Linear and Multilinear Algebra, 1992, Vol. 32, pp. 131–148
Reprints available directly from the publisher
Photocopying permitted by license only

Printed in the United States of America

Principal Components of Minus M-Matrices*

MICHAEL NEUMANN**
Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009

HANS SCHNEIDER†
Department of Mathematics, University of Wisconsin, Madison, WI 53706

(Received January 18, 1991; in final form July 16, 1991)

In this paper we determine the nonnegativity structure of the principal components of an $n \times n$ nonnegative matrix P in terms of the marked reduced graph $\mathcal{R}(A)$ of $A = P - \rho(P)I$, the minus M-matrix which can be associated with P. We then apply this result to consider various types of nonnegative bases for the Perron eigenspace of P which can be extracted from a certain nonnegative matrix which is a polynomial in P. We also obtain a characterization for the eigenprojection on the Perron eigenspace of P to be, itself, a nonnegative matrix. Our results provide new proofs and extensions of results of Friedland and Schneider and of Hartwig, Neumann, and Rose.

1. INTRODUCTION

In this paper we determine the nonnegativity structure of the principle components of an $n \times n$ nonnegative matrix P in terms of the marked reduced graph $\mathcal{R}(A)$ of the associated minus M-matrix $A = P - \rho(P)I$. We then apply this result to obtain a characterization for when the eigenprojection on the generalized eigenspace corresponding to the Perron root (subsequently called *the Perron eigenspace* of P or A) is itself a nonnegative matrix. We also apply this result to obtain new proofs and extensions of results of Hartwig, Neumann, and Rose [3] and Friedland and Schneider [5].

It is well known that the Perron eigenspace of P has a basis of nonnegative vectors with many specified properties, see Rothblum [11], Richman and Schneider [10], Schneider [13], and Hershkowitz and Schneider [6]. The approach taken in these papers is matrix combinatorial and uses the Frobenius normal form. More recently the authors of [3] gave an analytic proof for the existence of a nonnegative basis. They found a nonnegative matrix J which turns out to be a polynomial in P and whose columns contain a basis for the Perron eigenspace of P. This suggests that the two approaches are related. In this paper the relation is investigated.

*Dedicated to Henryk Minc on the occasion of his 70th birthday.

**Work supported by U.S. Air Force Grant AFOSR-88-0047 and NSF Grant DMS-9007030. Part of this author's work was carried out while visiting the Sonderforschungsbereich 343 "*Diskrete Strukturen in der Mathematik*", at the University of Bielefeld, Bielefeld, Germany.

†Work supported by NSF Grant DMS-8901445, by NSF Grant ECS-8718971, and by the Ministry of Education, Science and Culture, Japan.

We now describe the contents of the paper in more detail. Section 2 contains most of the notions which we use in the paper. In Section 3 we prove our first main result (Theorem 1): *Let P be an $n \times n$ nonnegative matrix in Frobenius normal form and let $Z^{(k)}$ be the kth principal component (associated with the spectral radius $\rho(P)$). Let i and j be vertices in $\mathcal{R}(A)$ and let $d = d(i,j)$ be the (singular) distance from i to j. Then the (i,j)th block of $Z^{(k)}$ is strictly positive if $d = k + 1$.* We mention that according to Lemma 2(i), the (i,j)th block of $Z^{(k)}$ is zero if $d < k + 1$. The results of [3] are a corollary of this result. Another outcome of this result is that a nonnegative basis can be extracted from the columns of J which is strongly combinatorial in the sense defined in Section 2.

In Section 4 we characterize when the eigenprojection $Z = Z^{(0)}$ on the Perron eigenspace of P is itself a nonnegative matrix. We begin by showing that a necessary condition for this to happen is that in $\mathcal{R}(A)$ no nonsingular vertex lies on the interior of a simple path connecting two singular vertices. This leads to our characterization (cf. Theorem 2) for Z to be nonnegative in terms of the existence of special nonnegative bases for the Perron eigenspaces of A and A^T which are bi-orthogonal and whose distinguishing feature, compared with other types of nonnegative bases which the Perron eigenspaces of these matrices possess, is their comparative sparseness.

In Section 5, using Theorem 1, we re-prove results contained in [5] on the asymptotic behavior of the powers of P. In the special case when all the vertices in $\mathcal{R}(A)$ come from primitive diagonal blocks in the Frobenius normal form of P, the result asserts that $(P^m)_{i,j}$ behaves asymptotically as $\gamma m^d s^m Z_s^{(\delta-1)}(P\{i,j\})$, where γ is a positive constant, $\delta = \delta(i,j)$ is the local distance from i and j, s is the local spectral radius of $P\{i,j\}$, the principal submatrix of P indexed by all vertices l which lie on a path from i to j in $\mathcal{R}(P)$, and $Z_s^{(\delta-1)}(P\{i,j\})$ is the $(\delta-1)$th principal component of $P\{i,j\}$ corresponding to the eigenvalue s. In the general case, without any primitivity assumption, we consider the sequence MP^m, where M is a smoothing matrix which is a polynomial in P.

In this introduction we have described our results in terms of the nonnegative matrix P. In Sections 2–4 it will be convenient to state our result in terms of the associated minus M-matrix A given above.

2. NOTATIONS AND PRELIMINARIES

For a positive integer n we denote by $\langle n\rangle$ the set $\{1,\ldots,n\}$.
For an $n \times n$ matrix A we denote by:

$N(A)$—the nullspace of A.
$E(A)$—the generalized nullspace of A, viz. $N(A^n)$.
$\nu(A)$—the index of 0 as an eigenvalue of A, viz. the size of the largest Jordan block associated with 0.

Let $\alpha \subseteq \langle n\rangle$. By $A[\alpha]$ we shall denote the principal submatrix of A whose rows and columns are determined by α. Similarly, for an n-vector x, we shall denote by $x[\alpha]$ the subvector of x whose entries are indexed by α. For an array C we shall use

$C \geq 0$ to denote when all its entries are nonnegative numbers. $C > 0$ shall denote the fact that $C \geq 0$, but $C \neq 0$. $C \gg 0$ shall denote the fact that all of the entries of C are positive numbers.

