LYAPUNOV REVISITED: VARIATIONS ON A MATRIX THEME

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Dedicated to Paul A. Fuhrmann on the occasion of his 60th birthday

Abstract

In this expository note it is shown that a cone version of the Perron–Frobenius theorem implies various generalizations of a matrix form of Lyapunov's famous theorem:

Si les équations différentielles du mouvement troublé sont telles qu'il est possible de trouver une fonction définie V, dont la dérivée $\dot{V}$ soit une fonction de signe fixe et contraire à celui de V, ou se réduise identiquement à zéro, le mouvement non troublé est stable.

Lyapunov's basic result on the stability of solutions of differential equations [Lyap, Ch.I, §16, Th.I] is here quoted from the French translation of his 1892 memoir. Lyapunov also investigated a more restrictive concept, that of asymptotic stability, in the case of linear differential equations with constant coefficients where V is a homogeneous form of degree m, see [Lyap, Ch. II]. Gantmacher [Gant, Ch.XV, §5] considered the case of constant coefficients $\dot{x} = Ax$, $x \in \mathbb{C}^n$, $A \in \mathbb{C}^{nn}$, and restricted V to a homogeneous quadratic form. Thus

$$V(x) = x^*Hx, \quad H^* = H$$

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and

$$\dot{V}(x) = \dot{x}^* H x + x^* H \dot{x} = x^*(A^* H + HA)x.$$

Putting

$$W(x) = x^* K x, \ K \gg 0,$$

where

$$K \gg 0 := K \text{ positive definite},$$

and noting that

$$x(t) \to 0 \text{ as } t \to \infty \Leftrightarrow \Re(\lambda) < 0, \text{ all } \lambda \in \text{spec}(A),$$

he obtained a result [Gant, Ch.XV, Th.3'] that is usually called "Lyapunov's theorem" by matrix theorists which I state in a slightly more general form:

Theorem 0 : Let $A \in \mathbb{C}^{nn}$ and let $K \gg 0$. Then there exists $H \gg 0$ such that $AH + HA^* = K$ if and only if A is positive stable (i.e. has all eigenvalues in the *right* half plane).

Gantmacher's reformulation, see also [Hahn, Kap. II, §8], had a deep influence on the inertia theory of matrices as developed in the 1960's and subsequently, but I shall pursue this topic no further. Note that in Lyapunov's original formulation the theorem concerned the *existence* of a function V with certain properties, while Gantmacher's version concerns *solving* a matrix equation. The two formulations are equivalent for we have:

$$\forall K \gg 0, \ \exists H \gg 0, \ AH + HA^* = K \Longleftrightarrow \exists H \gg 0, \ AH + HA^* \gg 0.$$

I had met this situation before in Perron–Frobenius theory. Thus we define, for $P \in \mathbb{R}^{mn}$,

$$P > 0 := p_{ij} > 0, \text{ all } (i,j),$$

$$P \geq 0 := p_{ij} \geq 0, \text{ all } (i,j)$$

and employ the spectral radius $\rho(P)$ defined as usual by

$$\rho(P) = \max\{|\lambda| : \lambda \in \text{spec}(P)\}.$$

If $P \geq 0$ it follows by Perron–Frobenius that $\rho(P)$ is an eigenvalue of P. We further have, see e.g. [BePl, Theorem 6.2.],

Theorem 1 : Let $A = \sigma I - P$ where $P \geq 0$. Then the following are equivalent:

1. $\sigma > \rho(P)$.

2. For all $y > 0$, there exists $x > 0$ such that $Ax = y$. (viz. $A^{-1} > 0$).

3. There exists $x > 0$ such that $Ax > 0$.

Again we have

$$\forall \Longleftrightarrow \exists \ .$$

In [Schn] I found a unified treatment and generalized Lyapunov's theorem. The key is a generalization of Perron–Frobenius to cones which is due to Krein–Rutman [KrRu] in a Banach space. We consider only the finite dimensional case here.

Definition: A subset $\mathcal{C}$ of a (finite dimensional) space V over $\mathbb{R}$ is a (pointed, full, closed)) *cone* if

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$, viz. $x + y \in \mathcal{C},\ \forall x, y \in \mathcal{C}$.

