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ONE-SIDED SIMULTANEOUS INEQUALITIES AND SANDWICH THEOREMS FOR DIAGONAL SIMILARITY AND DIAGONAL EQUIVALENCE OF NONNEGATIVE MATRICES*

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Abstract. Results on the simultaneous scaling of nonnegative matrices involving one sided inequalities are presented. These are applied to scalings involving two sided inequalities. The proofs are graph theoretic. The setting is generalized to matrices with elements in lattice ordered Abelian groups with 0.

Key words. Nonnegative matrices, Diagonal similarity scaling, Diagonal equivalence scaling, Digraph of a matrix, Cyclic product, Cycle mean, Lattice ordered commutative groups.

AMS subject classifications. 15A24, 15A12, 15A45, 65F35.

1. Introduction.

Let $\mathbb{R}_+$ be the set of nonnegative numbers and let $\mathbb{R}_+^{mn}$ be the set of all nonnegative $m \times n$ matrices. If $A \in \mathbb{R}_+^{nn}$ and X is a nonsingular positive diagonal matrix in $\mathbb{R}_+^{nn}$, then XAX^{-1} is called a *diagonal similarity scaling* of A. If $A \in \mathbb{R}_+^{mn}$ and X and Y are a nonsingular diagonal matrices in $\mathbb{R}_+^{mm}$ and $\mathbb{R}_+^{nn}$ respectively, then XAY is called a *diagonal equivalence scaling* of A.

Results on the diagonal scaling of matrices have a long history and they and their proofs are of several different types. In this paper we consider results that involve only scaling and inequalities (thus we do not consider results that also involve sums of elements of matrices). We generalize and apply a result found in [8] which was proved by means of a theorem of the alternative. In contrast, our proofs heavily involve graphs and cyclic products of elements. References to previous results will be found in the various sections.

Our basic results are to be found in Section 2 of this paper. Let $A^{(k)}, B^{(k)}$, $k = 1, \ldots, s$, be matrices in $\mathbb{R}_+^{nn}$. In Section 2 we prove necessary and sufficient conditions for the existence of a positive diagonal $n \times n$ matrix X such that

$$XA^{(k)}X^{-1} \leq B^{(k)}, k = 1, \ldots, s,$$

see Theorem 2.5. This result is applied to obtain necessary and sufficient conditions in Theorem 2.18 for the existence of a positive diagonal $n \times n$ matrix X such that

$$C^{(k)} \leq XA^{(k)}X^{-1} \leq B^{(k)}, \quad k = 1, \ldots, s,$$

where $C^{(1)}, \ldots, C^{(s)}$ are also matrices in $\mathbb{R}_+^{nn}$.

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In Section 3, we find necessary and sufficient conditions in Corollary 3.4 such that there exists a positive diagonal matrix X such that

$$lA^{(k)} \leq XB^{(k)}X^{-1}, k = 1, \ldots, s,$$

for a given positive number l and, in Theorem 3.8, we obtain conditions equivalent to

$$lB \leq XAX^{-1} \leq uB,$$

for given positive l and u.

In Section 4 we prove analogous results on diagonal equivalence. In Section 5 we generalize the setting of our results to lattice ordered commutative groups with an additional minimal element 0. Some results for the noncommutative case may be found in [4]. Results with the order relations reversed could be obtained by adjoining a maximal element in place of a minimal element.

2. Simultaneous diagonal similarity. NOTATION 2.1. Let A and B be matrices in $\mathbb{R}_+^{mn}$.

(i) We denote by A/B the $m \times n$ matrix defined by

$$(A/B)_{ij} = \begin{cases} a_{ij}/b_{ij} & \text{if} \quad b_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) We denote by $1/B$ the $m \times n$ matrix defined by

$$(1/B)_{ij} = \begin{cases} 1/b_{ij} & \text{if} \quad b_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

(iii) For matrices $A^{(1)}, \ldots, A^{(s)}, B^{(1)}, \ldots, B^{(s)}$ in $\mathbb{R}_+^{mn}$ we define the matrix

$$Q = \max\{(A^{(1)}/B^{(1)}), \ldots, (A^{(s)}/B^{(s)})\}$$

by

$$q_{ij} = \max\{(A^{(1)}/B^{(1)})_{ij}, \ldots, (A^{(s)}/B^{(s)})_{ij}\}, \quad i, j = 1, \ldots, n.$$

Similarly, we define the matrix

$$R = \min\{(A^{(1)}/B^{(1)}), \ldots, (A^{(s)}/B^{(s)})\}$$

by

$$r_{ij} = \min\{(A^{(1)}/B^{(1)})_{ij}, \ldots, (A^{(s)}/B^{(s)})_{ij}\}, \quad i, j = 1, \ldots, n.$$

NOTATION 2.2. Let Γ be a digraph.

(i) For a path α in Γ we denote by $|\alpha|$ the number of arcs in α.

(ii) For a vertex i in Γ we denote by $\mathcal{P}_i(\Gamma)$ be the set of all nonempty paths ending at i (whatever their length may be).

(iii) For a (simple) cycle γ in Γ we identify γ with the set of arcs that form the cycle.

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NOTATION 2.3. Let n be a positive integer.

(i) We denote by Γ_n the complete digraph with vertices $\{1,\dots,n\}$.

(ii) Let M be an $n\times n$ matrix. For a set α of arcs in Γ_n we denote by $\Pi_\alpha(M)$ the product $\Pi_{(i,j)\in\alpha} m_{ij}$. If α is a cycle in Γ_n, then the product $\Pi_\alpha(M)$ is said to be a *cyclic product* of the matrix M.

DEFINITION 2.4. Let M be an $n\times n$ matrix. The *digraph* $\Gamma(M)$ *of* M is defined to be the digraph with vertices $\{1,\dots,n\}$ and where there is an arc from i to j if and only if $m_{ij}\neq 0$.

THEOREM 2.5. *Let $A^{(k)}, B^{(k)}$, $k=1,\dots,s$, be matrices in $\mathbb{R}^{nn}_+$ with $\Gamma(A^{(k)})\subseteq \Gamma(B^{(k)})$, $k=1,\dots,s$. Let $Q=\max\{(A^{(1)}/B^{(1)}),\dots,(A^{(s)}/B^{(s)})\}$. Then the following are equivalent:*

(i) *The positive diagonal matrix $X=\operatorname{diag}(x_1,\dots,x_n)$, defined by*

$$x_i := \max\left\{\max_{\delta\in\mathcal{P}_i(\Gamma_n),\ 1\le|\delta|\le n-1}\Pi_\delta(Q)\ ,\ 1\right\},\quad i=1,\dots,n, \tag{2.6}$$

satisfies

$$XA^{(k)}X^{-1}\le B^{(k)},\ k=1,\dots,s. \tag{2.7}$$

(ii) *There exists a positive diagonal $n\times n$ matrix X such that (2.7) holds*

(iii) *There exists a positive diagonal $n\times n$ matrix X such that all entries of the matrix XQX^{-1} are less than or equal to* 1.

(iv) *All cyclic products of the matrix Q are less than or equal to* 1.

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii) is obvious, since it follows from (2.7) that

$$x_i q_{ij} x_j^{-1} = \max_{k=1,\dots,s} x_i(a_{ij}^{(k)}/b_{ij}^{(k)})x_j^{-1}\le 1. \tag{2.8}$$

(iii) ⇒ (iv). By (iii), all cyclic products of XQX^{-1} are less than or equal to 1. But corresponding cyclic products of Q and XQX^{-1} coincide, which yields (iv).

