, viz. Neumann–Plemmons [17, Lemma 2.2] or Frommer–Szyld [9, Theorem 4.1], see also [3, Theorem 2.5], where applications are given. The norm used in this theorem is a weighted max norm and the cone is the nonnegative orthant. Here, we put this theorem into a cone setting, and one of our principal purposes is to investigate to what extent its hypotheses are needed for its conclusion.

We now describe our paper in some detail. Among other preliminaries, in Section 2 we introduce the classes of *cone absolute* norms, *cone linear absolute* norms and *cone max* norms in order to put our results in a setting of a proper, but otherwise general, cone in finite dimensional real space. Our principal results are contained in Section 3. Since we here consider *left* weak regular splittings, the norms we employ are cone linear absolute. We prove a dual form of the theorem of [9,17] mentioned above in this setting (Theorem 3.3) and we show (Theorem 3.4) that a cone absolute norm must satisfy stringent conditions to yield a norm comparison theorem. A sequence of results is summed up in Theorem 3.7. In Section 4 we obtain some analogous results for right weak regular splittings which are the duals of the theorems in Section 3. In Section 5 we derive a comparison theorem for spectral radii, and, turning to a different approach, we give a proof of the well-known [22, Theorem 3.32] in a cone setting which relies on order and convergence properties of operators without any appeal to Perron–Frobenius theory.

As one motivating application for some of our present concerns, we can consider a processor serving a set of buffered input sources. If there is a setup time for switching tasks, a ‘clearing round-robin’ policy is reasonable and (cf., e.g., [10,12]) the analysis of system dynamics involves the convergence of powers of an iteration matrix for what is in effect a regular splitting. However, for related settings, e.g., consideration of a bank of processors (cf., e.g., [13]), one has a (somewhat random) product of matrices rather than powers of a fixed matrix. To show system stability for these more general settings (involving products of iteration matrices of different splittings) one now needs to have a uniform norm inequality for the iteration matrices to ensure suitable stability of these products; see the papers we have cited.

2. Preliminaries on cones and norms

In the following, $\mathscr{P}$ will always be a ‘general’ *proper cone*, not necessarily simplicial, in the real finite dimensional space $\mathscr{X}$, that is $\mathscr{P}$ is a closed and convex subset of $\mathscr{X}$ with $\mathscr{P} + (-\mathscr{P}) = \mathscr{X}$, $\mathscr{P} \cap (-\mathscr{P}) = \{0\}$. [We note that this implies that both $\mathscr{P}$ and the dual cone $\mathscr{P}^* = \{\varphi : \mathbf{u} \in \mathscr{P} \Rightarrow \varphi \cdot \mathbf{u} \geqslant 0\}$ in $\mathscr{X}^*$ have nonempty interiors.] Specification of $\mathscr{P}$ then induces a partial order $\geqslant = \geqslant_{\mathscr{P}}$ for the space $\mathscr{X}$ so $\mathbf{x} \geqslant \mathbf{y}$ means $\mathbf{x} - \mathbf{y} \in \mathscr{P}$. This also induces a partial order for $K \times K$ matrices (viewed as operators: $\mathscr{X} \to \mathscr{X}$) so $\mathbf{A} \geqslant 0$ means $\mathbf{Au} \in \mathscr{P}$ for all $\mathbf{u} \in \mathscr{P}$ and we have an induced proper cone $\mathscr{M}_+ = \{\mathbf{A} \geqslant 0\}$ for such matrices. [Note that the cones $\mathscr{M}_+$ and $\mathscr{M}_+^* = \{\mathbf{B} : \varphi \in \mathscr{P}^* \Rightarrow \varphi\mathbf{B} \in \mathscr{P}^*\} = \{\mathbf{A}^\top : \mathbf{A} \in \mathscr{M}_+\}$ are each again convex with nonempty interiors.]

It is natural that the best known case takes $\mathscr{P}$ to be the nonnegative orthant $\mathbb{R}_+^K \subset \mathscr{X} = \mathbb{R}^K$. (When we speak simply of $\mathbb{R}^K$ we shall always assume that $\mathscr{P} = \mathbb{R}_+^K$.) For this case $\mathbf{A} \geqslant 0$ just means that all the entries $(a_{j,k})$ are nonnegative. In this context, the relevant matrix norm for our questions will be that induced by the ℓ^1 norm on $\mathbb{R}^K$—or a weighted ℓ^1 norm given by

$$\nu(\mathbf{x}) := \sum_{k=1}^{K} w_k |x_k| \quad \text{for } \mathbf{x} = (x_1, \ldots, x_K) \in \mathbb{R}^K \tag{2.1}$$

with positive weights $w_k > 0$. These norms belong to a well-studied class of norms on $\mathbb{R}^K$ (including all ℓ_p norms) called *absolute norms*: by definition these are norms satisfying

$$\nu(\mathbf{x}) = \nu(|\mathbf{x}|), \tag{2.2}$$

where $|\mathbf{x}| = (|x_1|, \ldots, |x_K|)$. It is well-known (see, [2] or [11, Theorem 5.5.10]) that an absolute norm has the monotonicity property

$$|\mathbf{x}| \leqslant |\mathbf{y}| \Rightarrow \nu(\mathbf{x}) \leqslant \nu(\mathbf{y}). \tag{2.3}$$

One would define $|\mathbf{x}| \in \mathscr{P}$ as $\max\{\mathbf{x}, -\mathbf{x}\} = \min\{\mathbf{u} : -\mathbf{u} \leqslant \mathbf{x} \leqslant \mathbf{u}\}$ and use (2.2) if this min were always attained, but the max or min of two elements need not be available for the order defined by a general proper cone and in order to generalize to our setting we shall call a norm ν on $\mathscr{X}$ *cone absolute* (with respect to the proper cone $\mathscr{P}$) if, for all $\mathbf{x} \in \mathscr{X}$,

$$\nu(\mathbf{x}) = \inf\{\nu(\mathbf{u}) : -\mathbf{u} \leqslant \mathbf{x} \leqslant \mathbf{u} : \mathbf{u} \in \mathscr{P}\}. \tag{2.4}$$

We note that a norm ν is cone absolute if and only if

$$\nu(\mathbf{x}) = \inf\{\nu(\mathbf{v} + \mathbf{w}) : \mathbf{v}, \mathbf{w} \in \mathscr{P}; \mathbf{v} - \mathbf{w} = \mathbf{x}\} \tag{2.5}$$

as we may see by putting $2\mathbf{v} = \mathbf{u} + \mathbf{x}$, $2\mathbf{w} = \mathbf{u} - \mathbf{x}$ giving $\mathbf{u} = \mathbf{v} + \mathbf{w}$.

While we are here taking (2.5) as a property of an already specified norm, we remark at this point that (2.4) may be taken to *define* a norm ν on $\mathscr{X}$ once it is already given as a monotone, positively homogeneous, subadditive function on the cone $\mathscr{P}$.

It is easy to see that a cone absolute norm has the property

$$\nu(\mathbf{u}) \leqslant \nu(\mathbf{u}+\mathbf{v}) \quad \text{for } \mathbf{u}, \mathbf{v} \in \mathscr{P}, \tag{2.6}$$

which we shall call *cone monotonicity*. When $\mathscr{X} = \mathbb{R}^K$ (and $\mathscr{P} = \mathbb{R}^K_+$) we shall simply refer to absolute and monotonic norms.[1] Cone monotonicity of a norm does not imply its cone absoluteness as can be seen by considering on $\mathbb{R}^2$ the norm: $\max(|x_1|, |x_2|, |x_1 + x_2|)$.

The most striking property of a weighted ℓ_1 norm on $\mathbb{R}^K$ is that on $\mathscr{P} = \mathbb{R}^K_+$ it is linear. Thus we shall call a norm ν *cone linear* (with respect to a proper cone $\mathscr{P}$) if it satisfies

$$\text{For some fixed } \varphi \in \mathscr{P}^* \text{ one has:} \quad \nu(\mathbf{u}) = \varphi \cdot \mathbf{u} \quad \text{for all } \mathbf{u} \in \mathscr{P}. \tag{2.7}$$

Clearly this φ must be in the interior of $\mathscr{P}^*$ to ensure, as is required for a norm, that $\nu(\mathbf{u}) > 0$ for $0 \neq \mathbf{u} \in \mathscr{P}$. A norm ν will be called *cone linear absolute* if it is both cone absolute and cone linear.