2. $\mathbb{R}_+\mathcal{C} \subseteq \mathcal{C}$, viz. $\alpha x \in \mathcal{C},\ \forall \alpha \geq 0, x \in \mathcal{C}$.

3. $\mathcal{C} \cap -\mathcal{C} = \{0\}$, viz. $x, -x \in \mathcal{C} \Rightarrow x = 0$.

4. $\mathcal{C} - \mathcal{C} = V$ viz. $\forall z \in V,\ \exists x, y \in \mathcal{C},\ z = x - y$,
   equivalently, the interior $\mathcal{C}^0 \neq \ \phi$.

5. $\mathcal{C}$ is closed.

We now redefine for $x \in V$:

$$x \geq 0 \ : \ x \in \mathcal{C},$$

$$x > 0 \ : \ x \in \mathcal{C}^0,$$

and for $T \in \mathrm{Hom}(V)$:

$$T \geq 0 : \ T\mathcal{C} \subseteq \mathcal{C}.$$

Again, Perron–Frobenius (Krein–Rutman) applies: If $T \geq 0$ then $\rho(T) \in \mathrm{spec}(T)$, and as a consequence we obtain

Theorem 2 : Let $\mathcal{C}$ be a cone. Let $T = R - S \in \mathrm{Hom}(V)$:

$$T = R - S,\ R^{-1} \geq 0,\ S \geq 0.$$

Then the following are equivalent:

1. $\rho(R^{-1}S) < 1$.
2. $T^{-1}\mathcal{C}^0 \subseteq \mathcal{C}^0 \quad (T^{-1} \geq 0)$.
3. $T\mathcal{C}^0 \cap \mathcal{C}^0 \neq \phi$.

The obvious model is $\mathcal{C} = \mathbb{R}^n_+$, the set of all vectors with nonnegative components in $\mathbb{R}^n$. In this case Theorem 2 is a slight generalization of Theorem 1 to splittings of type $T = R - S$ which Varga exploited and called *regular splittings*, see [Varg, p. 88]. We, however, are interested in the following set up:

$$V = \mathcal{H}_n, \ \mathcal{C} = \mathcal{P}_n,$$

where

$$\mathcal{H}_n = \text{real space of Hermitians in} \mathbb{C}^{nn},$$

$$\mathcal{P}_n = \text{cone of positive semidefinite Hermitians in } \mathcal{H}_n.$$

If $R \in \text{Hom}(\mathcal{H}_n)$ is defined by $R(H) = AHA^*$, where $A \in C^{nn}$ is nonsingular, then $R \geq 0$ and $R^{-1} \geq 0$. If $S \in \text{Hom}(\mathcal{H}_n)$ is defined by $S(H) = \Sigma_k C_k^* H C_k$ then $S \geq 0$. The operator $R^{-1}S$ in (1.) of Theorem 2 now becomes

$$R^{-1}S \ = \ \Sigma_{k=1}^s \ (A^{-1}C_k \times \bar{A}^{-1}\bar{C}_k)$$

where $\times$ is the Kronecker (tensor) product. Thus Theorem 2 specializes to:

Theorem 3 : Let $A, \ C_k, \ k = 1, \ldots, s$, be complex $n \times n$ matrices. Let H be Hermitian. Then the following are equivalent:

1. A is nonsingular and
$$\rho(\Sigma_{k=1}^s A^{-1}C_k \times \bar{A}^{-1}\bar{C}_k) < 1.$$
2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that
$$AHA^* - \Sigma_{k=1}^s C_k H C_k^* = K.$$
3. There exists an $H \gg 0$ such that
$$AHA^* - \Sigma_{k=1}^s C_k H C_k^* \gg 0.$$

4. A is nonsingular and there exists an $H \gg 0$ such that

$$\rho((\Sigma_{k=1}^{s} A^{-1} C_k H C_k^* A^{*-1}) H^{-1}) < 1.$$

Condition (4.) of Theorem 3 is a consequence of (3.) and was pointed out to me by S. Friedland.

