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Integral Equations and Operator Theory

Equilibria of Pairs of Nonlinear Maps Associated with Cones

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Abstract. Let K_1, K_2 be closed, full, pointed convex cones in finite-dimensional real vector spaces of the same dimension, and let $F : K_1 \to \operatorname{span} K_2$ be a homogeneous, continuous, K_2-convex map that satisfies $F(\partial K_1) \cap \operatorname{int} K_2 = \emptyset$ and $FK_1 \cap \operatorname{int} K_2 \neq \emptyset$. Using an equivalent formulation of the Borsuk-Ulam theorem in algebraic topology, we show that we have $F(K_1 \backslash \{0\}) \cap (-K_2) = \emptyset$ and $K_2 \subseteq FK_1$. We also prove that if, in addition, $G : K_1 \to \operatorname{span} K_2$ is any homogeneous, continuous map which is (K_1, K_2)-positive and K_2-concave, then there exist a unique real scalar ω_0 and a (up to scalar multiples) unique nonzero vector $x_0 \in K_1$ such that $Gx_0 = \omega_0 Fx_0$, and moreover we have $\omega_0 > 0$ and $x_0 \in \operatorname{int} K_1$ and we also have a characterization of the scalar ω_0. Then, we reformulate the above result in the setting when K_1 is replaced by a compact convex set and recapture a classical result of Ky Fan on the equilibrium value of a finite system of convex and concave functions.

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1. Introduction

In this paper we prove equilibrium theorems of Perron-Frobenius type for a pair of nonlinear maps F and G from a proper cone K_1 in a finite dimensional real space to another finite dimensional real space ordered by another proper cone K_2; namely, we determine conditions under which there is a unique positive scalar ω_0 and a unique fixed vector x_0 (up to scalar multiples) in K_1 such that $Gx_0 = \omega_0 Fx_0$, see Theorem 2.3. We also show that ω_0 can be obtained as infimum or supremum of analogs of the Collatz-Wielandt sets further discussed in our last section. In

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Theorem 2.4 we derive a version of our equilibrium theorem with a compact convex set as the domain space.

Our motivation is [8] Theorem 1, due to Ky Fan on the equilibrium value of a finite system of convex and concave functions which we state at the beginning of next section. However, we do not use this theorem in deriving our main results, Theorems 2.3 and 2.4, which may be considered as its extensions. Instead, we use the Borsuk-Ulam theorem to establish a geometric result about a nonlinear map (see Theorem 2.1) and then use it to deduce our main results. Ky Fan's theorem can be recovered from our extension by means of Sperner's Lemma [15].

Our paper continues a long tradition of generalizations of the Perron-Frobenius theorem. While the setting of our work is strictly finite dimensional (which is natural in view of our use of the Borsuk-Ulam theorem and Sperner's Lemma), many generalizations are to operators in a Banach space which leave a cone invariant. We point to recent linear and nonlinear generalizations in [11], [12] and [13], and to the recent surveys [5], [18], [16] and books [2], [10] and [9] for different aspects of the theory and many further references.

2. Statements of Main Results

In [8, Theorem 1], Ky Fan obtained the following result discussed in our introduction.

Ky-Fan's Theorem. *Let S denote the standard $(n-1)$-simplex of $\mathbb{R}^n$, i.e., $S = \{(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_+ : \sum_{j=1}^n \xi_j = 1\}$, and let $S_i = \{(\xi_1, \ldots, \xi_n) \in S : \xi_i = 0\}$ for $i = 1, \ldots, n$. For $i = 1, \ldots, n$, also let $f_1, \ldots, f_n$, $g_1, \ldots, g_n$ be $2n$ real-valued functions defined on S that satisfy the following:*

(a) *Each f_i is continuous and convex on S;*

(b) *$f_i(x) \leq 0$ for each $x \in S_i$;*

(c) *For each $x \in S$ there is an index i for which $f_i(x) > 0$; and*

(d) *Each g_i is continuous, concave and positive on S.*

Then there exist a unique real number λ and a unique point $\hat{x} \in S$ such that for every i, $g_i(\hat{x}) = \lambda f_i(\hat{x})$. Moreover, we have $\lambda > 0$, $\hat{x}$ has positive components, and

$$\frac{1}{\lambda} = \min_{x \in S} \max_{1 \leq i \leq n} \frac{f_i(x)}{g_i(x)} = \max_{x \in S} \min_{1 \leq i \leq n} \frac{f_i(x)}{g_i(x)}.$$

Notice that under the hypotheses of Ky Fan's theorem, if we define a map $f : S \to \mathbb{R}^n$ by $f(x) = (f_1(x), \ldots, f_n(x))$, then f is a convex map in the sense that, for any scalar λ, $0 < \lambda < 1$, and $x,\ y \in S$, we have $f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$, where the ordering is componentwise. Similarly, if we define $g : S \to \mathbb{R}^n$ by $g(x) = (g_1(x), \ldots, g_n(x))$, then g is a concave map (i.e., $-g$ is a convex map). The conclusion of Ky Fan's theorem can now be restated as: $g(\hat{x}) = \lambda f(\hat{x})$ for some real number λ and $\hat{x} \in S$. In this case, we say that λ is an *equilibrium value* and $\hat{x}$ is an *equilibrium point* for the system (g, f). The concepts of equilibrium value and equilibrium point come from economic models (see, for instance, [2].

As already noted in [8], if $A = (a_{ij})$ is an $n \times n$ (entrywise) positive matrix, and if we define f_i, g_i $(1 \leq i \leq n)$ on S by $f_i(x) = \xi_i$ and $g_i(x) = \sum_{i=1}^{n} a_{ij}\xi_j$ for $x = (\xi_1, \ldots, \xi_n) \in S$, then conditions (a), (b), (c) and (d) of Ky Fan's theorem are satisfied. In this case, the first part of Ky Fan's theorem becomes the classical Perron's theorem on positive matrices (with λ being the spectral radius and $\hat{x}$ the Perron vector of A). The last part of Ky Fan's theorem becomes Wielandt's extremal characterization of the spectral radius.

In Aubin [1] one may find extensions or variants of Ky Fan's theorem in the setting of a pair of multi-valued maps. In [14, THeorem 4.1], Simons generalized the first part of Ky Fan's theorem in such a way that the finite systems of functions are replaced by two (single-valued) maps whose common range space is a real vector space with a given sublinear (i.e., positively homogeneous, convex) function, referred to as a sublineared space, and which are convex or concave in a certain generalized sense defined with respect to the sublinear structure, and moreover the domain space is not restricted to an $(n-1)$-simplex. In fact, Simons obtained first a continuity result about a pair of multi-valued maps that involve a sublineared space and used it to deduce the aforementioned result and also to obtain a result that generalizes [1, Theorem 2], and hence the last part of Ky Fan's theorem, in the setting of a pair of multi-valued maps. In this paper, we give a generalization in a different direction. We first examine conditions (a)–(c) of Ky Fan's theorem in the setting of a homogeneous map on a proper cone.

We call a nonempty subset K in a finite-dimensional real vector space V a *proper cone* of V if K is a convex cone (i.e. $\alpha K + \beta K \subseteq K$ for all $\alpha, \beta \geq 0$), which is pointed (i.e. $K \cap (-K) = \{0\}$), closed (with respect to the usual topology of V) and has nonempty interior (or equivalently, $\operatorname{span} K$, the linear span of K, is V). We use $\geq^K$ to denote the partial ordering on $\operatorname{span} K$ induced by the proper cone K, i.e. $x \geq^K y$ if and only if $x - y \in K$. For convenience, we also adopt the following notation:

$$\begin{aligned} & x >^K 0 \quad \text{if and only if } x \geq^K 0 \text{ and } x \neq 0, \\ \text{and} \quad & x \gg^K 0 \quad \text{if and only if } x \in \operatorname{int} K. \end{aligned}$$

Sometimes we also use $\geq$, $>$ and $\gg$ in place of $\geq^K$, $>^K$ and $\gg^K$, when there is no danger of confusion.