In all our considerations we shall assume that A is an $n \times n$ real matrix given in a block lower triangular form with p square diagonal blocks as follows

$$A = \begin{pmatrix} A_{1,1} & 0 & \cdots & 0 \\ A_{2,1} & A_{2,2} & & 0 \\ \vdots & & \ddots & \vdots \\ A_{p,1} & \cdots & \cdots & A_{p,p} \end{pmatrix}, \tag{2.1}$$

where each diagonal block is irreducible or the 1×1 null matrix. The above form is the so called *Frobenius normal form* of A. It is well known that any square matrix is symmetrically permutable to such a form. The *reduced graph of* A, $\mathcal{R}(A)$, is defined to be the graph with vertices $1,\ldots,p$, where (i,j) is an arc from i to j if $A_{i,j} \neq 0$. A vertex i in $\mathcal{R}(A)$ is said to be singular if $A_{i,i}$ is singular. Otherwise the vertex is called *nonsingular*. The set of all singular vertices in $\mathcal{R}(A)$ will be denoted by $\mathcal{S}(A)$. A sequence of vertices $(i_1,\ldots,i_k)$ in $\mathcal{R}(A)$ is said to be a *path* from i_1 to i_k if there is an arc in $\mathcal{R}(A)$ from i_j to i_{j+1}, $\forall j \in \langle k-1 \rangle$. The path is said to be *simple* if $i_1,\ldots,i_k$ are distinct. The *(singular) length of a simple path* is the number of singular vertices lying on it. The empty path will be considered a simple path linking every vertex $i \in \mathcal{R}(A)$ to itself. If there is a path (in $\mathcal{R}(A)$) from i to j we shall write that $i \succeq j$. If $i \neq j$ and there is a path from i to j, we shall write that $i \succ j$. If there is a path from i to j, define the *(singular) distance* $d(i,j)$ from i to j to be the maximal length of a simple path connecting i and j. If there is no path from i to j, we set $d(i,j) = -1$. In particular it follows from our definition that $d(i,i) = 0$ if i is a nonsingular vertex and $d(i,i) = 1$ if i is a singular one. Note that each vertex $i \in \mathcal{R}(A)$ can be thought of a subset of $\langle n \rangle$ which consists of those elements in $\langle n \rangle$ upon which the ith diagonal block in A is indexed.

If 0 is an eigenvalue of A, then in a punctured neighborhood of 0 which contains no other eigenvalues of A, the resolvent operator $(\epsilon I - A)^{-1}$ admits the Laurent expansion

$$(\epsilon I - A)^{-1} = \sum_{k=0}^{\nu(A)-1} \frac{A^k Z_A}{\epsilon^{k+1}} + T(\epsilon), \tag{2.2}$$

where Z_A is the eigenprojection of A on $E(A)$ (that is, the projection on $E(A)$ along the join of all eigenspaces of A corresponding to eigenvalues other than zero) and where $T(\epsilon)$ is an analytic operator in ϵ defined throughout the nonpunctured neighborhood of 0 and satisfying $Z_A T(\epsilon) = T(\epsilon) Z_A = 0$ (cf. Kato [8, pp. 34–43]). We call the matrices $Z^{(k)} := A^k Z_A$, $k = 0,1,\ldots,\nu(A)-1$, *the principal components of* A (*corresponding to the eigenvalue* 0). Recall, e.g. Lancaster and Tismenetsky [9], that all principal components of A are functions of A and hence also polynomials in A. Note, however, that the principal components as defined here differ by factorial multiples from those introduced in Lancaster and Tismenetsky (see [9, p. 314]).

Consider $\mathcal{R}(A)$. We shall let

$$\langle i,j\rangle = \{l \in \mathcal{R}(A) : i \succeq l \succeq j\}$$

and we shall let $A\{i,j\} = A[\langle i,j\rangle]$. The following lemma is readily obtained:

LEMMA 1

(i) *For every polynomial* $B = q(A)$ *we have that* $B[\langle i,j\rangle] = q(A\{i,j\})$. *In particular, as all the principal components of* A *are polynomials in* A, $(Z^{(k)})[\langle i,j\rangle] = (A\{i,j\})^k Z_{A\{i,j\}}$.
(ii) $\nu(A\{i,j\}) \leq \nu(A)$.

Proof (i) Since B is a polynomial in A it suffices to show that for any positive integer r, $(A^r)[\langle i,j\rangle] = (A[\langle i,j\rangle])^r$. Let $s,t \in \langle i,j\rangle$. Then

$$(A^r)_{s,t} = \sum_{1\leq\gamma_1\leq\cdots\leq\gamma_{r-1}\leq p} A_{\gamma_0,\gamma_1}\cdots A_{\gamma_{r-1},\gamma_r},$$

where $s = \gamma_0$ and $t = \gamma_r$. But $A_{h,k} \neq 0$ implies that $h \succeq k$. Hence

$$(A^r)_{s,t} = \sum_{s\succeq\gamma_1\succeq\cdots\succeq\gamma_{r-1}\succeq t} A_{\gamma_0,\gamma_1}\cdots A_{\gamma_{r-1},\gamma_r} = ((A^r)[\langle i,j\rangle])_{s,t}.$$

That $(Z^{(k)})[\langle i,j\rangle)] = (A\{i,j\})^k Z_{A\{i,j\}}$ now follows because, as mentioned earlier, the principal components of A are polynomials in A.