(iv) ⇒ (i). Define x_i, $i=1,\dots,n$, by (2.6). Since all cyclic products of Q are less than or equal to 1, it follows that actually

$$x_i := \max\left\{\max_{\delta\in\mathcal{P}_i(\Gamma_n)}\Pi_\delta(Q)\ ,\ 1\right\},\quad i=1,\dots,n. \tag{2.9}$$

Assume first that $q_{ij}>0$. if $x_i=\max_{\delta\in\mathcal{P}_i(\Gamma_n)}\Pi_\delta(Q)$ then, clearly, x_iq_{ij} is the maximum of path products ending at j whose last arc is (i,j). Else, we have $x_i=1$ and, again, $x_iq_{ij}=q_{ij}$ is the maximum of path products ending at j whose last arc is (i,j). Thus, we have $x_iq_{ij}\le x_j$, implying that $x_i(a_{ij}^{(k)}/b_{ij}^{(k)})x_j^{-1}\le x_iq_{ij}x_j^{-1}\le 1$, and it follows that

$$x_ia_{ij}^{(k)}x_j^{-1}\le x_ib_{ij}^{(k)}x_j^{-1}. \tag{2.10}$$

If $q_{ij}=0$ then it follows from the graph inclusions $\Gamma(A^{(k)})\subseteq\Gamma(B^{(k)})$ that $a_{ij}^{(k)}=0$, $b_{ij}^{(k)}\ge 0$, and so $x_ia_{ij}^{(k)}x_j^{-1}=0\le x_ib_{ij}^{(k)}x_j^{-1}$. It now follows that (2.10) holds for all i, j and k, and so the diagonal matrix $X=\operatorname{diag}(x_1,\dots,x_n)$ satisfies (i). ☐

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We observe that, for $k = 1$ and $B = E$, the all 1's matrix, a result equivalent to Theorem 2.5 was proven by Fiedler-Ptak [5, Theorem 2.2] using a graph theoretic argument close to our proof of Theorem 2.5. This was preceded by Afriat [1, Theorem 7.1] who essentially proved the equivalence of (ii) and (iii) in the same special case of Theorem 2.5 by means of a theorem of the alternative . In these papers the result is stated additively. For $k = 1$ and irreducible matrices A and B this result also appears in [6, Theorem 3.3], and in [4, Theorem 4.1] in the setting of lattice ordered (non-commutative) groups.

REMARK 2.11. Let $A^{(k)}, B^{(k)}$, $k = 1, \ldots, s$, be matrices in $\mathbb{R}_+^{nn}$, and let $R = \min\{(B^{(1)}/A^{(1)}), \ldots, (B^{(s)}/A^{(s)})\}$. By Notation 2.1 we have $(A^{(k)}/B^{(k)})_{ij} \neq 0$ if and only if $a_{ij}^{(k)} b_{ij}^{(k)} \neq 0$. Thus, $(A^{(k)}/B^{(k)})_{ij} = 0$ if and only if $(B^{(k)}/A^{(k)})_{ij} = 0$. It follows that if $r_{ij} > 0$, then $q_{ij} > 0$. Also, in this case we have $r_{ij} = 1/q_{ij}$. Therefore, Condition (iii) of Theorem 2.5 implies the following condition:
(v) *All nonzero cyclic products of the matrix R are greater than or equal to 1.*
The converse, however, does not hold in general, since $q_{ij} > 0$ does not necessarily imply that $r_{ij} > 0$. For example, if one of the matrices $A^{(k)}$ is a zero matrix then $R = 0$. Nevertheless, in the case that $s = 1$ we indeed have $q_{ij} > 0$ if and only if $r_{ij} > 0$, and thus in this case Condition (v) above is equivalent to the conditions of Theorem 2.5.

REMARK 2.12. Since $XA^{(k)}X^{-1} \leq B^{(k)}$ is equivalent to $A^{(k)} \leq X^{-1}B^{(k)}X$, it follows immediately that the second statement of Theorem 2.5 is also equivalent to the existence of a positive diagonal matrix $Y \in \mathbb{R}_+^{nn}$ such that $A^{(k)} \leq YB^{(k)}Y^{-1}$, $k = 1, \ldots, s$.

REMARK 2.13. In general, the scaling obtained by (2.6) is not the unique diagonal similarity that satisfies Condition (ii). For example, if we define

$$p_i = \max_{\delta \in \mathcal{P}_i(\Gamma_n),\ 1 \leq |\delta| \leq n} \Pi_\delta(Q), \quad i = 1, \ldots, n, \tag{2.14}$$

then any positive diagonal matrix X which satisfies

$$\begin{cases} x_i = p_i\ , & \text{if } p_i \geq 1, \\ p_i \leq x_i \leq 1\ , & \text{otherwise,} \end{cases} \tag{2.15}$$

also satisfies (2.7). However, in this paper the main focus is on the applications of the equivalence of Conditions (ii), (iii) and (iv) of Theorem 2.5, and therefore we do not attempt to determine the set of positive diagonal matrices which satisfy (2.7) or (equivalently) Condition (iii) of Theorem 2.5. We note that further inequality requirements stronger than Condition (iii) can be satisfied, and that these lead to a unique scaling matrix X (up to constants) when Q is irreducible, see [13] and [14]. Finally, we observe that conditions similar to (i) of Theorem 2.5 that lead to computational tests could be added to most of our subsequent theorems, though these are stated as existence theorems.

REMARK 2.16. We note that Theorem 2.5 yields an efficient $O(n^3)$ algorithm for testing simultaneous diagonal similarity to satisfy the inequalities (2.7). For if X is the matrix defined by (2.6), then the required scaling exists if and only if XQX^{-1}

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satisfies Condition (iii), in which case one such scaling is given by X. We now give more detail on the computation involved.

Step 1. Find the elementwise quotients of the matrices $A_i, B_i, i = 1, \ldots, s$ (treating 0/0 as 0) and then take the elementwise maximum.

Step 2. We next compute the maximal path products defined in 2.6 by starting with the vector $x^0 = [1, \ldots, 1] \in \mathbb{R}^n$ and successively compute the multiplicative max algebra products $x^i = x^{i-1} \otimes Q,\ i = 1, \ldots n-1$. We then put $x = \max(x^0, \ldots, x^{n-1})$, see [2, Algorithm 3.4] for a related algorithm and remarks following [2, Theorem 4.1] for further explanation.

Step 3. Put $X = \text{diag}(x)$. This is the matrix X of (2.7).

Then *either* all elements of XQX^{-1} are less than or equal to 1, in which case X satisfies (2.7) *or* no X satisfying (2.7) exist. It should be noted that in Step 2 we could equally put $x = \max(x^1, \ldots, x^n)$, and thus obtain $x_i = p_i, i = 1, \ldots, n$, where the p_i are given by (2.14). In general, we obtain a different scaling satisfying (2.7). Our algorithm can be applied to several of our theorems below. For example, it will find a matrix X that satisfies Condition (i) Theorem 2.18 below or determine that no such X exists. Note also, that by putting $C^k = B^k$, $k = 1, \ldots, s$, in Theorem 2.18 we can find a matrix X that simultaneous satisfies $XA^kX^{-1} = B^k$, $k = 1, ..., s$, if any such X exists.

THEOREM 2.17. *Let $A^{(k)}$, $B^{(k)}$, $C^{(k)}$ be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(C^{(k)}) \subseteq \Gamma(A^{(k)}) \subseteq \Gamma(B^{(k)})$, $k = 1, \ldots, s$. Let X be a positive diagonal $n \times n$ matrix. Then the following are equivalent:*

(i) $$C^{(k)} \leq XA^{(k)}X^{-1} \leq B^{(k)}, \quad k = 1, \ldots, s.$$

(ii) $$XA^{(k)}X^{-1} \leq B^{(k)}, \quad k = 1, \ldots, s$$

and $$X(C^{(k)})^tX^{-1} \leq (A^{(k)})^t, \quad k = 1, \ldots, s.$$

Proof. The result follows since $C^{(k)} \leq XA^{(k)}X^{-1}$ is equivalent to $X^{-1}C^{(k)}X \leq A^{(k)}$, which in turn is equivalent to $X(C^{(k)})^tX^{-1} \leq (A^{(k)})^t$. ◻

Combining Theorem 2.5 and Theorem 2.17 we now obtain one of our main results.