We obtain the following result:

Theorem 2.1. *Let K_1, K_2 be proper cones. Let $F : K_1 \to \operatorname{span} K_2$ be a homogeneous map that satisfies each of the following conditions:*

(a) *For any x, $y \in K_1$, there exist positive constants α, β (depending on x and y) such that $\alpha Fx + \beta Fy \geq^{K_2} F(x+y)$;*

(b) *$F(\partial K_1) \cap \operatorname{int} K_2 = \emptyset$; and*

(c) $FK_1 \cap \operatorname{int} K_2 \neq \emptyset$.

Then $F(K_1 \backslash \{0\}) \cap (-K_2) = \emptyset$. *If, in addition,* $\dim K_1 = \dim K_2$ *and* F *is continuous, then* $K_2 \subseteq FK_1$.

Here we use $\operatorname{int} S$ (respectively, ∂S) to denote the interior (respectively, boundary) of S. A map $T : D \subseteq V_1 \to V_2$, where V_1, V_2 are real vector spaces and D satisfies $\lambda D \subseteq D$ for all $\lambda > 0$, is said to be *homogeneous* (of degree one) if $T(\lambda x) = \lambda Tx$ for all $\lambda > 0$ and $x \in D$.

To avoid trivialities, we assume that the cones K_1, K_2 considered in Theorem 2.1 are nonzero. The same remark also applies (sometimes to K) in the remaining parts of the paper.

Note that, when $K_1 = K_2 = K$, condition (b) of Theorem 2.1 is weaker than the following natural extension of condition (b) of Ky Fan's theorem: For any $x \in \partial K$, $p \in \partial K^*$, where K^* denotes the dual cone of K, we have $p(Fx) \leq 0$ whenever $p(x) = 0$.

The proof of Theorem 2.1 relies on the use of an equivalent formulation of the Borsuk-Ulam theorem in algebraic topology. A modification of the argument used in the proof also leads to the following unexpected side-product:

Theorem 2.2. *Let* K_1, K_2 *be proper cones such that* $\dim K_1 > \dim K_2$. *Let* $F : K_1 \to \operatorname{span} K_2$ *be a homogeneous, continuous map with the property that for any* $x, y \in K_1$, *there exist* $\alpha, \beta > 0$ *such that* $\alpha Fx + \beta Fy \geq^{K_2} F(x+y)$. *If* $FK_1 \cap \operatorname{int} K_2 \neq \emptyset$, *then* $F(\partial K_1) \cap \operatorname{int} K_2 \neq \emptyset$ *and moreover we have either* $F(K_1 \backslash \{0\}) \cap (-K_2) \neq \emptyset$ *or* $\operatorname{int} K_2 \cap F(\operatorname{int} K_1) \subseteq F(\partial K_1)$.

Theorems 2.1 and 2.2 can be restated as results about solvability of nonlinear systems.

Using Theorem 2.1, we derive the following result which extends Ky Fan's theorem and also [8, Corollaries 1 and 2] in the setting of homogeneous maps on proper cones.

A map F: $K_1 \to \operatorname{span} K_2$ is said to be K_2-*convex* (respectively, K_2-concave) if $(1-\lambda)Fx + \lambda Fy \geq^{K_2} F((1-\lambda)x + \lambda y)$ (respectively, $(1-\lambda)Fx + \lambda Fy \;{}^{K_2}\!\!\leq F((1-\lambda)x + \lambda y))$ for all real scalar λ, $0 < \lambda < 1$, and $x, y \in K_1$; F is (K_1, K_2)-*nonnegative* (respectively, (K_1, K_2)-*positive*) if $FK_1 \subseteq K_2$ (respectively, $F(K_1 \backslash \{0\}) \subseteq \operatorname{int} K_2$); F is (K_1, K_2)-*monotone* (or, *order-preserving*, according to some authors) if $y \geq^{K_1} x$ implies $Fy \geq^{K_2} Fx$. Clearly, if F is homogeneous, K_2-convex, then F possesses the property that for any $x, y \in K_1$, there exist $\alpha, \beta > 0$ such that $\alpha F(x) + \beta F(y) \geq^{K_2} F(x+y)$.

Theorem 2.3. *Let* K_1, K_2 *be proper cones such that* $\dim K_1 = \dim K_2$. *Let* $F : K_1 \to \operatorname{span} K_2$ *be a homogeneous, continuous map that satisfies each of the following conditions*:

(a) F *is* K_2-*convex*;

(b) $F(\partial K_1) \cap \operatorname{int} K_2 = \emptyset$; *and*

(c) $FK_1 \cap \operatorname{int} K_2 \neq \emptyset$.

Then, for any homogeneous, continuous, K_2-*concave and* (K_1, K_2)-*positive map*

$G : K_1 \to \operatorname{span} K_2$, *there exist a unique scalar* ω_0 *and a (up to scalar multiples) unique nonzero vector* x_0 *of* K_1 *such that* $Gx_0 = \omega_0 F x_0$. *We have,* $\omega_0 > 0$, $x_0 \in \operatorname{int} K_1$ *and* $\sup \Omega = \inf \Sigma_1 = \omega_0$, *where*

$$\begin{aligned} \Omega &= \{\omega \geq 0 : \exists x >^{K_1} 0,\ Gx \geq^{K_2} \omega F x\} \\ \text{and } \Sigma_1 &= \{\sigma \geq 0 : \exists x \gg^{K_1} 0,\ Gx \ {}^{K_2}\!\!\leq \sigma F x\}. \end{aligned}$$

Moreover, for any $x >^{K_1} 0$ *and* $\omega, \sigma > 0$, *we have*

$$\begin{aligned} &\omega < \omega_0 \text{ whenever } Gx \geq^{K_2} \omega F x \text{ and } x \text{ is not a multiple of } x_0 \\ \text{and} \quad &\sigma > \omega_0 \text{ whenever } Gx \ {}^{K_2}\!\!\leq \sigma F x \text{ and } x \text{ is not a multiple of } x_0. \end{aligned}$$

In Theorem 2.4 below we give a reformulation of Theorem 2.3 in the setting when the common domain K_1 of F and G is replaced by a compact convex set.

For a convex set C, we use $\operatorname{ri} C$ and $\operatorname{rbd} C$ to denote respectively the relative interior and the relative boundary of C. A map $g : C \to W$ from a convex set C to a real vector space W ordered by a proper cone K is said to be (C, K)-*nonnegative* (respectively, (C, K)-*positive*) if $g(C) \subseteq K$ (respectively, $g(C) \subseteq \operatorname{int} K$); K-convexity and K-concavity of g are defined in the same way as in the case when C is a proper cone.

Theorem 2.4. *Let* C *be a compact convex set in a finite-dimensional real vector space, and let* $f : C \to W$ *be a continuous map from* C *to a finite-dimensional real vector space* W *ordered by a proper cone* K *such that* $\dim W = \dim C + 1$. *Suppose that* f *satisfies each of the following conditions*:

(a) f *is* K-*convex*;

(b) $f(\operatorname{rbd} C) \cap \operatorname{int} K = \emptyset$; *and*

(c) $f(C) \cap \operatorname{int} K \neq \emptyset$.