(ii) This index inequality follows from (i) and the fact that $\nu(A)$ is the largest integer q such that $Z^{(q)} \neq 0$. ∎

Let P be an $n \times n$ nonnegative matrix. The Perron Frobenius theory (cf. Berman and Plemmons [1]) tells us that the spectral radius of P, given by the quantity

$$\rho(P) = \max\{|\lambda| : \det(P - \lambda I) = 0\},$$

is an eigenvalue of P to which there corresponds a nonnegative eigenvector. In particular, if P is irreducible, then $\rho(P)$ is a simple eigenvalue and the corresponding eigenvector is, up to a multiple by a scalar, positive. The matrix $A = P - \rho(P)I$, which has all its off-diagonal entries nonnegative, is the $n \times n$ minus M-matrix which we associate with P and, in several sections of our paper, it will be convenient to work with A rather than with P. (We call A a *minus* M*-matrix* if $-A$ is an M-matrix. For the many equivalent conditions for a real matrix with nonpositive off-diagonal entries to be an M-matrix see Berman and Plemmons [1, Chp. 6].) Suppose now that $m = \dim(E(A))$. It is known that m is equal to the number of singular vertices in $\mathcal{R}(A)$, e.g., Cooper [2]. Rothblum [11] has shown that $\nu(A)$ is equal to the maximum over all lengths of the simple paths in $\mathcal{R}(A)$, a result to which we shall refer as the *Rothblum index theorem*. Let $S(A) = \{\alpha_1, \ldots, \alpha_m\}$. Rothblum [11] and, independently, Richman and Schneider [10] have shown that $E(A)$ possess a basis of nonnegative vectors $u^{(1)},\dots,u^{(m)}$ having the following properties:

$$u^{(j)}[i] = \begin{cases} \gg 0, & \text{iff} \quad i \succeq \alpha_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

We shall call a nonnegative basis for $E(A)$ satisfying (2.3) (*nonnegatively*) *strongly combinatorial*.

Let A be a minus M-matrix given in the form (2.1). Then A^T is now a minus M-matrix in upper triangular Frobenius normal form. If we were to introduce formally the concept of the reduced graph for matrices in upper triangular form, we would observe that in going from A to A^T, the direction of the access between the vertices is reversed. In any event, by possibly transforming A^T to a lower triangular form via similarity permutations, it follows, on applying the aforementioned results for nonnegative bases for matrices in lower triangular Frobenius normal form, that $E(A^T)$ has a nonnegative basis of vectors $v^{(1)},\dots,v^{(m)}$ such that for $i \in \langle p\rangle$ and $j \in \langle m\rangle$,

$$v^{(j)}[i] = \begin{cases} \gg 0, & \text{iff in} \quad \mathcal{R}(A), \quad \alpha_j \succeq i, \\ 0, & \text{otherwise.} \end{cases}$$

Motivated by this observation and by Hershkowitz and Schneider [6] we now introduce the notions of column and row proper combinatorial bases.

DEFINITION 1 *Let A be the $n \times n$ minus M-matrix given in* (2.1) *and consider $\mathcal{R}(A)$. A nonnegative basis $u^{(1)},\dots,u^{(m)}$ is called a* (nonnegatively) proper column combinatorial basis *for $E(A)$ if*

$$u^{(j)}[i] > 0 \Rightarrow i \succeq \alpha_j$$

and

$$u^{(j)}[\alpha_j] \gg 0$$

for all $i \in \langle p\rangle$ and $j \in \langle m\rangle$. Similarly, a nonnegative basis $v^{(1)},\dots,v^{(m)}$ for $E(A^T)$ is called a (nonnegatively) proper row combinatorial basis *for $E(A^T)$ if*

$$v^{(j)}[i] > 0 \Rightarrow \alpha_j \succeq i$$

and

$$v^{(j)}[\alpha_j] \gg 0,$$

for all $i \in \langle p\rangle$ and $j \in \langle m\rangle$.

We end this section with the following observation which will be useful for our results in the next sections:

LEMMA 2 *Let A be as in* (2.1) *and let i and j be vertices in $\mathcal{R}(A)$.*

(i) *For all $k \geq d(i,j)$, $(Z^{(k)})_{i,j} = 0$.*
(ii) *If, in addition, A is a minus singular M-matrix, then $Z^{(k-1)}[\{i,j\}] > 0$.*

Proof (i) By the result in Friedland and Hershkowitz [4] and Hershkowitz, Rothblum, and Schneider [7] the index of $A\{i,j\}$ does not exceed $d(i,j)$ and so it does not exceed k. Hence, by Lemma 1(ii), $Z^{(k)}[\{i,j\}] = 0$ and the result follows.

(ii) This is a consequence from the resolvent expansion of $A\{i,j\}$ in a sufficiently small punctured neighborhood of 0 considering the fact that $(\epsilon I - A\{i,j\})^{-1} \geq 0$, $\forall \epsilon > 0$. ∎

3. PRINCIPAL COMPONENTS

We mentioned in the introduction that the first approach to the questions of existence of a nonnegative basis to the Perron eigenspace of a nonnegative matrix P and the properties of such a basis were *matrix combinatorial* in the sense that they were based on the Frobenius normal form. This approach was developed by Rothblum [11] and Richman and Schneider [10]. A later approach to these questions, obtained by Hartwig, Neumann, and Rose [3], is analytic in the sense that it utilizes the resolvent expansion, but does not involve the Frobenius normal form. In [3] the authors show that if $Z_{P-\rho(P)I}$ is the eigenprojection on the Perron eigenspace of P, then for $\lambda - \rho(P) > 0$ sufficiently small, the matrix

$$J = (\lambda I - P)^{-1} Z_{P-\rho(P)I} \tag{3.1}$$

is nonnegative and its columns contain a basis for the Perron eigenspace. We now observe that if P is in lower block triangular Frobenius normal form, then since J is a polynomial in P, J is also a block lower triangular matrix whose partitioning conforms with that of P. This suggests that there is a connection between the combinatorial and the analytic approaches to the existence of a nonnegative basis described above. In this section we shall make the connection more precise. It will be convenient for us to work and state our results in terms of the associated minus M-matrix $A = P - \rho(P)I$.

Our first main result is the following:

THEOREM 1 *Let A be a minus M-matrix in Frobenius normal form given by* (2.1) *and set $Z = Z_A$. Suppose $i,j \in \mathcal{R}(A)$. If $d(i,j) = k \geq 1$, then $(Z^{(k-1)})_{i,j} \gg 0$.*

Proof The theorem is proved by induction on number of blocks p in the Frobenius normal form.

Let $p = 1$. Then $i = j = 1$. If 1 is a nonsingular node, then $d(1,1) = 0$ and hence of no interest. If 1 is a singular node, then $d(1,1) = 1$ and, as is well known, $Z^{(0)} \gg 0$. This proves the case $p = 1$.

Assume now that the result is true for all minus M-matrices with fewer than p blocks in their Frobenius normal form and suppose that A is a minus M-matrix with p blocks in its normal form. Let $i,j \in \mathcal{R}(A)$ be such that $d(i,j) = k \geq 1$. We need to distinguish four cases.