THEOREM 2.18. *Let $A^{(k)}$, $B^{(k)}$, $C^{(k)}$ be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(C^{(k)}) \subseteq \Gamma(A^{(k)}) \subseteq \Gamma(B^{(k)})$, $k = 1, \ldots, s$. Let X be a positive diagonal $n \times n$ matrix. Let $Q = \max\{(A^{(1)}/B^{(1)}), \ldots, (A^{(s)}/B^{(s)}), (C^{(1)})^t/(A^{(1)})^t, \ldots, (C^{(s)})^t/(A^{(s)})^t\}$. Then the following are equivalent:*

(i) $$C^{(k)} \leq XA^{(k)}X^{-1} \leq B^{(k)}, \quad k = 1, \ldots, s.$$

(ii) *All cyclic products of the matrix Q are less than or equal to* 1.

Let $A \in \mathbb{R}_+^{nn}$. In [10] and [8] the term "cycle" was defined in a more general manner in reciprocal pair with one member of this pair corresponding to a cycle of $\max\{A, 1/A^t\}$ in the present paper. Thus, as a corollary to Theorem 2.5 and to Theorem 2.18 we obtain a result for the case $s = 1$, proven in [8, Theorem 4.1] by means of theorems of the alternative.

COROLLARY 2.19. *Let A, B and C be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(C) \subseteq \Gamma(A) \subseteq \Gamma(B)$. Then the following are equivalent:*

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(i) *There exists a positive diagonal* $n \times n$ *matrix* X *such that*

$$C \le XAX^{-1} \le B.$$

(ii) *All cyclic products of the matrix* $\max\{A/B, C^t/A^t\}$ *are less than or equal to* 1.
(iii) *For every cycle* γ *in the graph* $\Gamma(A/B + C^t/A^t)$ *and every set of arcs* δ, $\delta \subseteq \gamma$, *we have*

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(C^t/A^t) \le 1.$$

Proof. (i) ⇔ (ii) follows immediately from Theorem 2.5 and Theorem 2.18.
(ii) ⇒ (iii). Let $Q = \max\{A/B, C^t/A^t\}$. Note that for every i and j, $1 \le i, j \le n$, we have $(A/B)_{ij}$, $(C^t/A^t)_{ij} \le q_{ij}$. Therefore, for every cycle γ in the graph $\Gamma(A/B + C^t/A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(C^t/A^t) \le \Pi_\gamma(Q),$$

and so (iii) follows from (ii).
(iii) ⇒ (ii). Let γ be a cycle in $\Gamma(Q)$. Let δ be the set of arcs, $\delta \subseteq \gamma$, defined by $\delta = \{(i,j) : (A/B)_{ij} > (C^t/A^t)_{ij}\}$. Note that

$$\Pi_\gamma(Q) = \Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(C^t/A^t),$$

and so (ii) follows from (iii). □

Example 2.20. Let

$$A = \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 9 & 1 & 2 \\ 3 & \frac{1}{2} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 3 & 1 & 3 \\ 3 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We have

$$Q = \max\{A/B, C^t/A^t\} = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} & \frac{2}{3} \\ 3 & 1 & 2 \\ \frac{3}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Applying (2.6) to the matrix Q we obtain the positive diagonal matrix

$$X = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence

$$XQX^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

It follows by an application of Theorem 2.7 that

$$S = XAX^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

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is a scaling that satisfies $C \leq S \leq B$. (Of course, this may be verified directly. In this particular example it is also easy to see that all cyclic products of Q are less than or equal to 1.)
However, if we replace the matrix C by the matrix

$$\overline{C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

then we have

$$\overline{Q} = \max\{A/B, \overline{C}^t/A^t\} = \begin{bmatrix} \frac{2}{3} & \frac{1}{9} & \frac{2}{3} \\ 3 & 1 & 2 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}.$$

Applying (2.6) to the matrix $\overline{Q}$ we obtain

$$\overline{X} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence

$$\overline{T} := \overline{XQX}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

Since some elements of T exceed 1 it follows that there cannot be a positive diagonal matrix Z with $\overline{C} \leq ZAZ^{-1} \leq B$, see Remark 2.16. (In this particular example, it is also clear that the cyclic products of $\overline{Q}$ corresponding to the cycles $(2, 3)$ and $(1, 3, 2)$ are greater than 1).

In order to proceed we prove the following easy lemma.

Lemma 2.21. *Let A and B be matrices in $\mathbb{R}_+^{nn}$, and let $Q = \max\{A/B, B^t/A^t\}$. Then $\Gamma(Q) = \Gamma(Q^t)$. Furthermore, whenever $q_{ij} \neq 0$ we have $q_{ij}q_{ji} \geq 1$.*

Proof. Since

$$q_{ij} \neq 0 \quad \Leftrightarrow \quad a_{ij}b_{ij} \neq 0 \text{ or } a_{ji}b_{ji} \neq 0 \quad \Leftrightarrow \quad q_{ji} \neq 0,$$

it follows that $\Gamma(Q) = \Gamma(Q^t)$. Assume now that $q_{ij} \neq 0$. Without loss of generality assume that $a_{ij}b_{ij} \neq 0$. If $a_{ji}b_{ji} = 0$ or if $a_{ij}/b_{ij} = b_{ji}/a_{ji}$ then $q_{ij} = a_{ij}/b_{ij}$ and $q_{ji} = b_{ij}/a_{ij}$, and it follows that $q_{ij}q_{ji} = 1$. Otherwise, we have $a_{ji}b_{ji} \neq 0$ and $a_{ij}/b_{ij} \neq b_{ji}/a_{ji}$. Without loss of generality assume that $a_{ij}/b_{ij} > b_{ji}/a_{ji} > 0$. It follows that $q_{ij} = a_{ij}/b_{ij}$ and $q_{ji} = a_{ji}/b_{ji} > b_{ij}/a_{ij}$, and so $q_{ij}q_{ji} > 1$. The same conclusion is obtained under the assumption that $a_{ij}/b_{ij} < b_{ji}/a_{ji}$. □

If we now let $B = C$ in Corollary 2.19, then we obtain the following result, originally proven in [10, Theorem 2.1], see also [8, Corollary 4.4]. These papers

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provide references to previous results for the special case when the matrices A and B are assumed to be irreducible (which leads to a considerable simplification).

COROLLARY 2.22. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$. Then the following are equivalent:*
(i) *There exists a positive diagonal $n \times n$ matrix X such that*

$$XAX^{-1} = B.$$

(ii) *All cyclic products of the matrix* $\max\{A/B, B^t/A^t\}$ *are less than or equal to* 1.
(iii) *All nonzero cyclic products of the matrix* $\max\{A/B, B^t/A^t\}$ *are equal to* 1.
(iv) *For every cycle γ in the graph $\Gamma(A + A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, such that $\Pi_{\gamma \setminus \delta}(A^t) \neq 0$, we have*

$$\frac{\Pi_\delta(A)}{\Pi_{\gamma\setminus\delta}(A^t)} = \frac{\Pi_\delta(B)}{\Pi_{\gamma\setminus\delta}(B^t)}.$$

Proof. (i) $\Leftrightarrow$ (ii) follows from Corollary 2.19.
(ii) $\Rightarrow$ (iii). Let $i, j \in \{1, \ldots, n\}$ be such that $q_{ij} \neq 0$. By Lemma 2.21 we have $q_{ij}q_{ji} \geq 1$, and so by (ii) we have $q_{ij}q_{ji} = 1$. It now follows that for every cycle γ in $\Gamma(Q)$ the reverse cycle $\hat{\gamma}$ is also a cycle in $\Gamma(Q)$ and $\Pi_\gamma(Q) = \frac{1}{\Pi_{\hat{\gamma}}(Q)}$. Since, by (ii), we have $\Pi_\gamma(Q), \Pi_{\hat{\gamma}}(Q) \leq 1$, it follows that $\Pi_\gamma(Q) = 1$.
(iii) $\Rightarrow$ (ii) is trivial.
(ii) $\Rightarrow$ (iv). By Corollary 2.19, (ii) is equivalent to statement (iii) of Corollary 2.19, that is, for every cycle γ in the graph $\Gamma(A/B + B^t/A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \leq 1. \tag{2.23}$$