The property (2.5) is sufficient to permit appropriate treatment of the induced matrix norms as well: for positive matrices the matrix norm may be computed with attention restricted to $\mathscr{P}$.

Theorem 2.1. *Let ν be a cone absolute norm on $\mathscr{X}$ (with respect to a proper cone $\mathscr{P}$) and let $\mathbf{A} : \mathscr{X} \to \mathscr{X}$ be positive ($\mathbf{A} \geqslant 0$ so $\mathbf{Au} \in \mathscr{P}$ when $\mathbf{u} \in \mathscr{P}$). Then*

$$\begin{aligned} \|\mathbf{A}\| &:= \sup\{\nu(\mathbf{Ax}) : \mathbf{x} \in \mathscr{X}; \nu(\mathbf{x}) \leqslant 1\} \\ &= \mathscr{N}_{\mathscr{P}}(\mathbf{A}) := \sup\{\nu(\mathbf{Au}) : \mathbf{u} \in \mathscr{P}; \nu(\mathbf{u}) \leqslant 1\}. \end{aligned} \tag{2.8}$$

Proof. Let $\mathbf{x} \in \mathscr{X}$ be such that $\nu(\mathbf{Ax}) = \|\mathbf{A}\|$ with $\nu(\mathbf{x}) = 1$. Since ν is an absolute norm, there exists $\mathbf{u} \in \mathscr{P}$ such that $-\mathbf{u} \leqslant \mathbf{x} \leqslant \mathbf{u}$ and $\nu(\mathbf{u}) = \nu(\mathbf{x}) = 1$ so $\nu(\mathbf{Au}) \leqslant \|\mathbf{A}\|$. Since $\mathbf{A} \geqslant 0$, we have $\mathbf{Au} \geqslant 0$ and $-\mathbf{Au} \leqslant \mathbf{Ax} \leqslant \mathbf{Au}$. Hence, by (2.4), $\|\mathbf{A}\| = \nu(\mathbf{Ax}) \leqslant \nu(\mathbf{Au})$ and it follows that $\nu(\mathbf{Au}) = \|\mathbf{A}\|$. □

Given (2.6), it follows from Theorem 2.1 that the matrix norm corresponding to any cone absolute norm will itself be cone monotone with respect to the matrix cone $\mathscr{M}_+$:

$$\|\mathbf{A}\| \leqslant \|\mathbf{A}+\mathbf{B}\| \quad \text{for } \mathbf{A}, \mathbf{B} \in \mathscr{M}_+. \tag{2.9}$$

However, a matrix norm satisfying (2.9) need not be cone absolute.

[1] This departs from the usual terminology in papers on matrix theory where the norms we simply call 'monotonic' are generally called 'orthant monotonic'.

This may be seen by considering the ℓ_2 norm on $\mathbb{R}^2$ and the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad |\mathbf{A}| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Although the ℓ_2 norm is absolute, the matrices $\mathbf{A}$, $|\mathbf{A}|$ have different norms (largest singular values): respectively $\sqrt{2}$ and 2.

We also observe that the conclusion of Theorem 2.1 need not hold for all cone monotonic norms.

To see this, consider the norm ν given by $\max\{|x_1|, |x_2|, |x_1 + x_2|/2\}$ on $\mathbb{R}^2$ and the transformation $\mathbf{T}(x_1, x_2) = (0, x_2)$. For $\mathbf{x} = (-2, 1)$ one would then have $\nu(\mathbf{Tx})/\nu(\mathbf{x}) = 2$ while $\nu(\mathbf{Tu})/\nu(\mathbf{u}) \leqslant 1$ for all $\mathbf{u} \geqslant 0$.

3. Inequalities for operator splittings

Throughout this section we assume that we have specified a proper cone $\mathscr{P}$ and vector inequalities will be with respect to this cone while matrix inequalities will be taken with respect to the induced cone $\mathscr{M}_+ = \mathscr{M}_+(\mathscr{P})$. We say that the pair $[\mathbf{M}, \mathbf{N}]$ is a *left weak regular splitting* of the $K \times K$ matrix $\mathbf{T}$ if $\mathbf{M}$ is invertible and

$$\mathbf{M} - \mathbf{N} = \mathbf{T} \quad \text{with } \mathbf{M}^{-1} \geqslant 0, \quad \mathbf{N}\mathbf{M}^{-1} \geqslant 0. \tag{3.1}$$

Similarly, the pair $[\mathbf{M}, \mathbf{N}]$ is called a right weak regular splitting of $\mathbf{T}$ if $\mathbf{M}$ is invertible and

$$\mathbf{M} - \mathbf{N} = \mathbf{T} \quad \text{with } \mathbf{M}^{-1} \geqslant 0, \quad \mathbf{M}^{-1}\mathbf{N} \geqslant 0. \tag{3.2}$$

These splittings in (3.1) and (3.2) are also known as weak nonnegative splittings of the first and second kind respectively; see [23] or [8]. The matrix $\mathbf{N}\mathbf{M}^{-1}$ (alternatively, $\mathbf{M}^{-1}\mathbf{N}$) is known as the *iteration matrix* of the splitting.

We shall assume that the norm ν is cone linear absolute with defining functional φ. Where needed, we impose on the functional φ of (2.7) the hypothesis that it is not only in the interior of $\mathscr{P}^*$, as is necessary for ν to be a norm, but that in addition we also have

$$\mathbf{T}^\top\varphi \geqslant 0: \quad \varphi \cdot \mathbf{Tv} \geqslant 0 \ \text{ for all } \mathbf{v} \in \mathscr{P}, \tag{3.3}$$

or the still stronger property

$$\mathbf{T}^\top\varphi > 0: \quad \varphi \cdot \mathbf{Tv} > 0 \ \text{ for all } \mathbf{v} \in \mathscr{P}, \mathbf{v} \neq 0. \tag{3.4}$$

Note that, without further assumptions, even (3.4) is much weaker than asking that $\mathbf{T}^\top \geqslant 0$ which would mean that $\mathbf{T}^\top\varphi \geqslant 0$ for *all* $\varphi \in \mathscr{P}^*$, not only the particular φ of (2.7). We do note, however, that when $\mathbf{T}$ is nonsingular and has a left weak regular splitting $[\mathbf{M}, \mathbf{N}]$ (whence $\mathbf{T}^\top$ has the right weak regular splitting $[\mathbf{M}^\top, \mathbf{N}^\top]$), then the existence of a functional satisfying (3.3) is equivalent to $(\mathbf{T}^{-1})^\top \geqslant 0$ and to $\rho(\mathbf{NM}^{-1}) \leqslant 1$, as is shown by an easy modification of the proof of [20, Lemma 1]. Further, the nonsingularity of $\mathbf{T}^\top$ is guaranteed under condition (3.4), which also yields $\rho(\mathbf{NM}^{-1}) < 1$; see [20, Lemma 1] and see [6, Theorem 2.11] for a related result. These remarks rest on Perron–Frobenius theory, but our next theorem shows that one can parallel the above results without any overt appeal to that theory.

Theorem 3.1. *Suppose the pair* $[\mathbf{M}, \mathbf{N}]$ *is a left weak regular splitting of a* $K \times K$ *matrix* $\mathbf{T}$ *so we have* (3.1) *with respect to the ordering of* $\mathscr{P}$. *Let the norm* ν *be a cone linear absolute norm given by the functional* φ (*so that* (2.4) *and* (2.7) *hold*).