Then, for any continuous, K-*concave and* (C, K)-*positive map* $g : C \to W$, *there exist a unique real scalar* ω_0 *and a unique point* x_0 *of* C *such that* $g(x_0) = \omega_0 f(x_0)$. *We have,* $\omega_0 > 0$, $x_0 \in \operatorname{ri} C$ *and* $\sup \Omega = \inf \Sigma_1 = \omega_0$, *where*

$$\begin{aligned} \Omega &= \{\omega \geq 0 : \exists x \in C : g(x) \geq^{K} \omega f(x)\} \\ \text{and } \Sigma_1 &= \{\sigma \geq 0 : \exists x \in \operatorname{ri} C : g(x) \ {}^{K}\!\!\leq \sigma f(x)\}. \end{aligned}$$

Moreover, for any $x \in C$ *and* $\omega, \sigma > 0$, *we have*

$$\begin{aligned} &\omega < \omega_0 \quad \text{whenever } g(x) \geq^{K} \omega f(x) \text{ and } x \neq x_0 \\ \text{and} \quad &\sigma > \omega_0 \quad \text{whenever } g(x) \ {}^{K}\!\!\leq \sigma f(x) \text{ and } x \neq x_0. \end{aligned}$$

3. Nonlinear Solvability Theorems

In this section we shall prove Theorems 2.1, 2.2 and make relevant remarks and illustrative examples. Before we begin, we recall some facts from topology, which we shall need.

We shall identify finite-dimensional real vector spaces with euclidean spaces. Let B^n, S^{n-1} denote respectively the euclidean unit ball and unit sphere of $\mathbb{R}^n$.

For a proper cone K in $\mathbb{R}^n$, $n \geq 2$, we define a map π_K from the set $\{(z,v) : z \in \operatorname{int} K \cap S^{n-1},\ v \in S^{n-1},\ v \neq z, -z\}$ to $\partial K \cap S^{n-1}$ as follows: Let $z, v \in S^{n-1}$ with $z \in \operatorname{int} K$ and $v \neq z, -z$. Then $\operatorname{span}\{z,v\} \cap S^{n-1}$ is a circle, and $\operatorname{span}\{z,v\} \cap K \cap S^{n-1}$ is a closed circular arc whose endpoints belong to opposite semicircles determined by z and $-z$ and constitute the set $\operatorname{span}\{z,v\} \cap (\partial K \cap S^{n-1})$. We denote by $\pi_K(z,v)$ the endpoint in the semicircle that contains v. Observe that the point $\pi_K(z,v)$ is uniquely determined by the property that it belongs to ∂K and can be expressed in the form $\frac{\lambda z + v}{\|\lambda z + v\|}$ for some $\lambda \in \mathbb{R}$. That π_K is a continuous map is probably known. We give a proof below, as we have not been able to find any suitable reference.

Assume to the contrary that π_K is not continuous at (z,v) for some $z \in \operatorname{int} K \cap S^{n-1}$ and $v \in S^{n-1}$, $v \neq z, -z$. Then there exist a sequence $(z_k)_{k\in\mathbb{N}}$ in $\operatorname{int} K \cap S^{n-1}$ converging to z and a sequence $(v_k)_{k\in\mathbb{N}}$ in S^{n-1} converging to v such that, for some fixed $\delta > 0$, we have $\|\pi_K(z_k, v_k) - \pi_K(z,v)\| \geq \delta$ for all k. Now, for each k, we have, $\pi_K(z_k, v_k) = \frac{\lambda_k z_k + v_k}{\|\lambda_k z_k + v_k\|}$ for some real scalar λ_k. Note that the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is bounded; otherwise, $(z_k + \lambda_k^{-1} v_k)_{k\in\mathbb{N}}$ is a sequence in ∂K with a subsequence converging to z, which is a contradiction, as $z \in \operatorname{int} K$. Replacing by a subsequence, if necessary, we may assume that $(\lambda_k)_{k\in\mathbb{N}}$ converges to λ. Then we have $\lim_{k\to\infty} \pi_K(z_k, v_k) = \frac{\lambda z + v}{\|\lambda z + v\|}$. But $\lim_{k\to\infty} \pi_K(z_k, v_k)$ belongs to ∂K, so it is, in fact, equal to $\pi_K(z,v)$, which is a contradiction.

If $z \in \operatorname{int} K$, and $x,\ \bar{x} \in \partial K \cap S^{n-1}$ are such that z can be expressed as a linear combination of x and $\bar{x}$ with positive coefficients, then we say that x and $\bar{x}$ form *a pair of antipodal points of $\partial K \cap S^{n-1}$ relative to z*. Notice that the map $\pi_K(z, \cdot)$ takes each pair of antipodal points of the sphere $(\operatorname{span}\{z\})^\perp \cap S^{n-1}$ (which can be identified with S^{n-2}) to a pair of antipodal points of $\partial K \cap S^{n-1}$ relative to z.

Recall that two continuous maps $f_0,\ f_1 : X \to Y$ between topological spaces X, Y are said to be *homotopic* if one can be deformed continuously to the other, i.e., f_0 and f_1 belong to a family of continuous maps $f_t : X \to Y$, $t \in [0,1]$, so that $\Phi : X \times [0,1] \to Y$ given by $\Phi(x,t) = f_t(x)$ is continuous.

We shall make use of the following known results from algebraic topology:

Lemma A. *A continuous map $f : S^{n-1} \to Y$, where Y is a topological space and $n \geq 1$, is homotopic to a constant map if and only if f can be extended to a continuous map from B^n to Y.*

Theorem A. *If $f : S^n \to S^n$, $n \geq 0$, is a continuous map which is homotopic to a constant map, then there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Corollary A. *If $f : S^n \to S^m$, where $0 \leq m < n$, is a continuous map, then there exists $x \in S^n$ such that $f(x) = f(-x)$.*

Lemma A is elementary and can be found in many textbooks of topology; see, for instance, [6, p.316, 1.2(2)]. Theorem A is equivalent to the Borsuk-Ulam theorem, which asserts that every continuous map $f : S^n \to \mathbb{R}^n$, $n \geq 1$, sends at least one pair of antipodal points to the same points, and, in fact, equivalent to them are also several other geometric results about the n-sphere, such as the Borsuk antipodal theorem, the Lusternik-Schnirelman-Borsuk theorem, etc. (see, for instance, [7, Theorems 5.2 and 6.1]). Corollary A can be deduced from Theorem A as follows: If $m < n$, we may regard S^m as lying in the equator of S^n and consider the map $\hat{f} : S^n \to S^n$ which is obtained from f by enlarging its range space to S^n. Since the image set $\hat{f}(S^n)$ is included in the upper hemisphere S^{n+} and S^{n+}, being homeomorphic to B^n, is a contractible space (i.e., one whose identity map is homotopic to a constant map), the map $\hat{f}$ is homotopic to a constant map. By Theorem A, it follows that there exists a pair of antipodal points of S^n with the same image under $\hat{f}$. Since $f(x) = \hat{f}(x)$ for all $x \in S^n$, we also have two antipodal points with the same image under f.

Proof of Theorem 2.1. Assume to the contrary that there exists $x > 0$ such that $Fx \in -K_2$. By conditions (c) and (b), there exists $u \gg 0$ such that $Fu \gg 0$. Since $u \in \operatorname{int} K_1$ and $-x \notin K_1$, there exists $\varepsilon > 0$ such that $u - \varepsilon x \in \partial K_1$. By the homogeneity of F and condition (a), we have

$$0 \ll Fu \leq \alpha F(u - \varepsilon x) + \beta \varepsilon F x$$

for some $\alpha,\ \beta > 0$. Thus, $F(u - \varepsilon x) \geq \alpha^{-1} F u - \alpha^{-1} \beta \varepsilon F x \gg 0$, as $-Fx \geq 0$ and $Fu \gg 0$. This contradicts condition (b).