If we choose $\hat{\gamma}$ to be the reverse cycle of γ and $\hat{\delta}$ to consist of the reverse arcs of the arcs in $\gamma \setminus \delta$, then we obtain

$$\Pi_{\hat{\delta}}(A/B)\Pi_{\hat{\gamma}\setminus\hat{\delta}}(B^t/A^t) \leq 1,$$

which is equivalent to

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \geq 1. \tag{2.24}$$

(2.23) and (2.24) yield that for every cycle γ in the graph $\Gamma(A + A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, such that $\Pi_{\gamma\setminus\delta}(A^t) \neq 0$, we have

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) = 1,$$

which is equivalent to (iv).
(iv) $\Rightarrow$ (i). Let γ be a cycle in the graph $\Gamma(A + A^t)\,(= \Gamma(A/B + B^t/A^t))$, and let δ be a subset of γ. It follows from (iv) that whenever

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \neq 0$$

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we have

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) = 1.$$

We thus have

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \le 1,$$

and by Corollary 2.19 applied to the case $B = C$ we obtain (i). □

Remark 2.25. Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$, let $Q^1 = \max\{A, 1/A^t\}$ and $Q^2 = \max\{B, 1/B^t\}$, and let γ be a cycle in $\Gamma(Q^1)\,(= \Gamma(Q^2))$. Since clearly

$$\Pi_\gamma(Q^1) = \max_{\delta,\,\delta\subseteq\gamma} \frac{\Pi_\delta(A)}{\Pi_{\gamma\setminus\delta}(A^t)}\ , \quad \Pi_\gamma(Q^2) = \max_{\delta,\,\delta\subseteq\gamma} \frac{\Pi_\delta(B)}{\Pi_{\gamma\setminus\delta}(B^t)},$$

Statement (ii) of Corollary 2.22 implies that all cyclic products of Q^1 are equal to the corresponding products of Q^2. The converse of this statement is, however, false in general, as is demonstrated by the matrices $A = \begin{bmatrix} 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \end{bmatrix}$. Clearly, we have $\max\{A, 1/A^t\} = \max\{B, 1/B^t\} = \begin{bmatrix} 2 \end{bmatrix}$. Nevertheless, obviously the matrices A and B are not similar.

We note that in [10, Theorem 2.1], [8, Corollary 4.4] additional equalities are asserted which however follow from the equalities in Statement (ii) of Corollary of 2.22.

If we add the further requirement that $\Gamma(C) = \Gamma(A) = \Gamma(A^t)$ to Corollary 2.19, then we get two additional equivalent conditions.

Corollary 2.26. *Let A, B and C be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(C) = \Gamma(A) = \Gamma(A^t) \subseteq \Gamma(B)$. Then the following are equivalent:*

(i) *There exists a positive diagonal $n \times n$ matrix X such that*

$$C \le XAX^{-1} \le B.$$

(ii) *All cyclic products of the matrix* $\max\{A/B, C^t/A^t\}$ *are less than or equal to* 1.

(iii) *All nonzero cyclic products of the matrix* $\min\{B/A, A^t/C^t\}$ *are greater than or equal to* 1.

(iv) *For every cycle γ in the graph $\Gamma(A/B + C^t/A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have*

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(C^t/A^t) \le 1.$$

(v) *For every nonzero cycle γ in the graph $\Gamma(A/B + C^t/A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have*

$$\Pi_\delta(B/A)\Pi_{\gamma\setminus\delta}(A^t/C^t) \ge 1.$$

Proof. (i) ⇔ (ii) ⇔ (iv) is in Corollary 2.19.

(ii) ⇔ (iii). Note that we have

$$(A/B)_{ij} \ne 0 \;\Leftrightarrow\; a_{ij}b_{ij} \ne 0.$$

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Since $\Gamma(A) \subseteq \Gamma(B)$, it follows that

$$a_{ij}b_{ij} \neq 0 \;\Leftrightarrow\; a_{ij} \neq 0.$$

Since $\Gamma(C) = \Gamma(A) = \Gamma(A^t)$, it follows that

$$a_{ij} \neq 0 \;\Leftrightarrow\; c^t_{ij}a^t_{ij} \neq 0.$$

Since

$$(C^t/A^t)_{ij} \neq 0 \;\Leftrightarrow\; c^t_{ij}a^t_{ij} \neq 0,$$

it now follows that

$$(A/B)_{ij} \neq 0 \;\Leftrightarrow\; (B/A)_{ij} \neq 0 \;\Leftrightarrow\; (C^t/A^t)_{ij} \neq 0 \;\Leftrightarrow\; (A^t/C^t)_{ij} \neq 0. \tag{2.27}$$

Therefore, we have

$$(\max\{A/B, C^t/A^t\})_{ij} \neq 0 \;\Leftrightarrow\; (\min\{B/A, A^t/C^t\})_{ij} \neq 0.$$

Furthermore, in this case we have

$$(\max\{A/B, C^t/A^t\})_{ij} = \frac{1}{(\min\{B/A, A^t/C^t\})_{ij}}.$$

The equivalence follows.
(iv) $\Leftrightarrow$ (v) follows easily, in view of (2.27). □

While, as is observed in Remark 2.11, the implication (ii) $\Rightarrow$ (iii) in Corollary 2.26 holds in general, even without the requirement $\Gamma(C) = \Gamma(A) = \Gamma(A^t)$, the reverse direction does not hold in general, as is demonstrated by the following examples.

Example 2.28. Let

$$A = \begin{bmatrix} 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 \end{bmatrix}.$$

Here we have $\Gamma(C) \neq \Gamma(A) = \Gamma(A^t)$. Note that

$$\max\{A/B, C^t/A^t\} = \begin{bmatrix} 2 \end{bmatrix}, \quad \min\{B/A, A^t/C^t\} = \begin{bmatrix} 0 \end{bmatrix},$$

and so, while Statement (iii) of Corollary 2.26 holds, Statement (ii) clearly does not hold.

Example 2.29. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$$

Here we have $\Gamma(C) = \Gamma(A) \neq \Gamma(A^t)$. Note that

$$\max\{A/B, B^t/A^t\} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ \frac{1}{2} & 1 & 1 \end{bmatrix}, \quad \min\{B/A, A^t/B^t\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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and so, while Statement (iii) of Corollary 2.26 clearly holds, Statement (ii) does not hold, since the cyclic product of the matrix $\max\{A/B, B^t/A^t\}$ corresponding to the cycle $(1, 3, 2)$ is greater than 1.

Similarly to Lemma 2.21 we prove the following result.

LEMMA 2.30. *Let A and B be matrices in $\mathbb{R}_+^{nn}$, and let $R = \min\{A/B, B^t/A^t\}$. Then $\Gamma(R) = \Gamma(R^t)$. Furthermore, whenever $r_{ij} \neq 0$ we have $r_{ij}r_{ji} \leq 1$.*

Proof. Since

$$r_{ij} \neq 0 \quad \Leftrightarrow \quad a_{ij}b_{ij} \neq 0 \text{ and } a_{ji}b_{ji} \neq 0 \quad \Leftrightarrow \quad r_{ji} \neq 0,$$

it follows that $\Gamma(R) = \Gamma(R^t)$. Assume now that $r_{ij} \neq 0$. If $a_{ij}/b_{ij} = b_{ji}/a_{ji}$ then $r_{ij} = a_{ij}/b_{ij}$ and $r_{ji} = b_{ij}/a_{ij}$, and it follows that $r_{ij}r_{ji} = 1$. Otherwise, without loss of generality assume that $a_{ij}/b_{ij} > b_{ji}/a_{ji} > 0$. It follows that $r_{ij} = b_{ji}/a_{ji}$ and $r_{ji} = b_{ij}/a_{ij} < a_{ji}/b_{ji}$, and so $r_{ij}r_{ji} < 1$. The same conclusion is obtained under the assumption that $a_{ij}/b_{ij} < b_{ji}/a_{ji}$. ◻

If we add the further requirement that $\Gamma(A) = \Gamma(A^t)$ to Corollary 2.22 then, by Corollary 2.26, we obtain an extension of Corollary 2.22.