1. *If the defining functional* φ *of* ν *satisfies* (3.3) *with respect to this* $\mathbf{T}$, *then, with the operator norm induced by* ν, *we have*
$$\|\mathbf{NM}^{-1}\| \leqslant 1. \tag{3.5}$$
2. *When* φ *satisfies* (3.4) *one has the strict inequality*
$$\|\mathbf{NM}^{-1}\| < 1. \tag{3.6}$$
3. *If either* (3.6) *holds or* (3.5) *holds and* $\mathbf{T}$ *is assumed to be nonsingular, then*
$$\mathbf{T}^{-1} \geqslant 0. \tag{3.7}$$
4. *If* $\mathbf{T}^{-1} \geqslant 0$, *then there exists a functional* $\varphi' > 0$ *such that the strict inequality* (3.4) *is satisfied for the cone linear absolute norm given by* φ' *and hence* (3.6) *holds for the induced norm.*

Proof. (1) Since (3.1) gives $\mathbf{NM}^{-1} \geqslant 0$, we may apply Theorem 2.1. Thus it is sufficient for (3.5) to show that $\nu(\mathbf{NM}^{-1}\mathbf{u}) \leqslant \nu(\mathbf{u})$ for each $\mathbf{u} \in \mathscr{P}$. Noting that $\mathbf{TM}^{-1} = \mathbf{I} - \mathbf{NM}^{-1}$ and using (2.7), we have
$$\nu(\mathbf{u}) - \nu(\mathbf{NM}^{-1}\mathbf{u}) = \varphi \cdot [\mathbf{u} - (\mathbf{I} - \mathbf{TM}^{-1})\mathbf{u}] = \varphi \cdot \mathbf{T}(\mathbf{M}^{-1}\mathbf{u}), \tag{3.8}$$
which is nonnegative by (3.3) since $\mathbf{M}^{-1} \geqslant 0$ gives $\mathbf{v} = \mathbf{M}^{-1}\mathbf{u} \in \mathscr{P}$.

(2) For this assertion we observe that in the argument above we need only consider the set $\mathscr{P}'$ consisting of $\mathbf{u} \in \mathscr{P}$ for which $\nu(\mathbf{u}) = 1$. This set is compact so, by continuity, $\varphi \cdot \mathbf{T}(\mathbf{M}^{-1}\mathbf{u})$ attains its minimum over $\mathbf{u} \in \mathscr{P}'$, necessarily positive since it follows from (3.4) that $\varphi \cdot \mathbf{T}(\mathbf{M}^{-1}\mathbf{u}) > 0$ for each such $\mathbf{u}$. Thus, (3.6) follows from (3.8).

(3) We now observe that if $\mathbf{P} = \mathbf{NM}^{-1}$ satisfies (3.6), then the Neumann series expansion of $(\mathbf{I} - \mathbf{P})^{-1}$ converges and we obtain $(\mathbf{I} - \mathbf{P})^{-1} \geqslant 0$ since the cone $\mathscr{M}_+$ is closed. If (3.5) holds then we can apply the same argument to $\alpha\mathbf{P}$ with $0 < \alpha < 1$ and use continuity of the inverse to obtain $(\mathbf{I} - \mathbf{P})^{-1} \geqslant 0$. In either case, $\mathbf{T}^{-1} = \mathbf{M}^{-1}(\mathbf{I} - \mathbf{P})^{-1} \geqslant 0$.

(4) Suppose $\mathbf{T}^{-1} \geqslant 0$ so also $(\mathbf{T}^\top)^{-1} \geqslant 0$. Then for any choice of $\psi > 0$ one has $\varphi' := (\mathbf{T}^\top)^{-1}\psi > 0$ and $\mathbf{T}\varphi' > 0$. The conclusions now follow by Part (2) of the theorem. $\square$

While we have used (3.3) to obtain (3.5) for every left weak regular splitting, we may remark that for any one such splitting it is sufficient that $(\mathbf{M}^{-1})^\top$ takes $\mathbf{T}^\top\varphi$ into $\mathscr{P}^*$.

On the other hand, if $\mathbf{T}$ is singular, then (3.5) does not imply the existence of a functional $\varphi > 0$ such that (3.3) holds, as is shown by the following example.

Example 3.2. Consider the left weak regular splitting

$$\mathbf{T} = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} = \mathbf{M} - \mathbf{N} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix}$$

$$\text{so } \mathbf{NM}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For the usual ℓ_1 norm, (3.5) is satisfied, yet there cannot be any positive functional φ for which $\mathbf{T}^\top\varphi$ is also positive, to yield (3.3).

Theorem 3.3. *Suppose* $[\mathbf{M}, \mathbf{N}]$ *and* $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ *are both left weak regular splittings of the same* $K \times K$ *matrix* $\mathbf{T}$. *Assume the defining functional* φ *of the cone linear absolute norm* ν *satisfies* (3.3). *Then, with the operator norm induced by* ν, *we have*

$$\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1} \quad \textit{implies} \quad \|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| \leqslant \|\mathbf{NM}^{-1}\|, \tag{3.9}$$

i.e., the mapping: $\mathbf{M}^{-1} \mapsto \|\mathbf{NM}^{-1}\|$ *is antitone for left weak regular splittings of* $\mathbf{T}$. *Further, if* φ *satisfies the stronger condition* (3.4) *and also*

$$\mathbf{u} \geqslant 0, \ \ [\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]\mathbf{u} = 0 \quad \textit{implies} \quad \mathbf{u} = 0, \tag{3.10}$$

then the conclusion of (3.9) *becomes the strict inequality*

$$\|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| < \|\mathbf{NM}^{-1}\|. \tag{3.11}$$

Proof. Applying Theorem 2.1 for each of the operator norms, it is sufficient for the desired norm inequality (3.9) to show that $\nu(\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{u}) \leqslant \nu(\mathbf{NM}^{-1}\mathbf{u})$ for each $\mathbf{u} \in \mathscr{P}$ when $\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1}$. Since $\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1},\ \mathbf{NM}^{-1} \geqslant 0$, this is equivalent by (2.7) to showing that $\varphi \cdot \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{u} \leqslant \varphi \cdot \mathbf{NM}^{-1}\mathbf{u}$ for all $\mathbf{u} \geqslant 0$.

We next observe that $\mathbf{TM}^{-1} = \mathbf{I} - \mathbf{NM}^{-1}$ and $\mathbf{T}\hat{\mathbf{M}}^{-1} = \mathbf{I} - \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}$ so we have the identity

$$\mathbf{NM}^{-1} - \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} = \mathbf{T}[\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}].$$

With $\hat{\mathbf{M}}^{-1} \geqslant \mathbf{M}^{-1}$ giving $\mathbf{v} = [\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]\mathbf{u} \in \mathscr{P}$ for $\mathbf{u} \in \mathscr{P}$, we have from (3.3) that $\varphi \cdot \mathbf{Tv} \geqslant 0$. Thus, as desired, for each such $\mathbf{u}$ we have $0 \leqslant \varphi \cdot \mathbf{T}[\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]\mathbf{u} = \varphi \cdot [\mathbf{NM}^{-1} - \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}]\mathbf{u}$.

If (3.4) and (3.10) hold then $\mathbf{v} = [\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]\mathbf{u}$ is nonnegative and nonzero for $\mathbf{u} \geqslant 0, \mathbf{u} \neq 0$ whence the above argument shows $\varphi \cdot \mathbf{Tv} > 0$. As in the last part of the proof of Theorem 3.1, this proves (3.11). $\square$

The condition (3.10) just asserts that the nullspace of $[\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]$ intersects $\mathscr{P}$ only trivially. Thus (3.10) holds if $\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1} > 0$ (in which case the inequality (3.11) follows even without the hypothesis (3.4). The hypothesis (3.10) occurs in the context of theorems of the alternative; see Lemma 4.1 below. For $\mathscr{X} = \mathbb{R}^K$ and $\mathscr{P} = \mathbb{R}^K_+$, condition (3.10) is satisfied if and only if $[\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]$ is a nonnegative matrix with at least one positive element in each column.

We can now show that for cone linear absolute norms the additional hypothesis (3.3) is necessary for (3.9)—at least if $\mathbf{T}$ has some left weak regular splitting $[\mathbf{M}, \mathbf{N}]$ as in (3.1) for which $\mathbf{NM}^{-1}$ is not only nonnegative, but is in the interior of the cone $\mathscr{M}_+$ of nonnegative matrices.

Theorem 3.4. *Suppose* $\mathbf{T}$ *has a left weak regular splitting* $[\mathbf{M}, \mathbf{N}]$ *with* $\mathbf{NM}^{-1}$ *strictly positive and that* ν *is a cone linear absolute norm with defining functional* φ. *Then* (3.9) *can hold only if* (3.3) *holds*.