Now suppose, in addition, that F is continuous and K_1, K_2 have the same dimension. There is no loss of generality in assuming that $\mathbb{R}^n = \operatorname{span} K_1 = \operatorname{span} K_2$. The case $n = 1$ is trivial. Hereafter, we assume that $n \geq 2$. Let $f : K_1 \cap S^{n-1} \to S^{n-1}$ be the map given by: $f(x) = Fx/\|Fx\|$, where $\|\cdot\|$ denotes the euclidean norm of $\mathbb{R}^n$. Note that f is well-defined, as $Fx \neq 0$ for all $x \in K_1 \backslash \{0\}$, and is also continuous. Since F is homogeneous, it suffices to show that $K_2 \cap S^{n-1} \subseteq f(K_1 \cap S^{n-1})$.

Assume to the contrary that there exists $y \in K_2 \cap S^{n-1}$ such that $y \notin f(K_1 \cap S^{n-1})$. Since the set $f(K_1 \cap S^{n-1})$ is compact and hence closed, we may choose y so that $y \in \operatorname{int} K_2$. Let $\theta_y : \partial K_1 \cap S^{n-1} \to \partial K_2 \cap S^{n-1}$ be the map defined by: $\theta_y(v) = \pi_{K_2}(y, fv)$, where $\pi_{K_2} : \{(z, v) : z \in \operatorname{int} K_2 \cap S^{n-1},\ v \in S^{n-1},\ v \neq z, -z\} \to \partial K_2 \cap S^{n-1}$ is the continuous map that we have introduced at the beginning of this section. Since $y, -y \notin f(K_1 \cap S^{n-1})$, θ_y is a well-defined map. Indeed, for the same reason, we can extend the domain of θ_y to $K_1 \cap S^{n-1}$, using the same formula for definition. Of course, θ_y and its extension are continuous maps. But there is a homeomorphism from $K_1 \cap S^{n-1}$ onto B^{n-1} which takes $\partial K_1 \cap S^{n-1}$ onto S^{n-2}, so by Lemma A, it follows that the map θ_y is homotopic to a constant map. Now we are going to obtain another map from $\partial K_1 \cap S^{n-1}$ to $\partial K_2 \cap S^{n-1}$, which is homotopic to θ_y, as follows. By conditions (c) and (b), there exists a vector $u \in \operatorname{int} K_1 \cap S^{n-1}$ such that $fu \in \operatorname{int} K_2 \cap S^{n-1}$. Denote fu by z and define the desired map θ_z by $\theta_z(v) = \pi_{K_2}(z, fv)$. Clearly, θ_z is well-defined and continuous. Moreover, the continuous map $\Phi : (\partial K_1 \cap S^{n-1}) \times [0, 1] \to \partial K_2 \cap S^{n-1}$ given by $\Phi(v, t) = \pi_{K_2}(y(t), fv)$, where $y(t) = \frac{(1-t)y+tz}{\|(1-ty)+tz\|}$, establishes a homotopy of θ_y to θ_z. Since θ_y is homotopic to a constant map, so is θ_z. On the other hand, the continuous map $\pi_{K_1}(u,\cdot)$ takes the compact set $(\mathrm{span}\{u\})^{\perp}\cap S^{n-1}$ one-to-one, and hence homeomorphically, onto $\partial K_1 \cap S^{n-1}$ and moreover it sends each pair of antipodal points of the sphere $(\mathrm{span}\{u\})^{\perp}\cap S^{n-1}$ (which can be identified with S^{n-2}) to a pair of antipodal points of $\partial K_1 \cap S^{n-1}$ relative to u. Also, $\partial K_2 \cap S^{n-1}$ is homeomorphic with S^{n-2}. In view of Theorem A, it follows that there exists a pair of antipodal points x, $\bar{x}$ of $\partial K_1 \cap S^{n-1}$ relative to u such that $\theta_z(x) = \theta_z(\bar{x})$. The fact that x, $\bar{x}$ are antipodes clearly implies that there exist $\nu,\ \eta > 0$ such that $u = \nu x + \eta \bar{x}$. By the homogeneity of F and condition (a), we have

$$\alpha\nu F(x) + \beta\eta F(\bar{x}) \geq F(\nu x + \eta\bar{x}) = F(u) \gg 0$$

for some $\alpha,\ \beta > 0$. On the other hand, the condition $\theta_z(x) = \theta_z(\bar{x})$, which amounts to $\pi_{K_2}(z, fx) = \pi_{K_2}(z, f\bar{x})$, together with the fact that $Fx, F\bar{x} \notin \mathrm{int}\, K_2$, clearly implies that $\lambda F(x) + \mu F(\bar{x}) \notin \mathrm{int}\, K_2$ for any $\lambda,\ \mu > 0$. So we arrive at a contradiction. $\square$

It can be readily checked that in Theorem 2.1 if we assume that F is homogeneous of degree p, where p is a positive number possibly different from 1, then the result is still valid.

The following examples illustrate the irredundancy of condition (a) of Theorem 2.1.

Example 3.1. Let K be the proper convex cone in $\mathbb{R}^2$ given by:

$$K = \{\lambda(\cos\theta, \sin\theta) : \lambda \geq 0, -\pi/4 \leq \theta \leq \pi/4\},$$

and let $F\colon K \to \mathbb{R}^2$ be the map defined by: $F(\lambda(\cos\theta, \sin\theta)) = \lambda(\cos 3\theta, \sin 3\theta)$. Then F is homogeneous, continuous and satisfies conditions (b) and (c) of Theorem 2.1 (with $K_1 = K_2 = K$). However, $F(K\backslash\{0\}) \cap (-K) \neq \emptyset$, as $F(1,1) = (1,-1) \in -K$. (But we do have $FK \supseteq K$ in this case.)

Example 3.2. Let g be any real-valued concave continuous function defined on the closed interval [0,1] such that $g(0) = g(1) = 0$ and $g(t) > 0$ for all $t \in (0,1)$. Let $F\colon \mathbb{R}^2_+ \to \mathbb{R}^2$ be the homogeneous map determined by: $F(1-t, t) = g(t)(\frac{1}{2}, \frac{1}{2})$ for all $t \in [0,1]$. Then F is continuous, $\mathbb{R}^2_+$-concave (but not $\mathbb{R}^2_+$-convex). Also, conditions (b) and (c) of Theorem 2.1 are satisfied. However, we have $F(\mathbb{R}^2_+\backslash\{0\}) \cap (-\mathbb{R}^2_+) = \{0\} \neq \emptyset$ and $\mathbb{R}^2_+ \not\subseteq F\mathbb{R}^2_+$.

Example 3.3. Let $F : \mathbb{R}^2_+ \to \mathbb{R}^2$ be defined by: $F(\xi_1, \xi_2)$ equals (ξ_1, ξ_2) if $\xi_1 \geq \xi_2$ and equals (ξ_2, ξ_1) if $\xi_1 < \xi_2$. Then F is homogeneous, continuous and we have $F(\partial\mathbb{R}^2_+) \cap \mathrm{int}\,\mathbb{R}^2_+ = \emptyset$, $F\mathbb{R}^2_+ \cap \mathrm{int}\,\mathbb{R}^2_+ \neq \emptyset$ and $F(\mathbb{R}^2_+\backslash\{0\}) \cap (-\mathbb{R}^2_+) = \emptyset$. However, $\mathbb{R}^2_+ \not\subset F\mathbb{R}^2_+$. So, in Theorem 2.1, when F is continuous and $\dim K_1 = \dim K_2$, without condition (a), we cannot infer that $K_2 \subseteq FK_1$, even if we add as an extra assumption the condition that $F(K_1\backslash\{0\}) \cap (-K_2) = \emptyset$.