COROLLARY 2.31. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(A^t) = \Gamma(B)$. Then the following are equivalent:*

(i) *There exists a positive diagonal $n \times n$ matrix X such that*

$$XAX^{-1} = B.$$

(ii) *All cyclic products of the matrix $\max\{A/B, B^t/A^t\}$ are less than or equal to 1.*

(iii) *All nonzero cyclic products of the matrix $\max\{A/B, B^t/A^t\}$ are equal to 1.*

(iv) *All nonzero cyclic products of the matrix $\min\{B/A, A^t/B^t\}$ are greater than or equal to 1.*

(v) *All nonzero cyclic products of the matrix $\min\{B/A, A^t/B^t\}$ are equal to 1.*

(vi) *For every cycle γ in the graph $\Gamma(A)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have*

$$\frac{\Pi_\delta(A)}{\Pi_{\gamma\setminus\delta}(A^t)} = \frac{\Pi_\delta(B)}{\Pi_{\gamma\setminus\delta}(B^t)}.$$

Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (vi) is in Corollary 2.22.

(ii) ⇔ (iv) is in Corollary 2.26.

(iv) ⇒ (v). Let $i, j \in \{1, \ldots, n\}$ be such that $r_{ij} \neq 0$. By Lemma 2.30 we have $0 < r_{ij}r_{ji} \leq 1$, and so by (i4) we have $r_{ij}r_{ji} = 1$. It now follows that for every cycle γ in $\Gamma(R)$ the reverse cycle $\hat{\gamma}$ is also a cycle in $\Gamma(R)$ and $\Pi_\gamma(R) = \frac{1}{\Pi_{\hat{\gamma}}(R)}$. Since, by (iv), we have $\Pi_\gamma(R), \Pi_{\hat{\gamma}}(R) \geq 1$, it follows that $\Pi_\gamma(R) = 1$.

(v) ⇒ (iv) is trivial. ◻

REMARK 2.32. While, as is observed in Remark 2.11, the implication (ii) ⇒ (iii) in Corollary 2.31 holds in general, even without the requirement $\Gamma(A) = \Gamma(A^t)$, the reverse direction does not hold in general, as is demonstrated Example 2.29

3. Diagonal similarities into bands.

DEFINITION 3.1. Let M be an $n \times n$ matrix, and let Γ_n be the complete digraph with vertices $\{1, \ldots, n\}$.

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(i) The *maximal cycle mean* $\overline{m}(M)$ of the matrix M is defined as

$$\overline{m}(M) = \max_{\text{cycles } \gamma \text{ in } \Gamma_n} \sqrt[|\gamma|]{\Pi_\gamma(M)} \quad \left(= \max_{\text{cycles } \gamma \text{ in } \Gamma(M)} \sqrt[|\gamma|]{\Pi_\gamma(M)} \right).$$

(ii) The *minimal cycle mean* $\underline{m}(M)$ of the matrix M is defined as

$$\underline{m}(M) = \begin{cases} \min_{\text{cycles } \gamma \text{ in } \Gamma(M)} \sqrt[|\gamma|]{\Pi_\gamma(M)} & \text{if } \overline{m}(M) > 0 \\ 0 & \text{otherwise} \end{cases}$$

We begin with a corollary to Theorem 2.5.

COROLLARY 3.2. *Let $A^{(k)}, B^{(k)}$, $k = 1, \ldots, s$, be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A^{(k)}) \subseteq \Gamma(B^{(k)})$, $k = 1, \ldots, s$, and let $Q = \max\{(A^{(1)}/B^{(1)}), \ldots, (A^{(s)}/B^{(s)})\}$.*

(i) *If $\overline{m}(Q) = 0$, then for every positive number u there exists a positive diagonal $n \times n$ matrix X such that*

$$XA^{(k)}X^{-1} \le uB^{(k)}, k = 1, \ldots, s. \tag{3.3}$$

(ii) *If $\overline{m}(Q) > 0$, then there exists a positive diagonal $n \times n$ matrix X such that (3.3) holds if and only if $u \ge \overline{m}(Q)$.*

Proof. By Theorem 2.5, we have (3.3) if and only if all cyclic products of the matrix $\frac{1}{u}Q$ are less than or equal to 1, which is equivalent to saying that u is greater than or equal to the maximal cycle mean of the matrix Q. □

We remark that for the case $B = E$, Corollary 3.2 was stated in a somewhat different form as Theorem 7.2 and Remark 7.3 in [4].

COROLLARY 3.4. *Let $A^{(k)}, B^{(k)}$, $k = 1, \ldots, s$, be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A^{(k)}) \subseteq \Gamma(B^{(k)})$, $k = 1, \ldots, s$, and let $Q = \max\{(A^{(1)}/B^{(1)}), \ldots, (A^{(s)}/B^{(s)})\}$. Then there exists a positive diagonal $n \times n$ matrix X such that*

$$lA^{(k)} \le XB^{(k)}X^{-1}, k = 1, \ldots, s,$$

if and only if $\overline{m}(Q)\, l \le 1$.

Proof. The claim follows immediately from Corollary 3.2, since $XA^{(k)}X^{-1} \le uB^{(k)}$ is equivalent to $\frac{1}{u}A^{(k)} \le X^{-1}B^{(k)}X$. □

In the case that $s = 1$ we obtain the following equivalent formulation of Corollary 3.4.

COROLLARY 3.5. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(B) \subseteq \Gamma(A)$.*

(i) *If $\overline{m}(A/B) = 0$, then for every positive number l there exists a positive diagonal $n \times n$ matrix X such that*

$$lB \le XAX^{-1}. \tag{3.6}$$

(ii) *If $\overline{m}(A/B) > 0$, then there exists a positive diagonal $n \times n$ matrix X such that (3.6) holds if and only if $l \le \underline{m}(A/B)$.*

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Proof. In view of Corollary 3.4 we have to consider only the case that $\overline{m}(B/A) > 0$. Note that in this case we also have $\overline{m}(A/B) > 0$. Furthermore, it easy to verify that in this case

$$\underline{m}(A/B) = \frac{1}{\overline{m}(B/A)}. \tag{3.7}$$

By Corollary 3.4 we have

$$l \leq \frac{1}{\overline{m}(B/A)},$$

and our claim now follows by (3.7). □

Let A and B be nonnegative matrices with the same digraph. In view of Corollaries 3.2 and 3.5 it is natural to look for conditions for a matrix A to be diagonally similar to a matrix between lB and uB for some positive numbers l and u. The following characterization is a consequence of Corollary 2.19.

THEOREM 3.8. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$, and let l and u be positive numbers. Then the following are equivalent:*

(i) *There exists a positive diagonal $n \times n$ matrix X such that*

$$lB \leq XAX^{-1} \leq uB. \tag{3.9}$$

(ii) *For every cycle γ in the graph $\Gamma(A + A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have*

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \leq \frac{u^{|\delta|}}{l^{|\gamma\setminus\delta|}}. \tag{3.10}$$

Proof. Since $\Gamma(A) = \Gamma(B)$, by Corollary 2.19 we have $lB \leq XAX^{-1} \leq uB$ if and only if for every cycle γ in the graph $\Gamma(A + A^t)$ and every set of arcs δ, $\delta \subseteq \gamma$, we have

$$\Pi_\delta(A/uB)\Pi_{\gamma\setminus\delta}(lB^t/A^t) \leq 1. \tag{3.11}$$

Since (3.11) is equivalent to (3.10), our claim follows. □

REMARK 3.12. Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$. If there exists a positive diagonal $n \times n$ matrix X such that (3.9) holds, and if $\overline{m}(A/B) > 0$, then by Corollary 3.5 we necessarily have $l \leq \underline{m}(A/B)$ and by Corollary 3.2 we necessarily have $u \geq \overline{m}(A/B)$. These inequalities also follows from Theorem 3.8. The inequality $l \leq \underline{m}(A/B)$ follows from (3.10) when considering empty δ's, and the inequality $u \geq \overline{m}(A/B)$ follows from (3.10) when considering $\delta = \gamma$. However, it will be shown in Example 3.19 below that A is not necessarily diagonally similar to a matrix between $\underline{m}(A/B)\,B$ and $\overline{m}(A/B)\,B$.