Proof. For (3.3) to be false there must exist some $\mathbf{v} \in \mathscr{P}$ such that $\varphi \cdot \mathbf{Tv} < 0$. Fixing $\mathbf{v}$, we define the dyad $\mathbf{D} = \mathbf{v} \otimes \varphi : \mathbf{x} \mapsto (\varphi \cdot \mathbf{x})\mathbf{v}$, noting that $\mathbf{D} \geqslant 0$ and $\|\mathbf{D}\| = \varphi \cdot \mathbf{v}$. For small enough $\varepsilon > 0$, we can then define $\hat{\mathbf{M}} := (\mathbf{I} + \varepsilon\mathbf{MD})^{-1}\mathbf{M}$, for which we have

$$\hat{\mathbf{M}}^{-1} = \mathbf{M}^{-1}(\mathbf{I} + \varepsilon\mathbf{MD}) = \mathbf{M}^{-1} + \varepsilon\mathbf{D} \quad \text{so } \hat{\mathbf{M}}^{-1} \geqslant \mathbf{M}^{-1} \geqslant 0.$$

Further,

$$\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} = \mathbf{I} - \mathbf{T}\hat{\mathbf{M}}^{-1} = \mathbf{I} - \mathbf{T}(\mathbf{M}^{-1} + \varepsilon\mathbf{D}) = \mathbf{NM}^{-1} - \varepsilon\mathbf{TD},$$

whence one has convergence $\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} \to \mathbf{NM}^{-1}$ as $\varepsilon \to 0$ so $\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} \geqslant 0$ for small enough $\varepsilon > 0$. Thus, $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ is again a left weak regular splitting of $\mathbf{T}$ and we have $\hat{\mathbf{M}}^{-1} \geqslant \mathbf{M}^{-1}$ as in (3.9).

Since $\mathbf{NM}^{-1} \geqslant 0$, we may apply Theorem 2.1 and note that, by continuity and compactness, the sup in the definition (2.8) of $\mathscr{N}_{\mathscr{P}}(\mathbf{NM}^{-1})$ must be attained. Thus there exists some $\mathbf{u} \in \mathscr{P}$ with $\varphi \cdot \mathbf{u} = \nu(\mathbf{u}) = 1$ and

$$\varphi \cdot \mathbf{NM}^{-1}\mathbf{u} = \nu(\mathbf{NM}^{-1}\mathbf{u}) = \mathscr{N}_{\mathscr{P}}(\mathbf{NM}^{-1}) = \|\mathbf{NM}^{-1}\|.$$

For this $\mathbf{u}$ we have $\mathbf{Du} = \mathbf{v}$ and

$$\begin{aligned}\|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| &\geqslant \nu(\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{u}) = \varphi \cdot \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{u} = \varphi \cdot [\mathbf{NM}^{-1}\mathbf{u} - \varepsilon\mathbf{TDu}] \\ &= \|\mathbf{NM}^{-1}\| - \varepsilon\varphi \cdot \mathbf{Tv} > \|\mathbf{NM}^{-1}\| \quad (\text{as } \varphi \cdot \mathbf{Tv} < 0)\end{aligned}$$

so (3.9) then fails. □

In view of the nonnegativity of $\mathbf{M}^{-1}$, $\hat{\mathbf{M}}^{-1}$ and the resolvent identity: $\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1} = \hat{\mathbf{M}}^{-1}(\mathbf{M} - \hat{\mathbf{M}})\mathbf{M}^{-1}$, we note that having $\hat{\mathbf{M}} \leqslant \mathbf{M}$ is a somewhat stronger hypothesis than having $\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1}$. We now show that this strengthening is just sufficient to compensate for the absence of (3.3).

Theorem 3.5. *Let both* $[\mathbf{M}, \mathbf{N}]$ *and* $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ *be left weak regular splittings of the same* $K \times K$ *matrix* $\mathbf{T}$. *With the operator norm induced by the cone linear absolute norm* ν, *we assume that* $\|\mathbf{N}\mathbf{M}^{-1}\| \leqslant 1$. *Then*

$$\hat{\mathbf{M}} \leqslant \mathbf{M} \quad \textit{implies} \quad \|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| \leqslant \|\mathbf{N}\mathbf{M}^{-1}\|, \tag{3.12}$$

i.e., subject to (3.5), *the mapping*: $\mathbf{M} \mapsto \|\mathbf{N}\mathbf{M}^{-1}\|$ *is isotone for left weak regular splittings of* $\mathbf{T}$ *even in the absence of* (3.3).

Proof. We begin by noting the identity

$$\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} - \mathbf{N}\mathbf{M}^{-1} = \mathbf{T}[\mathbf{M}^{-1} - \hat{\mathbf{M}}^{-1}] = (\mathbf{I} - \mathbf{N}\mathbf{M}^{-1})(\hat{\mathbf{M}} - \mathbf{M})\hat{\mathbf{M}}^{-1}$$

for splittings $[\mathbf{M}, \mathbf{N}]$, $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ of $\mathbf{T}$. Assuming $\hat{\mathbf{M}} \leqslant \mathbf{M}$ and $\|\mathbf{N}\mathbf{M}^{-1}\| \leqslant 1$, we wish to show that

$$\varphi \cdot \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{u} \leqslant \varphi \cdot \mathbf{N}\mathbf{M}^{-1}\mathbf{u} \quad \text{for each } \mathbf{u} \geqslant 0,$$

since this is sufficient for the conclusion of (3.12).

By the identity, we have $\varphi \cdot [\mathbf{N}\mathbf{M}^{-1} - \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}]\mathbf{u} = \varphi \cdot \mathbf{v} - \varphi \cdot \mathbf{N}\mathbf{M}^{-1}\mathbf{v}$ with $\mathbf{v} := (\mathbf{M} - \hat{\mathbf{M}})\hat{\mathbf{M}}^{-1}\mathbf{u} \in \mathscr{P}$. Since $\|\mathbf{N}\mathbf{M}^{-1}\| \leqslant 1$, we have $\varphi \cdot \mathbf{N}\mathbf{M}^{-1}\mathbf{v} = \nu(\mathbf{N}\mathbf{M}^{-1}\mathbf{v}) \leqslant \nu(\mathbf{v}) = \varphi \cdot \mathbf{v}$. Thus, $\varphi \cdot [\mathbf{N}\mathbf{M}^{-1} - \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}]\mathbf{u} \geqslant 0$ for each $\mathbf{u} \geqslant 0$ and it follows that $\|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| \leqslant \|\mathbf{N}\mathbf{M}^{-1}\|$ as desired. □

The following reformulation of a special case of Theorem 3.5 is of interest.

Corollary 3.6. *Suppose* $\mathbf{P} = \mathbf{A} + \mathbf{B}$ *with* $\|\mathbf{P}\| < 1$ *and with both* $\mathbf{A}, \mathbf{B} \geqslant 0$. *Then, subject to this, the map*: $\mathbf{A} \mapsto \|\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\|$ *is isotone*.

Proof. Let $\mathbf{T}$ be the matrix $\mathbf{I} - \mathbf{P}$, let $\mathbf{M} = (\mathbf{I} - \mathbf{A})$, and $\mathbf{N} = \mathbf{B} \geqslant 0$. One easily sees that $[\mathbf{M}, \mathbf{N}]$ is a left weak regular splitting of $\mathbf{T}$. We first apply Theorem 3.5 to compare this with the splitting $[\mathbf{I}, \mathbf{P}]$ and see that we always have $\|\mathbf{B}(\mathbf{I} - \mathbf{A})^{-1}\| \leqslant \|\mathbf{P}\| < 1$. Theorem 3.5 then applies to give the isotonicity. □

While the proof above of Theorem 3.5 does use (2.7), the possibility remains open that this apparent necessity is merely an artifact of the particular proof. We now show that this is not the case: for a cone absolute norm ν (and indeed for any cone monotone norm which satisfies (2.8)) the linearity on $\mathscr{P}$ is really needed for the isotonicity (3.12).

> In particular, when $\mathscr{P}$ is the usual positive orthant $\mathbb{R}_+^K$ we cannot have (3.12) when the matrix norm is induced by, e.g., a (weighted) ℓ^p norm with $p > 1$.

Theorem 3.7. *Assume the norm* $\nu(\cdot)$ *is cone absolute* (2.4). *Then* (3.12) *holds only if* $\nu(\cdot)$ *is cone linear, even if we restrict attention to left regular splittings* $[\mathbf{M}, \mathbf{N}]$ *with* $\|\mathbf{N}\mathbf{M}^{-1}\| \leqslant 1$.