CASE 1 i and j are nonsingular vertices.

(a) Since $d(i,j) = k$ there exists a simple path from i to j containing k singular vertices. Let (i,m) be the first arc in this path so that $A_{i,m} > 0$. Then $d(m,j) = k$.

Thus, by our inductive assumption and Lemma 2 applied to $A\{m,j\}$, we have that $(Z^{(k-1)})_{m,j} \gg 0$. Furthermore, again by Lemma 2, since the index of $A\{i,j\}$ is k, $Z^{(k-1)}[\langle i,j\rangle] > 0$ and $Z^{(k)}[\langle i,j\rangle] = 0$ so that $(Z^{(k)})_{i,j} = 0$. Now as

$$0 = (Z^{(k)})_{i,j} = A_{i,j}(Z^{(k-1)})_{j,j} + \cdots + A_{i,m}(Z^{(k-1)})_{m,j} + \cdots + A_{i,i}(Z^{(k-1)})_{i,j} \quad (3.2)$$

and as every term in this sum except for the last one is nonnegative with at least one of the terms being semipositive, it follows that

$$-A_{i,i}(Z^{(k-1)})_{i,j} > 0. \quad (3.3)$$

But $-A_{i,i}$ is a nonsingular and irreducible M-matrix and so its inverse is strictly positive. It follows that $(Z^{(k-1)})_{i,j} > 0$ and that each of its columns is either strictly positive or zero.

(b) Let (l,j) be the last arc in a path from i to j which contains k singular vertices. As in part (a) we obtain that $A_{l,j} > 0$ and it follows by our inductive assumption that $(Z^{(k-1)})_{i,l} \gg 0$. As before, from

$$0 = (Z^{(k)})_{i,j} = (Z^{(k-1)})_{i,j}A_{j,j} + \cdots + (Z_{i,l}^{(k-1)})A_{l,j} + \cdots + (Z^{(k-1)})_{i,i}A_{i,j}, \quad (3.4)$$

it follows that

$$-(Z^{(k-1)})_{i,j}A_{j,j} > 0. \quad (3.5)$$

By an argument similar to the one following (1.3) it follows that $(Z^{(k-1)})_{i,j} > 0$ and that each of its rows is either zero or positive.

Combining the results of (a) and (b) we obtain that $(Z^{(k-1)})_{i,j} \gg 0$.

CASE 2 i is a nonsingular vertex and j is a singular one.

(a) As in Case 1 we choose a simple path from i to j containing k singular vertices. By the argument of Case 1(a), which did not use the nonsingularity of the vertex j we prove that $(Z^{(k)})_{i,j} > 0$ and each of its columns is either positive or zero.

(b) Choose l as in Case 1(b). Since $d(i,l) = k-1$ it follows from our inductive assumptions and Lemma 2 that $Z^{(k-1)}[\langle i,l\rangle] = 0$ and that $(Z^{(k)})_{i,j} = 0$. Thus $(Z^{(k-1)})_{i,j}(-A_{j,j}) \geq 0$. But then, as $-A_{j,j}$ is a singular and irreducible M-matrix we must have (cf. Berman and Plemmons [1, Theorem 6.4.16]) that

$$(Z^{(k-1)})_{i,j}(-A_{j,j}) = 0. \quad (3.6)$$

Thus each row of $(Z^{(k-1)})_{i,j}$ is a left null vector of the irreducible M-matrix $-A_{j,j}$ and hence is either positive or zero.

Combining the results of (a) and (b) we obtain that $(Z^{(k-1)})_{i,j} \gg 0$.

CASE 3 i is a singular vertex and j is a nonsingular one.

The proof of this case is similar to the proof of Case 2 with the roles of the vertices i and j reversed.

CASE 4 i and j are singular vertices.

Observe that by Lemma 2 the trailing principal submatrix of $Z^{(k-1)}[\langle i,j\rangle]$ obtained by deleting the first block row and the first block column of $Z^{(k-1)}[\langle i,j\rangle]$ is zero. A similar argument shows that the leading principal submatrix of $Z^{(k-1)}[\langle i,j\rangle]$ obtained by deleting the last block row and the last block column of $Z^{(k-1)}[\langle i,j\rangle]$ is zero. Hence all the blocks of $Z^{(k-1)}[\langle i,j\rangle]$ with the possible exception of $Z_{i,j}^{(k-1)}$ are zero. However, as the index of $A\{i,j\}$ is k, $Z^{(k-1)}[\langle i,j\rangle] \neq 0$, so that by Lemma 2(ii) we must have that $(Z^{(k-1)})_{i,j} > 0$. Repeating the proof of Case 2(b) we obtain that each row of $(Z^{(k-1)})_{i,j}$ is either positive or zero. Similarly, by reversing the roles of i and j in this proof, we obtain that each column of $(Z^{(k-1)})_{i,j}$ is either positive or zero. Together these conclusions give that $(Z_{i,j}^{(k-1)}) \gg 0$. ■

Theorem 1 has the immediate consequence that on choosing $\epsilon = \lambda - \rho(P) > 0$ sufficiently small, the nonnegative basis for $E(A)$, $A = P - \rho(P)I$, which can be extracted from the columns of the matrix J in (3.1) is a strongly combinatorial basis. To see this note that from (2.2) and (3.1) it follows that

$$J = \sum_{k=0}^{\nu(A)-1} \frac{Z^{(k)}}{\epsilon^{k+1}} \tag{3.7}$$

and recall that as A is in lower triangular Frobenius normal form, J itself is lower triangular. The nonzero diagonal blocks of J are positive and occur, precisely, in positions corresponding to the singular blocks of A. The remaining diagonal blocks of J are zero. Next let α_k be a singular vertex in $\mathcal{R}(A)$ and let $i \in \mathcal{R}(A)$, $i \neq \alpha_k$, have access to α_k. Then $d(i,\alpha_k) \geq 1$ and, according to Theorem 1, $(Z^{(d(i,\alpha_k)-1)})_{i,\alpha_k} \gg 0$ while $(Z^{(s)})_{i,\alpha_k} = 0$ for $s \geq d(i,\alpha_k)$. Thus for sufficiently small $\epsilon > 0$ it follows from (3.7) that $J_{i,\alpha_k} \gg 0$. To select a strongly combinatorial basis choose first a sufficiently small $\epsilon > 0$ so that for each singular vertex $\alpha_k \in \mathcal{R}(A)$ and for each $i \in \mathcal{R}(A)$ which has access to α_k, $J_{i,\alpha_k} \gg 0$. Next from each **block column** of J containing a positive diagonal block of J pick a **column** of J. Clearly the m columns thus selected are linearly independent and form a basis for $E(A)$ which satisfies the requirements of (2.3) for being a strongly combinatorial basis.