REMARK 3.13. Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$. It immediately follows from Theorem 3.8 that for a given positive number l, $l \leq \underline{m}(A/B)$, the minimal positive number u such that there exists a positive diagonal $n \times n$ matrix X for which (3.9) holds is given by

$$u = \max_{\text{cycles } \gamma \text{ in } \Gamma(A+A^t),\ \phi\neq\delta\subseteq\gamma} \sqrt[|\delta|]{l^{|\gamma\setminus\delta|}\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t)}.$$

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Similarly, for a given positive number u, $u \geq \overline{m}(A/B)$, the maximal positive number l such that there exists a positive diagonal $n \times n$ matrix X for which (3.9) holds is given by

$$l = \min_{\text{cycles } \gamma \text{ in } \Gamma(A+A^t),\ \delta\subseteq\gamma,\, \delta\neq\gamma} \sqrt[-|\gamma\setminus\delta|]{u^{-|\delta|}\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t)}.$$

In the special case of Theorem 3.8 where $l = \underline{m}(A/B)$ and $u = \overline{m}(A/B)$ we have the following further result.

Proposition 3.14. *Let A and B be matrices in $\mathbb{R}^{nn}_+$ with $\Gamma(A) = \Gamma(B)$, and assume that $\overline{m}(A/B) > \underline{m}(A/B)$. If there exists a positive diagonal $n \times n$ matrix X such that*

$$\underline{m}(A/B)B \leq XAX^{-1} \leq \overline{m}(A/B)B, \tag{3.15}$$

then maximal mean cycles and minimal mean cycles of A/B do not have a common arc.

Proof. First notice that A/B and XAX^{-1}/B have the same corresponding cyclic products. Therefore, we have

$$\overline{m}(A/B) = \overline{m}(XAX^{-1}/B), \quad \underline{m}(A/B) = \underline{m}(XAX^{-1}/B).$$

Let m^* the maximal value of an element of XAX^{-1}/B. It follows from (3.15) that

$$m^* \leq \overline{m}(XAX^{-1}/B). \tag{3.16}$$

On the other hand, since every element XAX^{-1}/B is less than or equal to m^* it follows that

$$m^* \geq \overline{m}(XAX^{-1}/B). \tag{3.17}$$

It now follows from (3.16) and (3.17) that $m^* = \overline{m}(XAX^{-1}/B)$. Note that for every cycle γ in $\Gamma(XAX^{-1}/B)$ that contains an element smaller than m^* we have $\Pi_\gamma(XAX^{-1}/B) < (m^*)^{|\gamma|}$. Thus, in this case we have

$$\sqrt[|\gamma|]{\Pi_\gamma(XAX^{-1}/B)} < m^* = \overline{m}(XAX^{-1}/B), \tag{3.18}$$

and therefore it follows that every element on a maximal mean cycle of XAX^{-1}/B is equal to m^*. Similarly, one shows that for the minimal value m_* of an element of XAX^{-1}/B we have $m_* = \underline{m}(XAX^{-1}/B)$, and that every element on a minimal mean cycle of XAX^{-1}/B is equal to m_*. Since $\overline{m}(XAX^{-1}/B) > \underline{m}(XAX^{-1}/B)$, it follows that maximal mean cycles and minimal mean cycles of XAX^{-1}/B, and thus of A/B, do not have a common arc. ∎

In view of Proposition 3.14 we can construct a matrix A which is not diagonally similar to a matrix between $\underline{m}(A/B)\,B$ and $\overline{m}(A/B)\,B$.

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EXAMPLE 3.19. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 4 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Note that the only (simple) cycles in $\Gamma(A/B)$ are $(1,2,1)$ and $(1,2,3,1)$. Therefore, we have

$$\overline{m}(A/B) = \sqrt[3]{\Pi_{(1,2,3,1)}(A/B)} = 2, \quad \underline{m}(A/B) = \sqrt[2]{\Pi_{(1,2,1)}(A/B)} = 1.$$

Since the maximal mean cycle $(1,2,3,1)$ and the minimal mean cycle $(1,2,1)$ share the arc $(1,2)$, it follows from Proposition 3.14 that there exists no positive diagonal $n \times n$ matrix X such that (3.15) holds.

Theorem 3.8 allows us to obtain the following result that, in some sense, complements Corollaries 3.2 and 3.5.

DEFINITION 3.20. Let M be an $n \times n$ matrix, and let Γ_n be the complete digraph with vertices $\{1, \ldots, n\}$.

(i) A set α of arcs in $\Gamma(M)$ is said is said to be *relevant* if there exists a set β of arcs in $\Gamma(M^t)$ such that the union of α and β forms a cycle in Γ_n.

(ii) The *maximal relevant set mean* $\hat{u}(M)$ of the matrix M is defined as

$$\hat{u}(M) = \max_{\text{relevant sets } \delta \text{ in } \Gamma(M)} \sqrt[|\delta|]{\Pi_\delta(M)}.$$

(iii) The *minimal relevant set mean* $\hat{l}(M)$ of the matrix M is defined as

$$\hat{l}(M) = \min_{\text{relevant sets } \delta \text{ in } \Gamma(M)} \sqrt[|\delta|]{\Pi_\delta(M)}.$$

REMARK 3.21. It follows from Definition 3.20 that every cycle in $\Gamma(M)$ is a relevant set in $\Gamma(M)$. It thus follows that $\overline{m}(M) \leq \hat{u}(M)$. Also, if $\underline{m}(M) > 0$ then $\underline{m}(M) \geq \hat{l}(M)$.

THEOREM 3.22. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$. Then there exists a positive diagonal $n \times n$ matrix X such that*

$$\hat{l}(A/B)B \leq XAX^{-1} \leq \hat{u}(A/B)B.$$

Proof. Let γ be a cycle in $\Gamma(A + A^t)$ and let $\delta \subseteq \gamma$. Note that

$$\Pi_\delta(A/B)\Pi_{\gamma \setminus \delta}(B^t/A^t) > 0$$

if and only if δ is a relevant set in $\Gamma(A/B)$ and $\gamma \setminus \delta$ is a relevant set in $\Gamma(B^t/A^t)$, in which case it follows from Definition 3.20 that

$$\Pi_\delta(A/B)\Pi_{\gamma \setminus \delta}(B^t/A^t) \leq \hat{u}(A/B)^{|\delta|}\hat{u}(B^t/A^t)^{|\gamma \setminus \delta|}. \tag{3.23}$$

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Since clearly

$$\hat{u}(B^t/A^t) = \hat{u}(B/A) = \frac{1}{\hat{l}(A/B)}, \tag{3.24}$$

it now follows from (3.23) and (3.24) that

$$\Pi_\delta(A/B)\Pi_{\gamma\setminus\delta}(B^t/A^t) \le \frac{\hat{u}(A/B)^{|\delta|}}{\hat{l}(A/B)^{|\gamma\setminus\delta|}},$$

and our assertion follows by Theorem 3.8. ☐

We remark that the band given in Theorem 3.22 is not necessarily the narrowest possible band. Both bounds on both sides can sometimes be improved, as is demonstrated by the following example.

EXAMPLE 3.25. Let

$$A = \begin{bmatrix} 0 & 4 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{7}{5} & 0 \\ 0 & 0 & \frac{7}{3} \end{bmatrix}.$$

We have

$$XAX^{-1} = \begin{bmatrix} 0 & \frac{20}{7} & 0 \\ \frac{14}{5} & 0 & \frac{9}{5} \\ 0 & \frac{5}{3} & 0 \end{bmatrix},$$

and so, while $\hat{u}(A/B) = 4$ and $\hat{l}(A/B) = 1$, we have

$$\frac{5}{3}\, B \le XAX^{-1} \le \frac{20}{7}\, B.$$

A result concerning the narrowest possible band in (3.9) is given in Corollary 4.10 below.

4. Simultaneous diagonal equivalence. DEFINITION 4.1. Let M be an $m\times n$ matrix. The *bipartite graph* $\Delta(M)$ *of* M is defined to be the bipartite graph with vertex sets $\{1,\ldots,m\}$ and $\{1',\ldots,n'\}$, and where there is an edge between i and j' if and only if $m_{ij} \ne 0$.