We would also like to point out that the last part of Theorem 2.1 is invalid if we assume $\dim K_1 < \dim K_2$ instead of the equality. Indeed, in this case, for any map $F : K_1 \to \mathrm{span}\, K_2$ which is linear (i.e., $F(\alpha x + \beta y) = \alpha Fx + \beta Fy$ for all $\alpha, \beta \geq 0$ and $x, y \in K_1$) and satisfies conditions (a)–(c) of Theorem 2.1 (for instance, take $K_2 = \mathbb{R}^3_+$, $K_1 = \mathrm{pos}\{(1,0,0),(0,1,1)\}$, where we use $\mathrm{pos}\, S$ to denote the positive hull of S, i.e., the set of all (finite) nonnegative linear combinations of vectors in S, and $F : K_1 \to \mathrm{span}\, K_2$ to be the canonical injection), it is impossible that the inclusion $K_2 \subseteq FK_1$ holds.

On the other hand, if $\dim K_1 > \dim K_2$, then we have Theorem 2.2 which, rather surprisingly, indicates that for a homogeneous, continuous map $F : K_1 \to \mathrm{span}\, K_2$ which satisfies condition (a) of Theorem 2.1, conditions (b) and (c) of Theorem 2.1 are incompatible !

Proof of Theorem 2.2. First, assume to the contrary that $F(\partial K_1) \cap \mathrm{int}\, K_2 = \emptyset$. As done in the proof for the first part of Theorem 2.1, we have $F(K_1 \backslash \{0\}) \cap (-K_2) = \emptyset$. Then we borrow part of the arguments used in the proof of the last part of Theorem 2.1, now assuming instead that $\mathrm{span}\, K_1 = \mathbb{R}^n$ and $\mathrm{span}\, K_2 = \mathbb{R}^m$. The continuous map $f : K_1 \cap S^{n-1} \to S^{m-1}$ can be defined in the same way as before, but we do not introduce the map θ_y. We do choose a vector u from $\mathrm{int}\, K_1 \cap S^{n-1}$ such that $fu \in \mathrm{int}\, K_2 \cap S^{m-1}$, denote fu by z and define the map $\theta_z : \partial K_1 \cap S^{n-1} \to \partial K_2 \cap S^{m-1}$ by $\theta_z(v) = \pi_{K_2}(z, fv)$. Note that $z,\ -z \notin f(\partial K_1 \cap S^{n-1})$; so θ_z is well-defined, continuous. Since the sets $\partial K_1 \cap S^{n-1}$ and $\partial K_2 \cap S^{m-1}$ are homeomorphic to S^{n-2} and S^{m-2} respectively and $m < n$ by our assumption, we can now apply Corollary A to conclude that there exists a pair of antipodal points $x, \bar{x}$ of $\partial K_1 \cap S^{n-1}$ relative to u such that $\theta_z(x) = \theta_z(\bar{x})$. Then we can derive a contradiction in the same way as before. So we must have $F(\partial K_1) \cap \mathrm{int}\, K_2 \neq \emptyset$.

To prove the second half of the theorem, suppose that $F(K_1 \backslash \{0\}) \cap (-K_2) = \emptyset$. Then the map f is well-defined. If, in addition, we have $\mathrm{int}\, K_2 \cap F(\mathrm{int}\, K_1) \not\subset F(\partial K_1)$, then we can choose a vector u from $\mathrm{int}\, K_1 \cap S^{n-1}$ such that $0 \ll f(u) \notin f(\partial K_1 \cap S^{n-1})$. Then we denote $f(u)$ by z, introduce the continuous map $\theta_z : \partial K_1 \cap S^{n-1} \to \partial K_2 \cap S^{m-1}$, and derive a contradiction in the same way as done above. $\square$

Below we give some "natural" conditions on a map $F : K_1 \to \mathrm{span}\, K_2$, which guarantee that F satisfies condition (a) of Theorem 2.1. The proof is straightforward.

A subset F of K is called a *face* of K if it is a convex cone and in addition possesses the property that $x \geq^K y \geq^K 0$ and $x \in F$ imply $y \in F$. For any nonempty subset S of a closed, pointed convex cone K, we denote by $\Phi(S)$ the *face of K generated by S*, i.e., the intersection of all faces of K that include S; equivalently, we have, $\Phi(S) = \{y \in K : y \ {}^K\!\leq \alpha x$ for some $\alpha > 0$ and $x \in \mathrm{pos}\, S\}$, where $\mathrm{pos}\, S$ denotes the positive hull (i.e., the set of all nonnegative linear combinations of vectors) of S. If $S = \{x\}$, where $x \in K$, we denote $\Phi(S)$ simply by $\Phi(x)$.

Remark 3.4. Consider the following conditions on a map T: $K_1 \to \mathrm{span}\, K_2$, where $K_1,\ K_2$ are proper cones in finite-dimensional real vector spaces.

(a) T is K_2-convex.

(b) For any $S \subseteq K_1$, $T(\Phi(S)) \subseteq \Phi(TS)$.

(c) For any $S \subseteq K_1$, $T(\operatorname{pos} S) \subseteq \Phi(TS)$.
(d) For any $x,\ y \in K_1$ and $\lambda,\ \mu > 0$, there exist $\alpha,\ \beta > 0$ such that $\alpha Tx + \beta Ty \geq^{K_2} T(\lambda x + \mu y)$.
(e) For any $x,\ y \in K_1$, there exist $\alpha,\ \beta > 0$ such that $\alpha Tx + \beta Ty \geq^{K_2} T(x+y)$.

Conditions (c) and (d) are equivalent, and we always have the implications (b) $\Longrightarrow$ (c) $\Longrightarrow$ (e) and (a) $\Longrightarrow$ (e). When T is homogeneous, (d) and (e) are also equivalent. When T is homogeneous and satisfies the condition that $T(\Phi(x)) \subseteq \Phi(Tx)$ for all $x \in K_1$ (which is the case if T is (K_1, K_2)-monotone), we also have (a) $\Longrightarrow$ (b).

4. Extensions of Ky Fan's Theorem

We need the following, parts of which are undoubtedly known:

Remark 4.1. Let $T : K_1 \to \operatorname{span} K_2$ be a homogeneous map.
(i) If T is (K_1, K_2)-monotone, then $T(0) = 0$ and T is (K_1, K_2)-nonnegative.
(ii) The following are equivalent statements:
(a) T is (K_1, K_2)-convex.
(b) For any $x,\ y \in K_1$, $Tx + Ty \geq^{K_2} T(x+y)$.
(c) $x \geq^{K_1} y \geq^{K_1} 0$ implies $T(x-y) \geq^{K_2} Tx - Ty$.

A similar assertion also holds for K_2-concavity.

(iii) If T is K_2-concave and (K_1, K_2)-nonnegative, then T is (K_1, K_2)-monotone.

(iv) If T is (K_1, K_2)-monotone, then T is bounded, in the sense that it maps bounded sets to bounded sets, or equivalently, there exists a positive constant M such that $\|Tx\|_2 \leq M\|x\|_1$ for all $x >^{K_1} 0$ and for some (and hence, for all) norms $\|\cdot\|_1$ and $\|\cdot\|_2$ of $\operatorname{span} K_1$ and $\operatorname{span} K_2$ respectively.