Continuing, we mention that both Rothblum [11] and Richman and Schneider [10] show that nonnegative bases for $E(A)$ can be selected possessing stronger properties than just (2.3). In particular, Richman and Schneider show that $E(A)$ has a so called *preferred basis* which is a set of nonnegative vectors $u^{(1)},\ldots,u^{(m)}$ which forms a basis for $E(A)$ satisfying (2.3) with the added stipulation that

$$Au^{(k)} = \sum_{j=1}^{m} c_{k,j} u^{(j)},$$

where the $c_{k,j}$'s are scalars satisfying

$$c_{k,j} = \begin{cases} > 0, & \text{if } \alpha_j \succ \alpha_k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Thus, in view of the foregoing paragraph, it should be noted that one can find examples such that, even for $\epsilon > 0$ sufficiently small in (3.7), a basis extracted from the columns of J cannot be a preferred basis as the following example shows: Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

As $\dim(E(A)) = 3$, we know that the three columns of J form a basis for $E(A)$. Let $J^{[i]}$, $i = 1,2,3$, denote the columns of J. Now since $AJ = JA$, we see that

$$AJ^{[1]} = c_{1,1}J^{[1]} + c_{1,2}J^{[2]} + c_{1,3}J^{[3]}$$

with

$$c_{1,1} = 0, \qquad c_{1,2} = 1, \qquad \text{and} \qquad c_{1,3} = 0.$$

We observe therefore that, while $3 \succ 1$, $c_{1,3} \not> 0$ defying the requirements of (3.8). However, we can show that, for sufficiently small $\epsilon > 0$, a basis satisfying a somewhat weaker conditions than being preferred can be chosen.

DEFINITION 2 *Let A be an $n \times n$ minus M-matrix given in the form* (2.1). *A nonnegative basis $u^{(1)},\ldots,u^{(m)}$ for $E(A)$ is said to be* semi-preferred *if it satisfies the requirements of* (2.3) *and if the following conditions hold*:

$$Au^{(k)} = \sum_{j=1}^{m} c_{k,j}u^{(j)}, \qquad k = 1,\ldots,m \tag{3.9}$$

where $c_{k,j} \geq 0$, $k,j = 1,\ldots,m$. Further we have the implication:

$$k \neq j \qquad \text{and} \qquad c_{k,j} > 0 \Rightarrow \alpha_j \succ \alpha_k. \tag{3.10}$$

We now have the following corollary to Theorem 1:

COROLLARY 1 *Suppose that A is a minus M-matrix given in the form* (2.1). *Then for sufficiently small $\epsilon > 0$ a basis can be extracted from the columns of J given in* (3.7) *which is semi-preferred.*

Proof From (2.2), the definition of the $Z^{(k)}$'s in Section 2, and (3.7) we see that $\epsilon > 0$ sufficiently small can be chosen so that

$$AJ = \sum_{j=1}^{\nu(A)-1} \frac{Z^{(k)}}{\epsilon^k} \geq 0.$$

As before let $\alpha_1,\ldots,\alpha_m$ be the singular vertices in $\mathcal{R}(A)$. For each $j = 1,\ldots,m$, choose from the α_jth block column of J a column of J, say it is column μ_j of J. Denote the columns of J so chosen by $J^{[\mu_1]},\ldots,J^{[\mu_m]}$. Suppose now that $c_{k,1},\ldots,c_{k,m}$ are nonnegative numbers such that

$$AJ^{[\mu_k]} = \sum_{j=1}^{m} c_{k,j}J^{[\mu_j]}.$$

Let $q \neq k$ and suppose that $c_{k,q} > 0$. Then we can write that

$$\begin{aligned}
0 &\leq (AJ^{[\mu_k]})[\alpha_q] \\
&= \sum_{j=1}^{m} c_{k,j} J^{[\mu_j]}[\alpha_q] \\
&= \sum_{j=1}^{m} c_{k,j} \sum_{s=0}^{\nu(A)-1} \frac{(Z^{(s)})^{[\mu_j]}[\alpha_q]}{\epsilon^{s+1}} \\
&= \sum_{s=0}^{\nu(A)-1} \sum_{j=1}^{q} c_{k,j} \frac{(Z^{(s)})^{[\mu_j]}[\alpha_q]}{\epsilon^{s+1}} \\
&= \sum_{s=0}^{\nu(A)-1} c_{k,q} \frac{(Z^{(s)})^{[\mu_q]}[\alpha_q]}{\epsilon^{s+1}} + \sum_{s=0}^{\nu(A)-1} \sum_{j=1}^{q-1} c_{k,j} \frac{(Z^{(s)})^{[\mu_j]}[\alpha_q]}{\epsilon^{s+1}} \\
&= c_{k,q} \frac{(Z^{(0)})^{[\mu_q]}[\alpha_q]}{\epsilon} + \sum_{j=1}^{q-1} c_{k,j} \sum_{s=0}^{\nu(A)-1} \frac{(Z^{(s)})^{[\mu_j]}[\alpha_q]}{\epsilon^{s+1}},
\end{aligned}$$

where the last equality follows by Theorem 1 from the fact that $(Z^{(s)})_{\alpha_k,\alpha_k} = 0$ when $s \geq 1$ since $d(\alpha_k, \alpha_k) = 1$. Now let $j \in \langle q-1 \rangle$ and consider the inner summation sign in the second term of the expression immediately above. If α_j is a vertex in $\mathcal{R}(A)$ such that $\alpha_q \not\succ \alpha_j$, then all summands under this summation are 0 because $Z^{(s)}_{\alpha_q,\alpha_j} = 0$ for all $s = 0, \ldots, \nu(A)-1$. On the other hand, if $\alpha_q \succ \alpha_j$, then from Theorem 1 we can deduce that the nonzero coefficient of the lowest exponent in $\epsilon > 0$ is a positive vector for it is a column of a positive submatrix of an appropriate principal component of A. Hence, for each such j we can choose $\epsilon > 0$ sufficiently small so that

$$\sum_{s=0}^{\nu(A)-1} \frac{(Z^{(s)})^{[\mu_j]}[\alpha_q]}{\epsilon^{s+1}} \geq 0.$$