Proof. Choose any $\mathbf{u}$ in the interior of $\mathscr{P}$, normalized so $\nu(\mathbf{u}) = 1$. By the Hahn–Banach theorem, e.g. [18, p. 58], there must be some $\varphi \in \mathscr{X}^*$ such that

$$\varphi \cdot \mathbf{x} \leqslant \nu(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathscr{X} \text{ but } \varphi \cdot \mathbf{u} = \nu(\mathbf{u}) = 1. \tag{3.13}$$

We note that (3.13) together with (2.6) give $\varphi \in \mathscr{P}^*$:

> Given any $\mathbf{v} \geqslant 0$, one has $(\mathbf{u} - s\mathbf{v}) \in \mathscr{P}$ for small enough $s > 0$ since $\mathbf{u}$ is in the interior of $\mathscr{P}$. Then (2.6) gives $\nu(\mathbf{u} - s\mathbf{v}) \leqslant \nu(\mathbf{u} - s\mathbf{v} + s\mathbf{v}) = \nu(\mathbf{u}) = 1$, whence
>
> $$1 - s\varphi \cdot \mathbf{v} = \varphi \cdot (\mathbf{u} - s\mathbf{v}) \leqslant \nu(\mathbf{u} - s\mathbf{v}) \leqslant 1$$
>
> so one must have $\varphi \cdot \mathbf{v} \geqslant 0$.

If ν is not cone linear, then (2.7) fails for this φ so there must be some $\mathbf{v}$ in $\mathscr{P}$ for which $\varphi \cdot \mathbf{v} \neq \nu(\mathbf{v})$, and by the continuity of the norm there must be such a $\mathbf{v}$ in the interior of $\mathscr{P}$. Normalizing $\mathbf{v}$, this necessarily means $\varphi \cdot \mathbf{v} < \nu(\mathbf{v}) = 1$. Again by the Hahn–Banach theorem, there is then some $\psi \in \mathscr{P}^*$ with $\nu^*(\psi) = 1$ and $\psi \cdot \mathbf{v} = \nu(\mathbf{v}) = 1$.

To obtain our counterexample for (3.12) we again work with appropriate dyadic matrices. First, by our choice of $\mathbf{v}$ we can choose r so $0 \leqslant \varphi \cdot \mathbf{v} < r < 1$ and we then set

$$\mathbf{M} := \mathbf{I}, \quad \mathbf{N} := r\mathbf{u} \otimes \psi : \mathbf{x} \mapsto r(\psi \cdot \mathbf{x})\mathbf{u}.$$

Since $r > 0$, $\mathbf{u} \in \mathscr{P}$, and $\psi \in \mathscr{P}^*$ as noted above, this makes $\mathbf{N}\mathbf{M}^{-1} = \mathbf{N} \geqslant 0$ so we have (3.1). As $\nu(\mathbf{N}\mathbf{M}^{-1}\mathbf{x}) = r(\psi \cdot \mathbf{x}) \leqslant r\nu(\mathbf{x})$, we have $\|\mathbf{N}\mathbf{M}^{-1}\| = r$ so our choice of $r < 1$ gives (3.5).

Next, we choose $0 < s < 1$ small enough that $(\mathbf{u} - s\mathbf{v}) \in \mathscr{P}$—possible since $\mathbf{u}$ is in the interior of $\mathscr{P}$—and then set

$$\begin{aligned}
&\mathbf{D} := rs\mathbf{v} \otimes \psi : \mathbf{x} \mapsto rs(\psi \cdot \mathbf{x})\mathbf{v},\\
&\hat{\mathbf{M}} = \mathbf{M} - \mathbf{D} = \mathbf{I} - rs\mathbf{v} \otimes \psi,\\
&\hat{\mathbf{N}} = \mathbf{N} - \mathbf{D} = r(\mathbf{u} - s\mathbf{v}) \otimes \psi.
\end{aligned}$$

Note that our choice of s ensures that $\mathbf{D} \geqslant 0$ so $\hat{\mathbf{M}} \leqslant \mathbf{M}$ and that $rs < 1$ so $\hat{\mathbf{M}}^{-1} \geqslant 0$. Further, since $(\mathbf{u} - s\mathbf{v}) \in \mathscr{P}$ we have $\hat{\mathbf{N}} \geqslant 0$ so $\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} \geqslant 0$ and $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ is another left weak regular splitting of $\mathbf{T} = \mathbf{M} - \mathbf{N} = \mathbf{I} - r\mathbf{u} \otimes \psi$.

Finally, we must show that $\|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| > r = \|\mathbf{N}\mathbf{M}^{-1}\|$ to see that this is, indeed, a counterexample for (3.12). For any $\mathbf{x}$ we have

$$\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{x} = \hat{\mathbf{N}}\mathbf{y} = r(\psi \cdot \mathbf{y})(\mathbf{u} - s\mathbf{v}),$$

where $\mathbf{y} := \hat{\mathbf{M}}^{-1}\mathbf{x}$. This gives $\mathbf{x} = \hat{\mathbf{M}}\mathbf{y} = \mathbf{y} - rs(\psi \cdot \mathbf{y})\mathbf{v}$ so

$$\psi \cdot \mathbf{x} = (\psi \cdot \mathbf{y}) - rs(\psi \cdot \mathbf{y})\psi \cdot \mathbf{v} = (1 - rs)\psi \cdot \mathbf{y}$$

and $\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{x} = \frac{r}{1-rs}(\psi \cdot \mathbf{x})(\mathbf{u} - s\mathbf{v})$. Recalling that $\varphi \cdot \mathbf{u} = 1 = \psi \cdot \mathbf{v}$, we then have

$$\begin{aligned}\|\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\| &\geqslant \nu(\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{v}) \geqslant \varphi \cdot \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}\mathbf{v} \\ &= \frac{r}{1-rs}(\psi \cdot \mathbf{v})[\varphi \cdot (\mathbf{u} - s\mathbf{v})] = \frac{1 - s(\varphi \cdot \mathbf{v})}{1 - rs} r.\end{aligned}$$

Because we chose $r > \varphi \cdot \mathbf{v}$ we now have $1 - s(\varphi \cdot \mathbf{v}) > 1 - rs$ for $s > 0$, so this does give a counterexample for (3.12). □

Finally, we summarize some of the results in this section.

Theorem 3.8. *Let ν be a cone absolute norm and suppose that* $\mathbf{T}$ *has a left weak regular splitting* $[\tilde{\mathbf{M}}, \tilde{\mathbf{N}}]$ *with* $\tilde{\mathbf{N}}\tilde{\mathbf{M}}^{-1} > 0$. *Then the following are equivalent*:

1. *The norm ν is cone linear absolute defined by a functional φ such that* (3.3) *holds*.
2. *For left weak regular splittings* $[\mathbf{M}, \mathbf{N}]$ *and* $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ *with* $\|\mathbf{N}\mathbf{M}^{-1}\| \leqslant 1$, *the implication* (3.12) *holds*.

Proof. (1) $\Rightarrow$ (2). Since $\hat{\mathbf{M}} \leqslant \mathbf{M}$ implies $\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1}$, the result follows from Theorems 3.3 and 3.5.

(2) $\Rightarrow$ (1). By Theorems 3.7 and 3.4. □

We leave open the question as to whether (3.12) in (2) of Theorem 3.8 can be replaced by (3.9).

4. Dual results

We now wish to consider *right* weak regular splittings of $\mathbf{T}$:

$$\mathbf{M} - \mathbf{N} = \mathbf{T} \quad \text{with } \mathbf{M}^{-1} \geqslant 0, \quad \mathbf{M}^{-1}\mathbf{N} \geqslant 0. \tag{4.1}$$

By considering dual spaces, dual norms and dual cones, we obtain results analogous to those of the previous section since the induced norm of the transpose of an operator with respect to the dual of a norm equals the induced norm of the original operator with respect to the original norm.

Note that the dual norm for a cone linear absolute norm as in (2.4) and (2.7) is a *cone max norm* on $\mathscr{X}$ having the form

$$\nu(\mathbf{x}) = \inf\{t : -t\mathbf{w} \leqslant \mathbf{x} \leqslant t\mathbf{w}\}, \tag{4.2}$$

where $\mathbf{w}$ is a fixed vector in the interior of the proper cone $\mathscr{P} \subset \mathscr{X}$.