The spectral radius of the operator in (1.) of Theorem 3 can be evaluated in terms the eigenvalues of its constituent matrices (only) under special assumptions. One such assumption is that the matrices $A,\ C_k,\ k = 1, \ldots, s$, are *simultaneously triangulable*, viz. there exists $Q \in \mathbb{C}^{nn}$ such that $Q^{-1}AQ,\ Q^{-1}C_kQ$, $k = 1, \ldots, s$, are (upper) triangular. In this case there exists an obvious *natural correspondence* $(\alpha_i, \gamma_i^{(1)}, \ldots, \gamma_i^{(s)})$, $i = 1, \ldots, n$, of the eigenvalues of $A,\ C_k,\ k = 1, \ldots, s$, such that every (noncommutative) polynomial $p(A, C_1, \ldots, C_s)$ has eigenvalues $p(\alpha_i, \gamma_i^{(1)}, \ldots, \gamma_i^{(s)})$, $i = 1, \ldots, n$. (A theorem of McCoy's assert that this latter property is equivalent to simultaneous triangulability). It was known to Frobenius that a set of pairwise commutative matrices is simultaneously triangulable. In particular, if $C_k = C^k,\ k = 0, \ldots, s$, then the C_k can be simultaneously triangulated. See [Taus] for more information and references on this topic.

If $A,\ C_k$, $k = 1, \ldots, s$, are simultaneously triangulable, then for the operator $R^{-1}S$ in Theorem 3 we have

$$\mathrm{spec}(R^{-1}S) \ = \{\Sigma_{k=1}^{s} \alpha_i^{-1} \gamma_i^{(k)} \bar{\alpha}_j^{-1} \bar{\gamma}_j^{(k)} : i, j = 1, \ldots, n\}.$$

We may apply Cauchy's inequality to obtain [Schn, Theorem 1]:

Theorem 4 : Let $A,\ C_k,\ k = 1, \ldots, s$, be complex $n \times n$ matrices which can be simultaneously triangulated. Suppose the eigenvalues of $A,\ C_k$ under a natural correspondence are $\alpha_i,\ \gamma_i^{(k)},\ i = 1, \ldots, n,\ k = 1, \ldots, s$. For Hermitian H, let

$$T(H) = AHA^* - \Sigma_{k=1}^{s} C_k H C_k^*.$$

Then the following are equivalent:

1. $\epsilon_i := |\alpha_i|^2 - \Sigma_{k=1}^{s} |\gamma_i^{(k)}|^2 > 0,\ i = 1, \ldots, n.$

2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that T(H) = K.

3. There exists an $H \gg 0$ such that $T(H) \gg 0$.

We note the following special cases:

If

$$T(H) = (B+I)H(B+I)^* - (BHB^* + IHI^*) = BH + HB^*$$

then $\epsilon_i = \beta_i + \bar{\beta}_i$ and thus we obtain Lyapunov's Theorem.

If

$$T(H) = IHI^* - CHC^*$$

then $\epsilon_i = 1 - |\gamma_i|^2$ and thus we have a result due to Stein.

We now turn to a generalization due to D.H. Carlson, published in [Hill]. Let

$$\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$$

and consider the operator T defined by

$$T(H) = \Sigma_{h,k=0}^{s} \ \varphi_{hk} C_h H C_k^*$$

for $H \in \mathcal{H}_n$. We define the $n \times (s+1)n$ matrix

$$\underline{C} = [C_0, \ \ldots, \ C_s]$$

and we obtain

$$T(H) = \underline{C}(\Phi \times H)\underline{C}^* = \underline{C}(U \times I)(\Delta \times H)(U^* \times I)\underline{C}^* = \underline{B}(\Delta \times H)\underline{B}^*,$$

where U is a unitary matrix, $\Phi = U\Delta U^*$ and $\underline{B} = \underline{C}U \in \mathbb{C}^{n,(s+1)n}$. If $C_0, \ \ldots, \ C_s$ are simultaneously triangulable and we put

$$\underline{\gamma_i} = [\gamma_i^{(0)}, \ \ldots, \ \gamma_i^{(s)}], \ i = 1, \ldots, n,$$

where $(\gamma_i^{(0)} \ \ldots \ \gamma_i^{(s)}), \ i = 1, \ldots, n$, is a natural correspondence of the eigenvalues, $k = 0, \ldots, s$. Since the eigenvalues of $B_k, \ k = 0, \ldots, s$, are $\Sigma_{h=0}^{s} \gamma_h^{(i)} u_{hk}$, $i = 1, \ldots, n$, Theorem 4 can be generalized to the following result, where by $\pi(\Phi)$ we denote the number of positive eigenvalues of the Hermitian matrix Φ.