PROPOSITION 4.2. *Let $A^{(k)}, B^{(k)}, C^{(k)}$ be matrices in $\mathbb{R}_+^{mn}$ with $\Delta(C^{(k)}) \subseteq \Delta(A^{(k)}) \subseteq \Delta(B^{(k)})$, $k = 1,\ldots,s$. Define the $(m+n)\times(m+n)$ matrices $R^{(k)}$ and $S^{(k)}$ by*

$$R^{(k)} = \begin{bmatrix} 0 & A^{(k)} \\ (C^{(k)})^t & 0 \end{bmatrix}, \quad S^{(k)} = \begin{bmatrix} 0 & B^{(k)} \\ (A^{(k)})^t & 0 \end{bmatrix}, \quad ,k = 1,\ldots,s.$$

Let X and Y be a positive diagonal $m\times m$ and $n\times n$ matrices respectively, and let D be the positive diagonal matrix $D = X \oplus Y^{-1}$. Then the following are equivalent:

(i) $$C^{(k)} \le XA^{(k)}Y \le B^{(k)}, \quad k = 1,\ldots,s.$$

(ii) $$DR^{(k)}D^{-1} \le S^{(k)}, \quad k = 1,\ldots,s.$$

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Proof. The result follows since $C^{(k)} \le XA^{(k)}Y$ is equivalent to $X^{-1}C^{(k)}Y^{-1} \le A^{(k)}$, which in turn is equivalent to $Y^{-1}(C^{(k)})^tX^{-1} \le (A^{(k)})^t$. ☐

In view of Theorem 2.5 we now immediately obtain the following result.

THEOREM 4.3. *Let $A^{(k)}, B^{(k)}, C^{(k)}$ be matrices in $\mathbb{R}_+^{mn}$ with $\Delta(C^{(k)}) \subseteq \Delta(A^{(k)}) \subseteq \Delta(B^{(k)})$, $k = 1, \ldots, s$. Define the matrices $R^{(k)}$ and $S^{(k)}$, $k = 1, \ldots, s$ as in Theorem 4.2. Let $Q = \max\{R^{(1)}/S^{(1)}, \ldots, R^{(s)}/S^{(s)}\}$. Then the following are equivalent:*

(i) *There exists a positive diagonal $m \times m$ matrix X and a positive diagonal $n \times n$ matrix Y such that the matrices $A^{(k)}, B^{(k)}, C^{(k)}$, $k = 1, \ldots, s$, satisfy*

$$C^{(k)} \le XA^{(k)}Y \le B^{(k)}, \quad k = 1, \ldots, s.$$

(ii) *All cyclic products of the matrix Q are less than or equal to* 1.

Again we observe that Condition (i) in Theorem 2.5 leads to an algorithm for finding an equivalence scaling that satisfies Theorem 4.3(i) if it exists or for negating its existence.

THEOREM 4.4. *Let A and B be matrices in $\mathbb{R}_+^{mn}$ with $\Delta(A) = \Delta(B)$, and let T be the $(m + n) \times (m + n)$ matrix defined by*

$$T = \begin{bmatrix} 0 & A/B \\ B^t/A^t & 0 \end{bmatrix}. \tag{4.5}$$

Let l and u be nonnegative numbers, $l \le u$. Then the following are equivalent:

(i) *There exist a positive diagonal $m \times m$ matrix X and a positive diagonal $n \times n$ matrix Y such that*

$$lB \le XAY \le uB.$$

(ii) *For every cycle γ in $\Gamma(T)$ we have*

$$\Pi_\gamma(T) \le \left(\frac{u}{l}\right)^{\frac{|\gamma|}{2}}.$$

Proof. Let R and S be the $(m + n) \times (m + n)$ matrices defined by

$$R = \begin{bmatrix} 0 & A \\ lB^t & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & uB \\ A^t & 0 \end{bmatrix}.$$

By Proposition 4.2 there exist a positive diagonal $m \times m$ matrix X and a positive diagonal $n \times n$ matrix Y such that $lB \le XAY \le uB$ if and only if there exist a positive diagonal $(m + n) \times (m + n)$ matrix D such that $DRD^{-1} \le S$. By Theorem 2.5, the latter holds if and only if all cyclic products of the matrix R/S are less than or equal to 1. Note that for every cycle γ in $\Gamma(R/S)$ the corresponding cyclic product contains $\gamma/2$ elements of the upper right block A/uB and $\gamma/2$ elements of the lower left block lB^t/A^t. Therefore, we have

$$\Pi_\gamma(R/S) = \left(\frac{l}{u}\right)^{\frac{|\gamma|}{2}} \Pi_\gamma(T).$$

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Our claim follows. □

The following corollary follows immediately from Theorem 4.4.

COROLLARY 4.6. *Let A and B be matrices in $\mathbb{R}_+^{mn}$ with $\Delta(A) = \Delta(B)$, and let T be the $(m+n) \times (m+n)$ matrix defined by (4.5). The minimal ratio between two nonnegative numbers u and l such that for some positive diagonal $m \times m$ matrix X and some positive diagonal $n \times n$ matrix Y we have $lB \le XAY \le uB$ is equal to the square of the maximal cycle mean of the matrix T.*

If in Corollary 4.6 all nonzero elements of B are equal to 1 then cycles of the matrix T defined in (4.5) correspond to the polygons defined in [9]. Thus Corollary 4.6 generalizes [9, Theorem 6'] which we here restate as:

COROLLARY 4.7. *Let A and be a matrix in $\mathbb{R}_+^{mn}$ and let $m^*(A)$ be defined as*

$$m^*(A) = \max_{i_1,\ldots i_k,\ j_1,\ldots j_k} \sqrt[k]{\frac{a_{i_1 j_1} \cdots a_{i_k j_k}}{a_{i_1 j_2} \cdots a_{i_k j_1}}}. \tag{4.8}$$

Then

$$\min_{X \in D_m, Y \in D_n} \left(\max_{i,k=1,\ldots,m,\ j,l=1,\ldots,n} \frac{x_i a_{ij} y_j}{x_k a_{kl} y_l} \right) = m^*(A), \tag{4.9}$$

where D_m and D_n are the sets of positive diagonal matrices in $\mathbb{R}_+^{mm}$ and $\mathbb{R}_+^{nn}$ respectively.

In the special case of Corollary 4.6 of diagonal similarity we obtain the following corollary.

COROLLARY 4.10. *Let A and B be matrices in $\mathbb{R}_+^{nn}$ with $\Gamma(A) = \Gamma(B)$, and let T be the $2n \times 2n$ matrix defined by (4.5). The ratio between two nonnegative numbers u and l such that for some positive diagonal $n \times m$ matrix X we have $lB \le XAX^{-1} \le uB$ is greater than or equal to the square of the maximal cycle mean of the matrix T.*

We remark that unlike in the diagonal equivalence case, it is possible that in the case of diagonal similarity the ratio between any two nonnegative numbers u and l such that for some positive diagonal $n \times m$ matrix X we have $lB \le XAX^{-1} \le uB$ always exceeds the maximal cycle mean of the matrix T, as is demonstrated in the following example.

EXAMPLE 4.11. Let A and B be as in Example 3.19, and let l and u be positive numbers for which there exists a positive diagonal $n \times n$ matrix X such that (3.9) holds. As is commented in Remark 3.12 and Example 3.19, we have $u \ge \overline{m}(A/B) = 2$ and $l \le \underline{m}(A/B) = 1$. In fact, it is shown there that $u/l > 2$. However, we have

$$T = \begin{bmatrix} 0 & A/B \\ B^t/A^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.12}$$

and it is easy to check that all cyclic products of T are equal to 1. See [9, Theorem 6] for a precise result.

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If we let $l = u$ in Theorem 4.4 then we obtain the following result.

THEOREM 4.13. *Let A and B be matrices in $\mathbb{R}_+^{mn}$ with $\Delta(A) = \Delta(B)$, and let T be the $(m+n) \times (m+n)$ matrix defined by (4.5). Then the following are equivalent:*
(i) *The matrix A is positive diagonally equivalent to the matrix B.*
(ii) *All cyclic products of the matrix T are equal to* 1.