For ν given as in (2.4) and (2.7) the dual norm $\nu^*(\xi) := \sup\{\xi \cdot \mathbf{x} : \nu(\mathbf{x}) \leqslant 1\}$ is equivalently given by (4.2) with $\mathbf{w} = \varphi$. To see this, we need only note that to have $-t\varphi \leqslant \xi \leqslant t\varphi$ in the sense of the dual order just means that $\pm\xi \cdot \mathbf{u} \leqslant t\varphi \cdot$

$\mathbf{u} = t\nu(\mathbf{u})$. Writing $\mathbf{x} = \mathbf{v} - \mathbf{w}$ with $\mathbf{v}, \mathbf{w} \in \mathscr{P}$ and $\nu(\mathbf{x}) = \nu(\mathbf{v} + \mathbf{w})$ as in (2.5), we then have

$$\xi \cdot \mathbf{x} = \xi \cdot \mathbf{v} - \xi \cdot \mathbf{w} \leqslant t\varphi \cdot \mathbf{v} + t\varphi \cdot \mathbf{w} = t\nu(\mathbf{v} + \mathbf{w}) = t\nu(\mathbf{x}),$$

whence $\nu^*(\xi) \leqslant t$. We observe that a cone max norm is cone absolute. The terminology '*cone max norm*' comes from the fact that for the case of $\mathscr{P} = \mathbb{R}_+^K$ the weighted ℓ^1 norm (2.1) has, as dual, the weighted ℓ^∞ norm, $\nu^*(\xi) = \max_k\{|\xi_k|/w_k\}$.

This dual norm ν^* is not cone linear in the sense of (2.7) when ν is, but it is cone absolute with respect to $\mathscr{P}^*$ and hence does give (2.8). We note that $\mathbf{A}^\mathsf{T}$ is nonnegative with respect to the cone $\mathscr{P}^*$ if and only if $\mathbf{A} \geqslant 0$ with respect to $\mathscr{P}$.

In order to dualize the condition for strict inequality in Theorem 3.3, we need a theorem of the alternative:

Lemma 4.1. *Let $\mathscr{W}$ be a subspace of $\mathscr{X}$ and let $\mathscr{P}$ be a proper cone in $\mathscr{X}$. Then the interior of $\mathscr{W} \cap \mathscr{P}$ is nonempty if and only if $\mathscr{W}^\perp \cap \mathscr{P}^* = \{0\}$, where $\mathscr{W}^\perp$ is the orthogonal complement of $\mathscr{W}$ and $\mathscr{P}^*$ is the cone dual to $\mathscr{P}$.*

Proof. See, e.g., [4, Theorem 2.8] and the references given there. For the case $\mathscr{P} = \mathbb{R}_+^K$; see [21, Lemma 1.2]. □

We now state the dual of Theorem 3.3 as follows.

Theorem 4.2. *Let $\|\cdot\|$ be the matrix norm induced by a cone max norm determined by a positive vector $\mathbf{w}$ as in (4.2) and consider a right weak regular splitting $[\mathbf{M}, \mathbf{N}]$ of $\mathbf{T}$ as in (4.1).*

1. *If $\mathbf{Tw} \geqslant 0$, then $\|\mathbf{M}^{-1}\mathbf{N}\| \leqslant 1$ and the map: $\mathbf{M}^{-1} \mapsto \|\mathbf{M}^{-1}\mathbf{N}\|$ is antitone, i.e.,*

$$\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1} \quad \textit{implies} \quad \|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\| \leqslant \|\mathbf{M}^{-1}\mathbf{N}\|. \tag{4.3}$$

   If $\mathbf{T}$ is nonsingular, then $\mathbf{T}^{-1} \geqslant 0$.
2. *Further, suppose that $\mathbf{Tw} > 0$. Then $\mathbf{T}$ is nonsingular with $\mathbf{T}^{-1} \geqslant 0$. Also, we have the strict inequality $\|\mathbf{M}^{-1}\mathbf{N}\| < 1$ and, provided that*

$$\textit{for some } \mathbf{u} > 0, \quad [\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]\mathbf{u} > 0, \tag{4.4}$$

   we have

$$\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1} \quad \textit{implies} \quad \|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\| < \|\mathbf{M}^{-1}\mathbf{N}\|. \tag{4.5}$$

3. *Finally, if $\mathbf{T}^{-1} \geqslant 0$ then there is a $\mathbf{w}' > 0$ such that $\mathbf{Tw}' > 0$ and hence the inequality $\|\mathbf{M}^{-1}\mathbf{N}\| < 1$ holds for the induced norm corresponding to the cone max norm defined by $\mathbf{w}'$.*

Proof. Applying Lemma 4.1 with $\mathscr{W} = \mathrm{Range}[\hat{\mathbf{M}}^{-1} - \mathbf{M}^{-1}]$, shows that this theorem is precisely the dual of Theorems 3.1 and 3.3. □

Remark 4.3. In view of our proof of Part (2) of Theorem 3.1 we note that it is possibly to prove Part (2) of Theorem 4.2 without any reference to the compactness of the norm ball. We note as an immediate consequence of Theorem 2.1 that, for the cone max norm given by $\mathbf{w} > 0$, we have for a nonnegative matrix $\mathbf{P}$ that

$$\mathbf{P} \geqslant 0 \quad \text{implies} \quad \|\mathbf{P}\| = \nu(\mathbf{P}\mathbf{w}), \tag{4.6}$$

see [19]. Hence, if $\mathbf{T}\mathbf{w} > 0$, it follows that $(\mathbf{I} - \mathbf{M}^{-1}\mathbf{N})\mathbf{w} = \mathbf{M}^{-1}\mathbf{T}\mathbf{w} > 0$. Thus for some $\alpha > 1$, we have $\alpha\mathbf{M}^{-1}\mathbf{N}\mathbf{w} < \mathbf{w}$, which proves $\|\mathbf{M}^{-1}\mathbf{N}\| < 1$.

We note that the first part of Theorem 4.2 has been shown directly by Neumann–Plemmons [17, Lemma 2.2] and by Frommer and Szyld [9, Theorem 4.1] for the particular setting $\mathscr{P} = \mathbb{R}^K_+$; see also [3, Theorem 2.5]. For the strict inequality (4.5) these references require the assumption $\mathbf{M}^{-1} < \hat{\mathbf{M}}^{-1}$, obviously a stronger requirement than our condition (4.4).

We now state the duals of Theorems 3.4, 3.5, 3.7 and 3.8 without further proof.

Theorem 4.4. *Suppose* $\mathbf{T}$ *has a right weak regular splitting* $[\mathbf{M}, \mathbf{N}]$ *with* $\mathbf{M}^{-1}\mathbf{N}$ *strictly positive and that* ν *is a cone max norm with defining vector* $\mathbf{w}$ *as in* (4.2). *Then* (4.3) *can hold only if* $\mathbf{T}\mathbf{w} \geqslant 0$.

Theorem 4.5. *Let* $\|\cdot\|$ *be the matrix norm induced by a cone max norm* $\nu(\cdot)$ *as in* (4.2) *and consider splittings of* $\mathbf{T}$ *as in* (4.1). *If* $\|\mathbf{M}^{-1}\mathbf{N}\| \leqslant 1$, *then the map*: $\mathbf{M} \mapsto \|\mathbf{M}^{-1}\mathbf{N}\|$ *is isotone*, *viz*.

$$\hat{\mathbf{M}} \leqslant \mathbf{M} \quad \textit{implies} \quad \|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\| \leqslant \|\mathbf{M}^{-1}\mathbf{N}\|. \tag{4.7}$$

Theorem 4.6. *Assume the norm* $\nu(\cdot)$ *is cone absolute* (2.4). *Then* (4.7) *holds only if* $\nu(\cdot)$ *is a cone max norm*, *even if we restrict attention to right regular splittings* $[\mathbf{M}, \mathbf{N}]$ *with* $\|\mathbf{M}^{-1}\mathbf{N}| \leqslant 1$.

Theorem 4.7. *Let* ν *be a cone absolute norm and suppose that* $\mathbf{T}$ *has a right weak regular splitting* $[\tilde{\mathbf{M}}, \tilde{\mathbf{N}}]$ *with* $\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{N}} > 0$. *Then the following are equivalent*:

1. *The norm* ν *is cone max defined by a vector* $\mathbf{w}$ *such that* $\mathbf{T}\mathbf{w} \geqslant 0$.
2. *For right weak regular splittings* $[\mathbf{M}, \mathbf{N}]$ *and* $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ *with* $\|\mathbf{M}^{-1}\mathbf{N}\| \leqslant 1$, *the implication* (4.7) *holds*.