Notice that the (K_1, K_2)-monotonicity of T alone does not guarantee (K_1, K_2)-nonnegativity nor $T(0) = 0$. The point is, if T is a (K_1, K_2)-monotone map, then the map S defined by $Sx = Tx + y$, where y is any fixed vector of K_2, is still a (K_1, K_2)-monotone map. However, if T is homogeneous and (K_1, K_2)-monotone, then from $2.0 \geq 0$, we obtain $2T(0) \geq T(0)$ and hence $T(0) \geq 0$. On the other hand, from $\frac{1}{2}.0 \geq 0$, we also obtain $T(0) \leq 0$. Hence, we have, $T(0) = 0$, and then by the (K_1, K_2)-monotonicity of T, the (K_1, K_2)-nonnegativity of T follows. This proves part (i) of Remark 4.1.

Parts (ii) and (iii) of Remark 4.1 can be readily proved. To prove (iv), choose any vector $v \in \operatorname{int} K_1$. By definition of interior, there exists $\varepsilon > 0$ such that $v + \varepsilon x \in K_1$ for all $x \in V_1$ with $\|x\|_1 \leq 1$, where $\|\cdot\|_1$ is any norm of $\operatorname{span} K_1$. Now choose a norm $\|\cdot\|_2$ of $\operatorname{span} K_2$ which is monotonic with respect to K_2; that is, $0 \ {}^{K_2}\!\!\leq x \ {}^{K_2}\!\!\leq y$ implies $\|x\|_2 \leq \|y\|_2$. (For the existence of monotonic norms, see [BP, pp.5–6, Exercise 2.24].) Consider any vector $x \in K_1$ with $\|x\|_1 \leq 1$. Clearly, we have $v - \varepsilon x \in K_1$. Since T is homogeneous and (K_1, K_2)-monotone, we also have $0 \leq \varepsilon Tx \leq Tv$. By the monotonicity of $\|\cdot\|_2$, it follows that $\varepsilon\|Tx\|_2 \leq \|Tv\|_2$ and $\varepsilon^{-1}\|Tv\|_2$ is the desired constant for the boundedness of T.

Proof of Theorem 2.3. First, we show that the set Ω contains some positive elements. Take any $u >^{K_1} 0$. By the positivity of G, $Gu \gg 0$. So, for $\varepsilon > 0$ sufficiently small, we have $Gu - \varepsilon Fu \geq 0$, i.e., $\varepsilon \in \Omega$. Also, note that Ω is bounded. Otherwise, choose $x_k \geq 0$, $\omega_k > 0$ for $k = 1, 2, \ldots$ such that $\lim_{k\to\infty} \omega_k = \infty$ and $Gx_k - \omega_k Fx_k \geq 0$ for each k. By the homogeneity of G and F, we may assume that each x_k is a unit vector (with respect to some norm of span K_1). Replacing by a subsequence, if necessary, we may also assume that $(x_k)_{k\in\mathbb{N}}$ converges to $\bar{x}$. By Remark 4.1(iii) and (iv), the sequence $(Gx_k)_{k\in\mathbb{N}}$ is bounded. Rewriting the above inequalities, we have $\omega_k^{-1} Gx_k \geq Fx_k$ for each k. Letting $k \to \infty$ and making use of the continuity of F at $\bar{x}$, we obtain $-F\bar{x} \geq 0$. On the other hand, since F is K_2-convex, by Remark 3.4, F satisfies condition (a) and hence the assumptions of Theorem 2.1. So by Theorem 2.1, we have $F(K_1\backslash\{0\}) \cap (-K_2) = \emptyset$. Hence, we arrive at a contradiction.

Denote $\sup \Omega$ by ω_0. Clearly $\omega_0 > 0$. By a modification of the above argument, it is clear that there exists $x_0 > 0$ such that $Gx_0 - \omega_0 Fx_0 \in \partial K_2$. We are going to show that $Gx_0 = \omega_0 Fx_0$.

In view of the last part of Theorem 2.1, there exists $z > 0$ such that $Fz = Gx_0$. By the positivity of G and condition (b), clearly $z \gg 0$. Since $-x_0 \notin K_1$, there exists $\lambda > 0$ such that $z - \lambda x_0 \in \partial K_1$. If $\lambda < \omega_0$, then by the convexity and homogeneity of F and the choice of z, we have

$$F(z - \lambda x_0) \geq Fz - \lambda Fx_0 = Gx_0 - \lambda Fx_0 = \left(1 - \frac{\lambda}{\omega_0}\right) Gx_0 + \frac{\lambda}{\omega_0}(Gx_0 - \omega Fx_0) \gg 0,$$

which contradicts condition (b). So, we must have $\lambda \geq \omega_0$. Then, since G is concave, positive, and $z - \omega_0 x_0 \geq z - \lambda x_0 \geq 0$, we have

$$Gz - \omega_0 Fz = Gz - \omega_0 Gx_0 \geq G(z - \omega_0 x_0) \geq 0.$$

If $z - \omega_0 x_0 > 0$, then by the positivity of G and the above, we would obtain $Gz - \omega_0 Fz \gg 0$, which clearly contradicts the maximality of ω_0. So we must have $z - \omega_0 x_0 = 0$, and from the above we obtain $\lambda = \omega_0$ and $z = \omega_0 x_0$. Hence,

$$Gx_0 - \omega_0 Fx_0 = Gx_0 - \omega_0 F(\omega_0^{-1} z) = Gx_0 - Fz = 0,$$

which is what we want. Since x_0 is a positive scalar multiple of z, we also have $x_0 \gg 0$.

From the above, clearly $\omega_0 \in \Omega \cap \Sigma_1$. In order to establish the equalities $\sup \Omega = \inf \Sigma_1 = \omega_0$, it suffices to prove that $\sigma \geq \omega$ for any $\sigma \in \Sigma_1$ and $\omega \in \Omega$. We are going to show that the latter assertion is true even if we replace Σ_1 by Σ, which is defined by $\Sigma = \{\sigma \geq 0 : \exists x >^{K_1} 0,\ Gx \ {}^{K_2}\!\!\leq \sigma Fx\}$ (and, in fact, as the proof will show, in this case we have $\Sigma_1 = \Sigma$). Let $x >^{K_1} 0$, $y >^{K_1} 0$ be such that $Gx \ {}^{K_2}\!\!\leq \sigma Fx$ and $Gy \geq^{K_2} \omega Fy$. By the (K_1, K_2)-positivity of G and condition (b), the first inequality clearly implies that $\sigma > 0$ and $x \in \operatorname{int} K_1$. So there exists $\varepsilon > 0$ such that $x - \varepsilon y \in \partial K_1$. Assume to the contrary that $\sigma < \omega$. Then $x - \varepsilon y \neq 0$ (otherwise, we would have $\sigma = \omega$) and by the given properties of F and G, we have

$$F(x-\varepsilon y) \geq Fx - \varepsilon Fy \geq \sigma^{-1}(Gx - \varepsilon\sigma\omega^{-1}Gy) \geq \sigma^{-1}(Gx - \varepsilon Gy) \geq \sigma^{-1}G(x-\varepsilon y),$$

which is a contradiction, as $G(x-\varepsilon y) \in \operatorname{int} K_2$ and $F(x-\varepsilon y) \notin \operatorname{int} K_2$,

The uniqueness of ω_0 and x_0 (up to positive multiples) will follow once we establish the last part of our result.