Overall, we can choose $\epsilon > 0$ sufficiently small so that the entire second term in the above expression is nonnegative. As $(Z^{(0)})_{\alpha_q,\alpha_q} \gg 0$, $(Z^{(0)})^{[\mu_q]}[\alpha_q] \gg 0$, showing, in turn, that for some $s \geq 1$, $(Z^{(s)})_{\alpha_q,\alpha_k} \gg 0$. Hence $d(\alpha_q, \alpha_k) \neq 0$ so that α_q has access to α_k. This completes the proof. ∎

We finally comment that Theorem 1 and Corollary 1 strengthen the results of Theorem 1 of Hartwig, Neumann, and Rose [3] which were briefly described in the beginning of the section. There the authors showed that a nonnegative basis for $E(A)$ could be extracted from the columns of J. Theorem 1 and Corollary 1 above now show that such a basis can be chosen which possesses a variety of combinatorial properties.

4. NONNEGATIVE EIGENPROJECTIONS

In this section we shall develop necessary and sufficient conditions for the eigenprojection on the Perron eigenspace of an $n \times n$ nonnegative matrix P to be, itself, a nonnegative matrix. We shall see that the nonnegativity of the eigenprojection is tantamount to the existence of certain "sparse" nonnegative bases for the Perron eigenspaces of P and P^T, respectively. We mention that the appearance of the Perron eigenspace of P^T as well as that of P's is of no surprise since $Z_A \geq 0$ if and only if $Z_{A^T} = (Z_A)^T \geq 0$. Again it will be convenient to state and prove our results in terms of the associated minus M-matrix $A = P - \rho(P)I$.

To begin with recall that A is in lower triangular Frobenius normal form with p diagonal blocks. Consider $\mathcal{R}(A)$. By Lemma 1, for any vertex $i \in \mathcal{R}(A)$, $Z_A[i] = Z_{A\{i\}} = Z_{A[i]}$. Thus, as generally known, but confirmed by our Theorem 1,

$$Z_A[i] = \begin{cases} \gg 0, & \text{if } \ i \text{ is a singular vertex in } \mathcal{R}(A), \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the matrix $Z_A - I$ is a minus M-matrix in (lower triangular) Frobenius normal form with $q \geq p$ diagonal blocks. The singular diagonal blocks of $Z_A - I$ are of same size and occur in same positions as the singular diagonal blocks of A. The nonsingular diagonal blocks of $Z_A - I$, which are now all 1×1, occur at positions which correspond to diagonal entries within nonsingular diagonal blocks of A. The implications of these facts can be expressed relative to the reduced graphs of A and $Z_A - I$. For that purpose, as before, denote by $\alpha_1, \ldots, \alpha_m$ the singular vertices of $\mathcal{R}(A)$ and now introduce $\beta_1, \ldots, \beta_m$ to denote the singular vertices of $\mathcal{R}(Z_A - I)$. Then as subsets of indices from $\langle n \rangle$, $\alpha_i = \beta_i$, $i = 1, \ldots, m$. Furthermore, as subsets of indices from $\langle n \rangle$, every nonsingular vertex of $\mathcal{R}(Z_A - I)$ is contained in some nonsingular vertex of $\mathcal{R}(A)$.

Next we find a simple lemma which gives a necessary condition for Z_A to be nonnegative.

LEMMA 3 *Let A be an $n \times n$ minus M-matrix. Then a necessary condition for $Z = Z_A$ to be nonnegative is that in $\mathcal{R}(A)$ no nonsingular vertex lies on the interior of a simple path connecting two singular vertices.*

Proof Suppose that the statement is false. Then there exist vertices $i, j \in \mathcal{S}(A)$ and $k \notin \mathcal{S}(A)$ such that

$$i \succ k \succ j$$

and such that

$$d(i,k) = d(k,j) = 1.$$

But then, by Theorem 1,

$$Z_{i,k} \gg 0 \qquad \text{and} \qquad Z_{k,j} \gg 0.$$

This means, according to the explanation preceding the lemma, that in $\mathcal{R}(Z - I)$ there is a path of length at least 2, contradicting the fact that Z is a nonnegative matrix whose Perron root has index 1. ∎

We are now ready to state the main result of this section.

THEOREM 2 *Let A be an $n \times n$ minus M-matrix. Then necessary and sufficient conditions for $Z = Z_A$ to be nonnegative are:*

(i) *In $\mathcal{R}(A)$ no nonsingular vertex lies on the interior of a simple path connecting two singular vertices.*

(ii) *$E(A)$ and $E(A^T)$ have proper nonnegative column and row combinatorial bases $u^{(1)},\ldots,u^{(m)}$ and $v^{(1)},\ldots,v^{(m)}$, respectively, which, for $i \in \langle p \rangle$ and $j \in \langle m \rangle$, possess the following properties:*
   (a) *$u^{(j)}[i] > 0 \Rightarrow i = \alpha_j$ or i is a nonsingular vertex.*
   (b) *$v^{(j)}[i] > 0 \Rightarrow i = \alpha_j$ or i is a nonsingular vertex.*