Theorem 5 : Let $C_k, \ 0 = 1, \ldots, s$, be complex $n \times n$ matrices which can be simultaneously triangulated. Suppose the eigenvalues of $C_0, \ \ldots, \ C_s$ under a natural correspondence are $\gamma_i^{(0)}, \ \ldots \gamma_i^{(s)}, \ i = 1, \ \ldots, \ n$. Let $\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$, where $\pi(\Phi) = 1$. For Hermitian H, let

$$T(H) = \Sigma_{h,k=0}^{s} \ \varphi_{hk} C_h H C_k^*.$$

Then the following are equivalent:

1. $\underline{\gamma_i}\Phi\underline{\gamma_i}^* > 0, \ i = 1, \ldots, n.$
2. For all $K \gg 0$, there exists a unique $H \gg 0$ such that T(H) = K.
3. There exists an $H \gg 0$ such that $T(H) \gg 0$.

Clearly the assumptions of Theorem 5 are satisfied if $A \in \mathbb{C}^{nn}$ and $C_k = A^k, \ k = 1, \ldots, n$. Thus we derive a result independently due to Kharitonov [Khar], see also [Gutm, Theorem 6.1].

Theorem 6 : Let $A \in \mathbb{C}^{nn}$ have eigenvalues $\alpha_i, \ i = 1, \ldots, n$. Let $K \in \mathbb{C}^{nn}$ be positive definite and suppose that Φ is a Hermitian matrix in $\mathbb{C}^{s+1,s+1}$ with $\pi(\Phi) = 1$. Then the following are equivalent:

1. $\Sigma_{h,k=0}^{s} \ \alpha_i^h \varphi_{hk} \bar{\alpha}_i^k > 0, \ i = 1, \ldots, n.$
2. The (unique) solution H of $\Sigma_{h,k=0}^{s} \ \varphi_{hk} A^h H A^{k*} = K$ is positive definite.

Theorems 5 and 6 do not hold without the assumption that $\pi(\Phi) = 1$, as is shown by the following example with $\pi(\Phi) = 2$. Let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$AHA^* + CHC^* = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}.$$

A perturbation argument shows that we can find H with $\pi(H) = 1$ or $\pi(H) = 2$ such that $AHA^* + CHC^* \gg 0$. However, a weaker result holds, see [Hill] for remarks on this topic. We need an additional assumption, which is again satisfied if the $C_k, \ k = 0, \ldots, s$, commute pairwise and hence if $C_k = A^k, \ k = 0, \ldots s$, see [CaPi] for information. Dropping the assumption that $\pi(\Phi) = 1$, we still have

Theorem 7 : Let $C_k, \ k = 0, \ldots, s$, be complex $n \times n$ matrices which can be simultaneously triangulated . Assume that for each distinct sequence of corresponding eigenvalues $(\gamma_i^{(0)}, \ \ldots, \ \gamma_i^{(s)}), \ i = 1, \ \ldots, \ n$, of $C_0, \ \ldots, \ C_s$ there exists a common eigenvector. Let $\Phi = \Phi^* \in \mathbb{C}^{s+1,s+1}$. Then the following are equivalent:

1. $\underline{\gamma_i}\Phi\underline{\gamma_i}^* > 0,\ i = 1, \dots, n.$

3. There exists an $H \gg 0$ such that $\Sigma_{h,k=0}^s \varphi_{hk} C_h H C_k^* \gg 0.$

As a special case of Theorem 7 we state

Theorem 8 : Let $A \in \mathbb{C}^{nn}$ have eigenvalues $\alpha_i,\ i = 1, \dots, n$. Suppose that Φ is a Hermitian matrix in $\mathbb{C}^{s+1,s+1}$. Then the following are equivalent:

1. $\Sigma_{h,k=0}^s \alpha_i^h \varphi_{hk} \bar{\alpha}_i^k > 0,\ i = 1, \dots, n.$

3. There exists $H \gg 0$ such that $\Sigma_{h,k=0}^s \varphi_{hk} A^h H A^{k*} \gg 0.$

Proofs of the last two theorems are implicit in [Hill] or [Khar].

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