Proof. Note that if γ is a cycle in $\Gamma(T)$ then so is the reverse cycle $\overline{\gamma}$. Furthermore, we have

$$\Pi_\gamma(T) = \frac{1}{\Pi_{\overline{\gamma}}(T)}.$$

Our claim now follows from Theorem 4.4. □

We remark that in the special case that A is fully indecomposable, Theorem 4.13 may be reduced to [3, Corollary 4.11].

REMARK 4.14. We again observe that our results can be put in a computational form, and we illustrate this be discussing the case of the diagonal equivalence of matrices A and B with $\Delta(A) = \Delta(B)$. Let T be defined by (4.5) and let $D = \text{diag}\,(d_1, \ldots, d_{m+n})$ be the positive diagonal matrix defined by

$$d_i := \max_\delta \ \Pi_\delta(T), \quad i = 1, \ldots, m+n,$$

where the maximum is taken over all paths δ in $\Gamma(T)$ of length less than or equal to $m + n - 1$. Then it follows by arguments similar to the proof Theorem 2.5 that all cyclic products of T are 1 if and only if all nonzero entries of DTD^{-1} are 1, cf. Remark 2.11. In this case, we obtain $B = XAY$ where $X = \text{diag}\,(d_1, \ldots, d_m)$ and $Y = \text{diag}\,(d_{m+1}, \ldots, d_{m+n})^{-1}$. If we apply this remark to the matrices A and B of Examples 3.19 and 4.11, then we obtain from the matrix T of (4.12) that $D = \text{diag}(1, 4, 1, 1, 1, 8)$. Since all nonzero elements of DTD^{-1} equal 1, it follows that $XAY = B$ for $X = \text{diag}(1, 4, 1)$ and $Y = \text{diag}(\frac{1}{4}, 1, \frac{1}{8})$.

For a related result on diagonal equivalence see [10, Theorem 3.1].

5. Generalizations to lattice ordered Abelian groups with 0.

In this section we show that our results may be generalized by replacing R^{nn} by a lattice ordered Abelian group with 0, in some instances, by a complete lattice ordered Abelian group with 0, cf. [4], where the non-commutative case is considered. Definitions are given below.

DEFINITION 5.1.
(i) A nonempty set G is called a *lattice ordered Abelian group* if it is a (multiplicative) Abelian group and a lattice, and for any nonempty finite subset U of G and for all $a \in G$ we have

$$a \sup\{U\} = \sup\{ax : \ x \in U\} \tag{5.2}$$

and

$$a \inf\{U\} = \inf\{ax : \ x \in U\}. \tag{5.3}$$

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(ii) A lattice ordered Abelian group G is called *complete* if the lattice order is complete and is compatible with multiplication, viz. $\sup\{U\}$ exists for infinite subsets of G which are bounded above and $\inf\{U\}$ exist in G for infinite subsets U of G which are bounded below, and (5.2) and (5.3) also hold for such subsets.

(iii) We call G_0 a *[complete] lattice ordered Abelian group with* 0 if G is a [complete] lattice ordered Abelian group and 0 is an additional element which satisfies

$$0 < a, \text{ for all } a \in G$$

and

$$0 = 0a, \text{ for all } a \in G.$$

Let G_0 be lattice ordered Abelian group with 0. We note that Notation 2.3.ii and Definition 2.4 can be applied to matrices in G_0^{nn}. Results that do not involve the maximum or minimum cycle means or the quantities $\hat{u}(M)$ or $\hat{l}(M)$ generalize to this setting with essentially the same proofs.

To generalize the remaining results, we let G_0 be a complete lattice ordered Abelian group with 0. We require new definitions for the maximal and minimal cycle means $\overline{m}(M)$ and $\underline{m}(M)$ to replace Definition 3.1, and new definitions for the maximal and minimal relevant set mean $\hat{u}(M)$ and $\hat{l}(M)$ to replace Definition 3.20.

For $M \in G_0^{nn}$ we define the set $U(M)$ by

$$U(M) = \{u \in G_0;\ u^{|\gamma|} \geq \Pi_\gamma(M) \text{ for all cycles } \gamma \text{ in } \Gamma(M)\}.$$

It can easily be shown that $U(M)$ is a nonempty set, containing the supremum of the elements of M, which is bounded below by the infimum of the elements of M. Therefore, we can define the *maximal cycle mean* $\overline{m}(M)$ of the matrix M by

$$\overline{m}(M) = \inf\{U(M)\}.$$

Similarly we define

$$L(M) = \{l \in G_0;\ l^{|\gamma|} \leq \gamma(M) \text{ for all cycles } \ \gamma \text{ in} \Gamma(M)\}$$

and the *minimal cycle mean* $\underline{m}(M)$ of the matrix M by

$$\underline{m}(M) = \sup\{L(M)\}.$$

With these definitions and similar definitions for $\hat{u}(M)$ and $\hat{l}(M)$ (omitted here) the remaining results in the previous sections generalize to matrices with elements in a complete lattice ordered Abelian group with 0, with the exception of Proposition 3.14. The following example demonstrates that Proposition 3.14 does not generalize even to matrices with elements in a complete fully ordered group with 0 that does not contain all square roots of its elements.

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EXAMPLE 5.4. Let G be the set of all integral powers of 2, and let

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is to verify that $m^* = \overline{m}(A/B) = 2$ and $m_* = \underline{m}(A/B) = 1$. However, both $\overline{m}(A/B)$ and $\underline{m}(A/B)$ are achieved for the same cycle $(1, 2, 1)$.

REMARK 5.5. In our results and, with some minor modifications, in definitions and proofs it is possible to interchange $\leq$ and $\geq$ and consequently also max and min.

REFERENCES

[1] S. N. Afriat. The system of inequalities $a_{rs} > X_r - X_s$. *Proc. Cambridge Philos. Soc.*, 59:125–133, 1963.
[2] L. Elsner and P. van den Driessche. Modifying the power method in max algebra. *Linear Algebra Appl.*, 332-334:3–13, 2001.
[3] G. M. Engel and H. Schneider. Cyclic and diagonal products on a matrix. *Linear Algebra Appl.*, 7:301–335, 1973.
[4] G. M. Engel and H. Schneider. Diagonal similarity and diagonal equivalence for matrices over groups with 0. *Czechoslovak Math. J.*, 25 (100):387–403, 1975.
[5] M. Fiedler and V. Ptak. Diagonally dominant matrices. *Czechoslovak Math. J.*, 17 (92):420–433, 1967.
[6] M. Fiedler and V. Ptak. Cyclic products and an inequality for determinants. *Czechoslovak Math. J.*, 19 (94):428–450, 1969.
[7] L. Fuchs. *Partially ordered algebraic systems.* Pergamon, Oxford, 1963.
[8] M. v. Golitschek, U. G. Rothblum, and H. Schneider. A conforming decomposition theorem, a piecewise nonlinear theorem of the alternative and scalings of matrices satisfying lower and upper bounds. *Math. Program.*, 27:291–306, 1983.
[9] U. G. Rothblum and H. Schneider. Characterizations of optimal scalings of matrices. *Math. Program.*, 19:121–136, 1980.
[10] B. D. Saunders and H. Schneider. Flows of graphs applied to diagonal similarity and equivalence of matrices. *Discrete Math.*, 24:139–162, 1978.
[11] B. D. Saunders and H. Schneider. Applications of the Gordan-Stiemke theorem in combinatorial matrix theory. *SIAM Review*, 21:528–541, 1979.
[12] B. D. Saunders and H. Schneider. Cones, graphs and optimal scalings of matrices. *Linear Multilinear Algebra*, 8:121–135, 1979.
[13] H. Schneider and M. H. Schneider. Max-balancing weighted directed graphs and matrix scaling. *Math. Oper. Res.*, 16:208–222, 1991.
[14] H. Schneider and M. H. Schneider. Towers and cycle covers for max-balanced graphs. *Congr. Numer.*, 73:159–170, 1990.