Proof We begin by proving the necessity part. Suppose therefore that $Z \geq 0$. Then (i) holds by Lemma 3. Consider now the minus M-matrix $Z - I$. Because Z is a projection, we know that $\nu(Z - I) \leq 1$. The Rothblum index theorem now implies that the length of any simple path in $\mathcal{R}(Z - I)$ is at most 1. Hence for any singular vertex $\alpha_j \in \mathcal{R}(Z - I)$, a vertex $i \in \mathcal{R}(Z - I)$ satisfies that $d(i,\alpha_j) > 0$ if and only if $i = \alpha_j$ or i is a nonsingular vertex. Thus, by (2.3), $E(Z - I)$ has a nonnegative proper column combinatorial basis $u^{(1)},\ldots,u^{(m)}$ satisfying the requirements in (a) with respect to $\mathcal{R}(Z - I)$. From $E(A) = E(Z - I)$ and from the relation between $\mathcal{R}(Z - I)$ and $\mathcal{R}(A)$ observed above, it now follows that $u^{(1)},\ldots,u^{(m)}$ must (also) be a nonnegative proper combinatorial basis for $E(A)$ satisfying the requirements of (a) with respect to $\mathcal{R}(A)$. Next using the fact that $Z_{A^T} = Z^T \geq 0$, we can follow arguments of a similar spirit to those used in showing (a), to exhibit that $E(A^T)$ has a nonnegative proper row combinatorial basis $v^{(1)},\ldots,v^{(m)}$ satisfying the requirements of (b).

We come now to the proof of the sufficiency of conditions (i) and (ii). Let $u^{(1)},\ldots,u^{(m)}$ and $v^{(1)},\ldots,v^{(m)}$ be nonnegative proper column and row combinatorial bases for $E(A)$ and $E(A^T)$ satisfying, respectively, the requirements of (a) and (b). First note that for any $i \neq \alpha_j, \alpha_k$, where $i \in \langle p \rangle$ and $j,k \in \langle m \rangle$, the subvectors $u^{(j)}[i]$ and $v^{(k)}[i]$ **cannot** be concurrently nonzero vectors. For if that were possible, then, by (ii), i is must be a nonsingular vertex such that, in $\mathcal{R}(A)$, $\alpha_k \succ i \succ \alpha_j$, which is not compatible with condition (i). This shows that the sets of vectors $\{u^{(1)},\ldots,u^{(m)}\}$ and $\{v^{(1)},\ldots,v^{(m)}\}$ are bi-orthogonal. In particular, we can a priori scale the vectors in these sets so that they become bi-orthonormal, that is,

$$(v^k)^T u^{(l)} = \delta_{k,l}, \qquad k,l \in \langle m \rangle. \tag{4.1}$$

Define the matrix

$$Y = \sum_{j=1}^{m} u^{(j)}(v^{(j)})^T. \tag{4.2}$$

Then, as can be readily ascertained using (4.1), $Y \geq 0$, $Y = Y^2$, and the columnspace of Y equals, precisely, $E(A)$. Moreover, as $(v^{(j)})^T = (v^{(j)})^T Z$ for all $j \in \langle m \rangle$, we see that $Yw = 0$ for any generalized eigenvector of A corresponding to an eigenvalue other than 0. Whence $Z = Y$. This completes the proof. ∎

Our theorem shows that for Z_A to be nonnegative it is necessary that $E(A)$ and $E(A^T)$ have, respectively, nonnegative column and row combinatorial bases satisfying its requirements. Therefore, for interest's sake we provide here an example of a minus M-matrix A such that $E(A)$ has a nonnegative proper column combinatorial basis satisfying conditions (ii)(a) of the theorem, but $E(A^T)$ does not have a nonnegative proper combinatorial row basis satisfying conditions (ii)(b) and so, as expected, Z_A is not nonnegative. Consider therefore

$$A_1 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here

$$\nu(A_1) = 2 \qquad \text{and} \qquad E(A_1) = \operatorname{span}\{(1 \quad 1 \quad 0)^T, (0 \quad 0 \quad 1)^T\},$$

whereas

$$E(A_1^T) = \operatorname{span}\{(1 \quad 1 \quad 0)^T, (-1/2 \quad 0 \quad 1)^T\}.$$

A computation now shows that

$$Z_{A_1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ -1/4 & 1/4 & 1 \end{pmatrix}.$$

We next comment that Theorem 2 also illustrates that the conditions for a minus M-matrix A to possess a nonnegative Perron eigenprojection are not purely combinatorial in the sense that two minus M-matrices can have the same reduced graph, yet one possesses a nonnegative Perron eigenprojection while the other one does not. To see this contrast A_1 above with the matrix

$$A_2 = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then both A_1 and A_2 have the same reduced graph, but whereas Z_{A_1} is not nonnegative, we find that

$$Z_{A_2} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Additionally we mention that if A is a minus M-matrix for which Z is nonnegative and in $\mathcal{R}(A)$ there is a vertex $i \notin \mathcal{S}(A)$ with access to a vertex $\alpha_j \in \mathcal{S}(A)$, then our characterization in the above theorem of the special basis which $E(A)$ must possess **does not**, a priori, tell us whether the ith subvector of the jth basis vector, namely, $u^{(j)}[i]$ is zero or not. That depends on whether Z_{i,α_j} is zero or not. Consider the following examples:

$$A_3 = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & -2 & 1 \\ 1 & 1 & 0 & 1 & 1 & -2 \end{pmatrix}$$

and

$$A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Observe that in $\mathcal{R}(A_3)$, $d(3,1) = 2$, and that in $\mathcal{R}(A_4)$, $d(3,1) = 2$. However, a computation shows that

$$Z_{A_3} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

and

$$Z_{A_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1 & 0 \end{pmatrix}.$$

Thus we see that in $\mathcal{R}(Z_{A_3} - I)$, $4 \not\succ 1$ and $3 \not\succ 1$, while in $\mathcal{R}(Z_{A_4} - I)$, $3 \succ 1$. This means that in the basis for $E(A_3)$ provided for by Theorem 2, $u^{(1)}[3] = 0$, whereas in the basis for $E(A_4)$ provided by the theorem, $u^{(1)}[3] \gg 0$.

We finally remark that in the introduction we described the nonnegative basis for $E(A)$ which exists under the conditions of Theorem 2 as "sparse". This is because when contrasted with other nonnegative bases which exist for $E(A)$, such as the strongly combinatorial basis given in (2.3), we see that, at least, wherever a subvector of a vector in the latter basis is positive because of an access from a singular class other than the one whose index equals to the index of the basis vector, the corresponding subvector of the basis vector with same index provided under the conditions of the theorem is 0.