Last part. Let $y > 0$ and $\omega \geq 0$ be such that $Gy \geq \omega Fy$. Then $\omega \in \Omega$ and, by what we have proved, $\omega \leq \omega_0$. If the strict inequality does not hold, then from the above argument (with $x = x_0$ and $y = y$), we obtain $F(x_0 - \varepsilon y) \geq \omega_0^{-1} G(x_0 - \varepsilon y)$ and with $x_0 - \varepsilon y \in \partial K_1$ for some $\varepsilon > 0$, which is not possible, unless $x_0 = \varepsilon y$.

Similarly, we can also show that if $\sigma \geq 0$ is such that $Gx \leq \sigma Fx$ for some $x \in K_1 \backslash \{0\}$, which is not a multiple of x_0, then $\sigma > \omega_0$. □

With some hindsight, we can give a few remarks on the relevance of conditions (a)–(c) of Theorem 2.3. First, the conclusion of Theorem 2.3, namely, $Gx_0 = \omega F x_0$, where $x_0 > 0$, $\omega > 0$, together with the assumption that G is positive, forces the necessity of condition (c). But conditions (a), (b) together do not guarantee condition (c); for instance, if we take $K_1 = K_2 = K$ and F to be a linear map that maps K into ∂K, then F satisfies (a) and (b) but not (c). That is why we impose the condition. Next, according to Theorem 2.1, conditions (a)–(c) and the assumption that $\dim K_1 = \dim K_2$, together with the continuity and homogeneity of F, guarantee two conditions, namely, $F(K_1 \backslash \{0\}) \cap (-K_2) = \emptyset$ and $FK_1 \supseteq K_2$. In the proof of Theorem 2.3, the former condition is needed to guarantee the boundedness of Ω. The latter condition is also crucial for our desirable conclusion. For, if $FK_1 \not\supseteq \operatorname{int} K_2$, then we can choose $z \in \operatorname{int} K_2 \backslash FK_1$ and find a positive linear map G which maps K_1 onto the ray generated by z. For any such G, it is clear that the system (F, G) has no equilibrium point.

Remark 4.2. Let K be a proper cone. If $F : K \to \operatorname{span} K$ is linear and satisfies conditions (b) and (c) of Theorem 2.3, then for any homogeneous, continuous, (K, K)-nonnegative map $G : K \to \operatorname{span} K$, there exist a positive scalar ω and a nonzero vector x of K such that $Gx = \omega Fx$. However, the uniqueness of the equilibrium point is not guaranteed, even if we assume, in addition, that G is linear and K-irreducible (i.e. $GK \subseteq K$ and G leaves invariant no faces of K other than $\{0\}$ and K itself).

To show the existence of an equilibrium point for the system (F, G), we first note that F can be readily extended to a linear map on $\operatorname{span} K$. We still use the same symbol to denote its extension map. By Theorem 2.1, we have, $FK \supseteq K$. Since K is a full cone in $\operatorname{span} K$, this implies that F is nonsingular and we have $F^{-1}K \subseteq K$. Then one can readily verify that the map $F^{-1}G : K \to \operatorname{span} K$ is homogeneous, continuous and (K, K)-nonnegative. But any such map has a (necessarily, nonnegative) eigenvalue and a corresponding eigenvector in K (as can be proved by applying the Brouwer fixed-point theorem to the continuous map $T : C \to C$ given by $Tx = (f(F^{-1}Gx))^{-1} F^{-1}Gx$, where f is any fixed vector chosen from the interior of the dual cone of K and C is the compact convex full cross-section of K given by $C = \{x \in K : fx = 1\}$, assuming that $Gx \neq 0$ for all $x \in K\backslash\{0\}$). If ω is an eigenvalue and $x >^K 0$ is a corresponding eigenvector of $F^{-1}G$, then ω is an equilibrium value and x is an equilibrium point for the original system (F, G).

To see that uniqueness of the equilibrium point is not guaranteed, just take $K = \mathbb{R}^2_+$ and choose F, G to be the same and be the restriction to $\mathbb{R}^2_+$ of the linear map determined by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We would also like to add the following, which extends [8, Theorem 3]:

Corollary 4.3. *Let $K_1,\ K_2$ be proper cones such that* $\dim K_1 = \dim K_2$. *Let $F, G : K_1 \to \operatorname{span} K_2$ be maps that satisfy the hypotheses of Theorem 2.3. Also let ω_0 denote the positive number which has the same meaning as given in the theorem. Then the following are equivalent conditions on a real number σ :*

(a) $\sigma > \omega_0$;

(b) *For all $y >^{K_2} 0$, there exists $x \in K_1$ (which, necessarily, lies in* $\operatorname{int} K_1$*) such that $(\sigma F - G)x = y$;*

(c) *For some $y >^{K_2} 0$, there exists $x \in K_1$ (which, necessarily, lies in* $\operatorname{int} K_1$*) such that $(\sigma F - G)x = y$.*

Proof. (a) $\Rightarrow$ (b): It is easy to see that, when $\sigma > \omega_0$, the map $\sigma F - G$ is homogeneous, continuous and satisfies conditions (a), (b) of Theorem 2.1. Since $(\sigma F - G)x_0 = (\sigma - \omega_0)Fx_0 \gg 0$, the map also satisfies condition (c). So, by the last part of Theorem 2.1, our assertion follows. [Since G is (K_1, K_2)-positive and F satisfies condition (b) of Theorem 2.3, it is clear that the solution vector x must lie in $\operatorname{int} K_1$.]

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): Suppose that condition (c) holds. If x is a multiple of x_0, then we have $0 < (\sigma F - G)x = (\sigma - \omega_0)Fx$, which implies $\sigma > \omega_0$, as $Fx = \omega_0^{-1}Gx \gg 0$. If x is not a multiple of x_0, then by the last part of Theorem 2.3 we also obtain $\sigma > \omega_0$. □

In order to obtain Theorem 2.4 from Theorem 2.3, we need to make use of the following lemma (except for its last part, which has interest of its own).

Lemma 4.4. *Let C be a compact convex set in a finite-dimensional real vector space and let $f : C \to W$ be a map from C to a finite-dimensional real vector space W ordered by a proper cone K. Suppose that 0 is not in the affine hull of C and let $F : \operatorname{pos} C \to W$ be the homogeneous map defined by $F(\lambda x) = \lambda f(x)$ for $x \in C$ and $\lambda \geq 0$. Then f is continuous (respectively, K-convex, K-concave, (C, K)-nonnegative, (C, K)-positive) if and only if F is continuous (respectively, K-convex, K-concave, $(\operatorname{pos} C, K)$-nonnegative, $(\operatorname{pos} C, K)$-positive). Furthermore, F is (C, K)-monotone if and only if for any $x,\ y \in C$ and $t > 1$, $(1-t)x + ty \in C$ implies $f(y) \geq (1 - \frac{1}{t})f(x)$.*

Proof. First, note that since $0 \notin \operatorname{aff} C$, each nonzero vector y of $\operatorname{pos} C$ can be expressed uniquely as λx, where $x \in C$ and $\lambda > 0$. So F is a well-defined map. By definition of F, it is clear that F is always homogeneous. Since f is the restriction of F to C, clearly f is continuous (or, convex, concave, nonnegative, positive), whenever F is. It is also easy to show that if f is continuous (respectively, nonnegative, positive), then so is F. We are going to show that if f is convex, then so is F, the proof for the corresponding concavity part being similar.