Again the question arises as to whether (4.7) in (2) of Theorem 4.7 can be replaced by the assumption (4.3).

5. Comparison for spectral radii

We begin this section with a corollary to Theorem 4.2. For $\mathscr{P} = \mathbb{R}^K_+$, this is found in [17,9], except that (as already noted after Theorem 4.2) we have weakened the hypothesis required for the strict inequality.

Corollary 5.1. *Consider a splitting* $[\mathbf{M}, \mathbf{N}]$ *of* $\mathbf{T}$ *as in* (4.1) *for which* $\mathbf{M}^{-1}\mathbf{N}$ *has a Perron vector* $\mathbf{w}$ (*i.e.*, $\mathbf{M}^{-1}\mathbf{N}\mathbf{w} = \rho(\mathbf{M}^{-1}\mathbf{N})\mathbf{w}$) *such that*

$$\mathbf{w} > 0 \quad \textit{with } \mathbf{T}\mathbf{w} \geqslant 0. \tag{5.1}$$

Then

$$\mathbf{M}^{-1} \leqslant \hat{\mathbf{M}}^{-1} \quad \textit{implies} \quad \rho(\hat{\mathbf{M}}^{-1}\hat{N}) \leqslant \rho(\mathbf{M}^{-1}\mathbf{N}), \tag{5.2}$$

i.e., the map: $\mathbf{M}^{-1} \mapsto \rho(\mathbf{M}^{-1}\mathbf{N})$ *is antitone. The conditions for strict inequality are the same as in Theorem* 4.2.

Proof. Let $\rho = \rho(\mathbf{M}^{-1}\mathbf{N})$ and let $\|\cdot\|$ be the cone max norm determined by $\mathbf{w}$ as in (4.2). For $0 \leqslant \mathbf{x} \leqslant \mathbf{w}$, we have $0 \leqslant \mathbf{M}^{-1}\mathbf{N}\mathbf{x} \leqslant \mathbf{M}^{-1}\mathbf{N}\mathbf{w}$ and $\mathbf{M}^{-1}\mathbf{N}\mathbf{w} = \rho\mathbf{w}$. It then follows from Theorem 2.1 that $\|\mathbf{M}^{-1}\mathbf{N}\| = \rho$ and as $\rho(\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}) \leqslant \|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\|$, we obtain (5.2).

The strict inequality follows as in Theorem 3.1. □

The additional conditions (5.1) imposed in Corollary 5.1 to obtain the spectral radius inequality correspond to the condition (3.3) used in Theorem 3.1. In view of Theorem 3.4 we expect that this cannot simply be omitted and adapt here an interesting example due to Elsner [7, p. 283]. In our example the Perron vector of $\mathbf{M}^{-1}\mathbf{N}$ does not satisfy (5.1), but, for a different $\mathbf{w}$, (3.3) is satisfied, which leads to inequality of spectral radius and norm in opposing directions. The last part of this example illustrates Theorem 4.4 and shows that the assumption $\mathbf{T}\mathbf{w} \geqslant 0$ is really needed for the conclusion (4.3).

Example 5.2. Consider the right weak regular splittings

$$\mathbf{T} = \begin{pmatrix} 1 & -1 \\ -1/2 & 1 \end{pmatrix} = \mathbf{M} - \mathbf{N} = \begin{pmatrix} 3/2 & -1 \\ -3/4 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 \\ -1/4 & 0 \end{pmatrix}$$
$$= \hat{\mathbf{M}} - \hat{\mathbf{N}} = \begin{pmatrix} 7/5 & -3/5 \\ -7/10 & 4/5 \end{pmatrix} - \begin{pmatrix} 2/5 & 2/5 \\ -1/5 & -1/5 \end{pmatrix}$$

for which we have

$$\mathbf{M}^{-1} = \begin{pmatrix} 4/3 & 4/3 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{M}^{-1}\mathbf{N} = \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\hat{\mathbf{M}}^{-1} = \begin{pmatrix} 8/7 & 6/7 \\ 1 & 2 \end{pmatrix}, \quad \hat{\mathbf{M}}^{-1}\hat{\mathbf{N}} = \begin{pmatrix} 2/7 & 2/7 \\ 0 & 0 \end{pmatrix}.$$

Here the Perron eigenvector $\mathbf{w} := (1, 0)^\top$ for $\mathbf{M}^{-1}\mathbf{N}$ is not strictly positive and, more significantly, $\mathbf{Tw} = (1, -1/2)^\top \not\geqslant (0, 0)^\top$ so the hypothesis (5.1) does not hold for this example. We then observe that $\hat{\mathbf{M}}^{-1} \leqslant \mathbf{M}^{-1}$ in the sense of the usual $\mathbb{R}^2_+$ ordering, but $\rho(\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}) = 2/7 \not\geqslant \rho(\mathbf{M}^{-1}\mathbf{N}) = 1/3$ so (5.2) fails. Of course, (consistent with Theorem 3.3, noting that $\mathbf{T}(1, 1)^\top$ is nonnegative) we do have $\|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\| > \|\mathbf{M}^{-1}\mathbf{N}\|$ with the usual ℓ^∞ norm (even though (4.4) does not hold). However, if we use $\mathbf{w}' = [1, 100]^\top$ as defining vector for our norm, and $\mathbf{P} = \mathrm{diag}(\mathbf{w}')$, we obtain

$$\|\hat{\mathbf{M}}^{-1}\hat{\mathbf{N}}\|' = 1/3 > 2/7 = \|\mathbf{M}^{-1}\mathbf{N}\|',$$

where $\|\cdot\|' = \|\mathbf{P}\cdot\mathbf{P}^{-1}\|$ is the operator norm induced by the weighted max norm given by $\mathbf{w}'$. We note that $\mathbf{Tw}' \not\geqslant 0$.

Comparison theorems for spectral radii of splittings of matrices have been generalized to bounded operators in Banach space, see [15,16,5]. In these papers additional conditions are imposed on positive operators, specifically the existence of a Perron vector is assumed.

We here adopt a different approach. For our final result, we use series domination in a Banach algebra setting to generalize the well-known result [22, Theorem 3.32]. We employ convergence properties of series so this applies to infinite dimensions without any appeal to the Perron–Frobenius theory of positive operators.

Thus we consider a real Banach algebra $\mathscr{A}$ (see [18, p. 245]) partially ordered by a proper cone $\mathscr{P}$ (with interior) consistent with addition and multiplication, viz.

$$\mathbf{P}, \mathbf{Q} \in \mathscr{P} \quad \text{implies} \quad \mathbf{P}+\mathbf{Q}, \mathbf{PQ} \in \mathscr{P}.$$

We also assume that the norm on $\mathscr{A}$ is monotone[2] on $\mathscr{P}$.

In this setting we note that the spectral radius ρ is given by the formula

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}, \tag{5.3}$$

which is consistent with the usual definition on $\mathscr{M}_+$. It is well known (see, e.g., [18, p. 263]) that (5.3) is equivalent to

$$\rho(A) = \inf_{n\geqslant 1} \|A^n\|^{1/n} \tag{5.4}$$

and easy to see that it is also equivalent to

$$1/\rho(\mathbf{A}) = \sup\{\alpha : \{\|[\alpha\mathbf{A}]^k\|\} \text{ bounded in } k\}. \tag{5.5}$$

[2] With small verbal changes, our proof below holds if the norm is semi-monotone, i.e.,

There is a $c > 0$ such that $0 \leqslant \mathbf{x} \leqslant \mathbf{y} \Rightarrow \|\mathbf{x}\| \leqslant c\|\mathbf{y}\|$.

This is a common assumption in this area; see, e.g., [14, p. 37] or [6].

A splitting $[\mathbf{M}, \mathbf{N}]$ of $\mathbf{T} \in \mathscr{A}$ is called *regular* if $\mathbf{T} = \mathbf{M} - \mathbf{N}$ where $\mathbf{M}^{-1} \geqslant 0$ and $\mathbf{N} \geqslant 0$.

Theorem 5.3. *With the properties of the Banach algebra $\mathscr{A}$ as above, either suppose that $\rho(\mathbf{N}\mathbf{M}^{-1}) < 1$ or suppose that $\mathbf{T}$ is invertible with $\mathbf{T}^{-1}$ nonnegative.*

1. *If $[\mathbf{M}, \mathbf{N}]$, $[\hat{\mathbf{M}}, \hat{\mathbf{N}}]$ are regular splittings of $\mathbf{T}$, then*
$$\hat{\mathbf{M}} \leqslant \mathbf{M} \quad \textit{implies} \quad \rho(\hat{\mathbf{N}}\hat{\mathbf{M}}^{-1}) \leqslant \rho(\mathbf{N}\mathbf{M}^{-1}) \leqslant 1, \tag{5.6}$$
i.e., the spectral radius of the iteration matrix is then isotone with respect to $\mathbf{M}$.
2. *Further, suppose that, for $\mathbf{P} \geqslant 0$, the sequence $\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^k + \cdots$ converges whenever its partial sums are uniformly bounded (viz., $\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^k \leqslant \mathbf{Q}$ for some $\mathbf{Q}$ and all k). Then $\mathbf{T}^{-1} \geqslant 0$ already implies that $\rho(\mathbf{N}\mathbf{M}^{-1}) < 1$.*

Proof. (1) We begin by setting $\mathbf{P} = \mathbf{N}\mathbf{M}^{-1}$ and $\overline{\rho} = \rho(\mathbf{P})$. With $\mathbf{M} - \hat{\mathbf{M}} \geqslant 0$, we also introduce
$$\mathbf{A} = \mathbf{I} - \hat{\mathbf{M}}\mathbf{M}^{-1} = (\mathbf{M} - \hat{\mathbf{M}})\mathbf{M}^{-1} \geqslant 0$$
and note that $\mathbf{A} = (\mathbf{N} - \hat{\mathbf{N}})\mathbf{M}^{-1}$ so $\mathbf{B} = \mathbf{P} - \mathbf{A} = \hat{\mathbf{N}}\mathbf{M}^{-1} \geqslant 0$—i.e., $0 \leqslant \mathbf{A} \leqslant \mathbf{P}$. Finally, we introduce
$$\mathbf{C} = (\mathbf{P} - \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{N}}\hat{\mathbf{M}}^{-1} \geqslant 0.$$

Our first observation is that the condition $\mathbf{T}^{-1} \geqslant 0$ implies $\overline{\rho} \leqslant 1$. To see this, note that $\mathbf{T} = (\mathbf{I} - \mathbf{P})\mathbf{M}$ so invertibility of $\mathbf{T}$ (and of $\mathbf{M}$) gives existence of $(\mathbf{I} - \mathbf{P})^{-1}$. Since $(\mathbf{I} - \mathbf{P})(\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^k) = \mathbf{I} - \mathbf{P}^{k+1}$ it follows that
$$\mathbf{P}^k \leqslant [\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^k] + \mathbf{P}^{k+1}(\mathbf{I} - \mathbf{P})^{-1} = (\mathbf{I} - \mathbf{P})^{-1} \tag{5.7}$$
for all k. Thus, each $\mathbf{P}^k \leqslant (\mathbf{I} - \mathbf{P})^{-1}$ whence, since the norm is monotonic, we deduce that for $\alpha = 1$, $\{\|[\alpha\mathbf{P}]^k\|\}$ is uniformly bounded by $\|(\mathbf{I} - \mathbf{P})^{-1}\|$, giving $\overline{\rho} \leqslant 1$ by (5.5). We can now choose $\alpha \geqslant 1$ arbitrarily close to $1/\overline{\rho}$—of course taking $\alpha = 1$ if $\overline{\rho} = 1$—and have $\{\|[\alpha\mathbf{P}]^k\|\}$ uniformly bounded.

To show (5.6) as desired means showing $\rho(\mathbf{C}) \leqslant \overline{\rho}$ for which, by (5.5), it is sufficient to show that $\{[\alpha\mathbf{C}]^k\}$ is uniformly bounded for α as here. We now define the nonnegative matrices
$$\mathbf{Z}_\alpha := (\mathbf{I} - \alpha\mathbf{P})^{-1}, \quad \mathbf{C}_\alpha := \mathbf{B}(\mathbf{I} - \alpha\mathbf{A})^{-1}.$$
When $\overline{\rho} < 1$ we always take $\alpha < 1/\overline{\rho}$ so we have convergence of the Neumann series in each case, noting the comparison $0 \leqslant \mathbf{A} \leqslant \mathbf{P}$. Note that comparison of the Neumann series gives $\mathbf{C} = \mathbf{C}_1 \leqslant \mathbf{C}_\alpha$ for $\alpha \geqslant 1$, as we are assuming. [If we were to have $\overline{\rho} = 1$ so $\alpha = 1$, then $\mathbf{Z}_\alpha = \mathbf{Z}_1 = (\mathbf{I} - \mathbf{P})^{-1}$, whose existence and nonnegativity have been assumed, and $\mathbf{C}_\alpha = \mathbf{C} \geqslant 0$.]

One can immediately compute the identities
$$\mathbf{Z}_\alpha = \mathbf{I} + \alpha\mathbf{A}\mathbf{Z}_\alpha + \alpha\mathbf{B}\mathbf{Z}_\alpha, \quad \mathbf{B}\mathbf{Z}_\alpha = \mathbf{C}_\alpha(\mathbf{I} + \alpha\mathbf{B}\mathbf{Z}_\alpha)$$

—multiply $\mathbf{I} = [(\mathbf{I} - \alpha[\mathbf{A} + \mathbf{B}]) + \alpha\mathbf{A} + \alpha\mathbf{B}]$ by $\mathbf{Z}_\alpha$ and, after noting that $\mathbf{B} = \mathbf{C}_\alpha(\mathbf{I} - \alpha\mathbf{A})$, multiply $\mathbf{I} - \alpha\mathbf{A} = [(\mathbf{I} - \alpha[\mathbf{A} + \mathbf{B}]) + \alpha\mathbf{B}]$ on the left by $\mathbf{C}_\alpha$ and on the right by $\mathbf{Z}_\alpha$. The first of these identities is the case $N = 0$ of the induction

$$\begin{aligned}\mathbf{Z}_\alpha &= \sum_0^N [\alpha\mathbf{C}_\alpha]^k + \alpha\mathbf{A}\mathbf{Z}_\alpha + \alpha[\alpha\mathbf{C}_\alpha]^N\mathbf{B}\mathbf{Z}_\alpha \\ &= \sum_0^N [\alpha\mathbf{C}_\alpha]^k + \alpha\mathbf{A}\mathbf{Z}_\alpha + \alpha[\alpha\mathbf{C}_\alpha]^N\mathbf{C}_\alpha(\mathbf{I} + \alpha\mathbf{B}\mathbf{Z}_\alpha) \\ &= \sum_0^{N+1} [\alpha\mathbf{C}_\alpha]^k + \alpha\mathbf{A}\mathbf{Z}_\alpha + \alpha[\alpha\mathbf{C}_\alpha]^{N+1}\mathbf{B}\mathbf{Z}_\alpha. \end{aligned} \tag{5.8}$$

Since each term is in $\mathscr{M}_+$, this shows that

$$[\alpha\mathbf{C}]^k \leqslant [\alpha\mathbf{C}_\alpha]^k \leqslant \mathbf{Z}_\alpha \quad \text{independently of } k$$

for each such α so, as above, $\{\|[\alpha\mathbf{C}]^k\|\}_0^\infty$ is bounded. Choosing α arbitrarily close to $1/\overline{\rho}$, this gives $\rho(\mathbf{C}) \leqslant \overline{\rho}$, completing the proof of 1.

(2) We have noted that the partial sums $[\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^k]$ are uniformly (order) bounded by $(\mathbf{I} - \mathbf{P})^{-1}$ so, under our additional hypothesis, the series converges. Of course the individual terms then go to 0, so certainly $\|\mathbf{P}^k\| < 1$ for large k whence, using (5.4), we have $\rho(\mathbf{P}) = \inf_k\{\|\mathbf{P}^k\|^{1/k}\} < 1$. $\square$

We remark that the condition in 2 of Theorem 5.3 is satisfied when $\mathscr{A} = \mathscr{M}_+$, since all linear topologies are equivalent to the Euclidean norm in finite dimensions. In this finite dimensional case we have thus obtained the full force of Varga's Theorem 3.32 in [22] without any appeal to Perron–Frobenius theory.

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