Suppose that f is convex. Since F is homogeneous, to establish the convexity of F, it suffices to show that for any $v,\ w \in \operatorname{pos} C\backslash\{0\}$, we have $F(v+w) \leq F(v) + F(w)$. Express v, w and $v+w$ in terms of vectors in C, say, $v = \alpha x$, $w = \beta y$ and $v + w = \gamma z$, where $\alpha,\ \beta,\ \gamma > 0$ and $x,\ y,\ z \in C$. Rewriting, we have $z = ax + by$, where $a = \alpha/\gamma$, $b = \beta/\gamma$ are both positive. Since $\operatorname{aff} C$ does not contain the origin 0, we can choose a nonzero vector e such that the inner product between e and each vector in $\operatorname{aff} C$ equals 1. Taking inner product of e with vectors on opposite side of the relation $\alpha x + \beta y = \gamma z$, we obtain $a + b = 1$. So by definition of F and the convexity of f, we have

$$F(v+w) = F(\gamma z) = \gamma f(z) = \gamma f(ax+by) \leq \gamma a f(x) + \gamma b f(y) = F(v) + F(w).$$

Last Part. Suppose that F is monotone. Let $x,\ y \in C$, $t > 1$ be such that $(1-t)x + ty \in C$. Then $y \geq (1-\frac{1}{t})x$ and by the homogeneity and monotonicity of F, we have, $F(y) \geq (1-\frac{1}{t})F(x)$, hence $f(y) \geq (1-\frac{1}{t})f(x)$.

Conversely, suppose that f possesses the given property. Consider any vectors $u,\ v \in \operatorname{pos} C\backslash\{0\}$ with $v \geq u$. Express v, u and $v - u$ in terms of vectors in C, say, $v = \beta y$, $u = \alpha x$ and $v - u = \gamma z$ where $x,\ y,\ z \in C$ and $\alpha,\ \beta,\ \gamma > 0$. Set $t = \beta/\gamma$. After some manipulations (and again making use of the fact that $\langle x, e\rangle = \langle y, e\rangle = \langle z, e\rangle = 1$, where the vector e has the same meaning as above), we obtain $(1-t)x + ty = z \in C$ and $t > 1$. By the property of f, we have $f(y) \geq (1-\frac{1}{t})f(x)$. Rewriting the latter inequality in terms of u, v (and α, β, γ) and simplifying, we obtain $F(v) \geq F(u)$. This shows that F is monotone. $\square$

Proof of Theorem 2.4. We may assume that $0 \notin \operatorname{aff} C$. Otherwise, choose a one-to-one affine map that takes C onto some compact convex set $\widetilde{C}$ for which $0 \notin \operatorname{aff} \widetilde{C}$, define maps $\tilde{f}$, $\tilde{g}$ corresponding to f, g in the natural way, and work with $\widetilde{C}$, $\tilde{f}$ and $\tilde{g}$ instead.

Let $F : \operatorname{pos} C \to W$ be the map defined by $F(y) = \lambda f(x)$ for $y \in \operatorname{pos} C$, where $y = \lambda x$, $x \in C$ and $\lambda \geq 0$. Since f is continuous, convex on C, by Lemma 4.4, F is continuous, convex on $\operatorname{pos} C$. In view of (b) and (c) (and the homogeneity of F), it is clear that, we have, $F(\partial(\operatorname{pos} C)) \cap K = \emptyset$ and $F(\operatorname{pos} C) \cap \operatorname{int} K \neq \emptyset$. Now let $G : \operatorname{pos} C \to W$ be the homogeneous map defined in a similar way (in terms of g). By Lemma 4.4 again, G is a continuous, concave positive map. Since the restriction of F (respectively, G) to C equals f (respectively, g) and $0 \notin \operatorname{aff} C$, we can apply Theorem 2.3 to the pair (F, G) to draw the desired conclusions. $\square$

With the aid of Sperner's Lemma (and by adapting the proof of [8, Theorem 1]), one can derive the first part of Ky Fan's theorem from the first part of Theorem 2.4. The last part of Ky Fan's theorem can also be deduced from the identity $\sup\Omega = \inf\Sigma_1 = \omega_0$ (of Theorem 2.4) by making use of the following readily-proved facts: $\sup\Omega = \max_{x\in S} r(x)$, $\inf\Sigma = \min_{x\in S} R(x)$, $\Sigma = \Sigma_1$ in this case, and for any $x \in S$, $r(x)^{-1} = \max_{1\le i\le n} f_i(x)/g_i(x)$ and $R(x)^{-1} = \min_{1\le i\le n} f_i(x)/g_i(x)$, where

$$\begin{aligned} r(x) &= \max\{\omega \ge 0 : g(x) \ge \omega f(x)\}, \\ R(x) &= \min\{\sigma \ge 0 : g(x) \le \sigma f(x)\} \text{ (by convention } \min\emptyset = \infty), \end{aligned}$$

and $\Sigma = \{\sigma \ge 0 : \exists x \in C, g(x) \overset{K}{\le} \sigma f(x)\}$.

Actually, Theorems 2.3 and 2.4 are equivalent. Also, Theorem 2.1 admits the following equivalent formulation with K_1 replaced by a compact convex set:

Theorem 2.1'. *Let C be a compact convex set in a finite-dimensional real vector space, and let $f : C \to W$ be a continuous map from C to a finite-dimensional real vector space W ordered by a proper cone K such that* $\dim W = \dim C + 1$. *Suppose that f satisfies each of the following conditions*:

(a) *f is K-convex*;

(b) $f(\operatorname{rbd} C) \cap K = \emptyset$; *and*

(c) $f(C) \cap \operatorname{int} K \neq \emptyset$.

Then $f(C) \cap (-K) = \emptyset$ and $K \subseteq \bigcup_{\lambda\ge 0} \lambda f(C)$.

Note that if C is an $(n-1)$-dimensional compact convex set whose affine hull does not contain the origin, then $\operatorname{pos} C$ is an n-dimensional closed, pointed convex cone. Then C (respectively, $\operatorname{rbd} C$) is homeomorphic with $(\operatorname{pos} C) \cap S^{n-1}$ (respectively, $\partial(\operatorname{pos} C) \cap S^{n-1}$), after identifying $\operatorname{span} C$ ($= \operatorname{span}(\operatorname{pos} C)$) with $\mathbb{R}^n$. Indeed, we could have introduced the concept of a pair of antipodal points of $\operatorname{rbd} C$ relative to a relative interior point of C, and also could have derived Theorem $1'$ directly (using an argument similar to that for Theorem 2.1) and then used it to prove Theorem 2.4.

Certainly we can also reformulate Corollary 4.3 in the setting when the common domain of F and G is a compact convex set.

5. Final Remarks

In Theorem 2.3, if K_1, K_2 are the same and equal to a proper cone K, F equals the identity map on $\operatorname{span} K$ and G equals a linear map A that preserves K (i.e. $AK \subseteq K$), then the sets Ω and Σ_1 considered in the theorem become two of the four Collatz-Wielandt sets associated with the cone-preserving map A. Collatz-Wielandt sets were first introduced by Barker and Schneider [4]. The greatest lower bound and the least upper bound of the Collatz-Wielandt sets are studied in [17]; in particular, it is proved that, for any linear map A that preserves K, we have $\sup\Omega = \inf\Sigma_1 = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A. For more recent developments of the topic, we refer the reader to the review paper [16]. In the book [12] chapter 11, Aubin has also elaborated on the results of [8] in the setting of a pair of maps F, G from the standard simplex of $\mathbb{R}^n$ to $\mathbb{R}^m$ and with the continuity assumptions on F, G replaced respectively by the lower and upper semi-continuity assumptions. The study of the Collatz-Wielandt sets associated with a pair of nonlinear maps (in particular, the determination of when $\sup \Omega$ and $\inf \Sigma_1$ are the same and equal to an equilibrium value, etc.), and also the introduction and study of the concepts of lower or upper semicontinuity of a map with respect to a proper cone seem worthwhile and will form the subject matter of future work.

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