5. ASYMPTOTIC BEHAVIOR OF POWERS OF A NONNEGATIVE MATRIX

In this section we apply our Theorem 1 to obtain a simple proof of Theorem 5.10 of [5] (see also Theorem 9.8 of [13].) The proof given in [5] depends on analytical results concerning convergent and summable series. We shall work with an $n \times n$ nonnegative matrix P which we shall assume to be, without loss of generality, in block lower triangular Frobenius normal form, viz.,

$$P = \begin{pmatrix} P_{1,1} & 0 & \cdots & 0 \\ P_{2,1} & P_{2,2} & & 0 \\ \vdots & & \ddots & \vdots \\ P_{p,1} & \cdots & \cdots & P_{p,p} \end{pmatrix}, \tag{5.1}$$

where the diagonal blocks are irreducible or the 1×1 null matrix.

We shall use here the terminology and notations of [13, Section 9] with the exceptions as noted below. We call a vertex i of $\mathcal{R}(P)$ a *λ-vertex* if λ is an eigenvalue of $P_{i,i}$. Now let i and j be vertices of $\mathcal{R}(P)$. We denote the spectral radius of $P\{i,j\}$ by $s(i,j)$ and we put $A = P\{i,j\} - s(i,j)I$. The *local distance* $\delta(i,j)$ is defined to be the (singular) distance $d(i,j)$ in $\mathcal{R}(A)$. In other words, $\delta(i,j)$ is the maximal number of $s(i,j)$-vertices in any path from i to j in $\mathcal{R}(P)$. (Note that $\delta(i,j)$ as defined in this paper equals $d(i,j)-1$ as defined in [13] or [5].) **For convenience we shall let** $Q = P\{i,j\}$. For an eigenvalue λ of Q we shall denote by $Z_\lambda^{(k)}(Q)$, $0 \le k \le \nu_\lambda(Q) - 1$, the kth *principal component of Q corresponding to* λ, where $\nu_\lambda(Q)$ is the index of λ as an eigenvalue of Q, that is, its multiplicity in the minimal polynomial. (Recall, c.f., [9, p. 314] that $Z_\lambda^{(k)}(Q) = A^k Z_\lambda^{(0)}(Q)$, $0 \le k \le \nu_\lambda(Q) - 1$, where $Z_\lambda^{(0)}(Q)$ is the eigenprojection of Q corresponding to λ.)

If P is an irreducible nonnegative matrix, the *cycle index* $c(P)$ of P is the greatest common divisor (g.c.d.) of the lengths of all simple cycles in the directed graph of P (which is defined in [13] and many other papers.) It is well known that $c(P)$ equals the number of eigenvalues of P on the spectral circle of P. Now suppose again that P is possibly a reducible nonnegative matrix in Frobenius normal form given in (5.1). Let i and j be vertices in $\mathcal{R}(P)$. We now define the quantity $g(i,j)$. If $s(i,j) = 0$, we put $g(i,j) = 1$. Suppose that $s(i,j) > 0$. Let $\Pi(i,j)$ be the set of all paths from i to j in $\mathcal{R}(P)$ such that each path contains $\delta(i,j)$ vertices that are $s(i,j)$-vertices. Then for each path η in $\Pi(i,j)$, let $g(\eta)$ be the g.c.d. of all cycle indices $c(P_{k,k})$, where k ranges over all $s(i,j)$-vertices of η. In this case we let $g(i,j)$ be the least common multiple of all $g(\eta)$ for paths $\eta \in \Pi(i,j)$. Suppose $s = s(i,j) > 0$ and put $g = g(i,j)$. Define now the *smoothing matrix*

$$M(i,j) = p(P),$$

where $p(z)$ is the polynomial

$$p(z) = (1 + z/s + \cdots + (z/s)^{g-1})/g. \tag{5.2}$$

Thus if $s = 1$, the smoothing matrix $M(i,j)$ as defined in this paper equals $M(i,j)/g$ as defined in [13]. (We take here the opportunity to point out two misstated definitions in [13]. Definition (9.7) of [13] is incorrect, P there should be replaced by P/s. Also, the definition of $g(i,j)$ given in [13] has an omission.)

We first prove the following lemma:

LEMMA 4 *Let P be a nonnegative matrix given in* (5.1). *Let $i,j \in \langle p \rangle$ and let $Q = P\{i,j\}$. Suppose that $\delta = \delta(i,j)$, $s = s(i,j)$, and $g = g(i,j)$. Suppose that $\lambda = s\omega \in \sigma(Q)$, where $|\omega| = 1$. Then*:

(i) $\nu_\lambda(Q) \leq \delta = \nu_s(Q)$

and

(ii) *if* $\nu_\lambda(Q) = \nu_s(Q)$, *then* $\omega^g = 1$.

Remark We comment that the inequality $\nu\lambda(Q) \leq \nu_s(Q)$ appears in Schaeffer [12, Appendix 2, Statement 2.4].

Proof (i) Let ζ be a path from i to j in $\mathcal{R}(P)$ which contains a maximal number of λ-vertices. Let q and p be, respectively, the number of λ-vertices and s-vertices on ζ. We shall show that

$$\nu_\lambda(Q) \leq q \leq p \leq \delta = \nu_s(Q).$$

The first of these inequalities is immediate from Theorem (5.9) of [7], see also [4]. To prove the second inequality, we note that it follows from the Perron–Frobenius theory applied to the irreducible blocks of Q that every λ-vertex of $\mathcal{R}(Q)$ is also an s-vertex since $s = \rho(Q)$. The third inequality is immediate, since by definition δ is the maximal number of s-vertices on any path from i to j in $\mathcal{R}(Q)$. The last equality follows from the Rothblum index theorem.

(ii) By the Perron–Frobenius theory, ω is a root of unity, say it is the kth root of 1. Since $p = q$, every s-vertex on ζ is also a λ-vertex. Hence (by Perron–Frobenius) the cycle index of every vertex on ζ is a multiple of k. It follows that the g.c.d. of the cycle indices of every s-vertex on ζ is a multiple of k. But, since $p = q$, we have that $\zeta \in \Pi(i,j)$ (which was defined previously in this section). Hence, by the definition of g, it follows that g is a multiple of k, and we are done. ■

We are now ready to use the results of Section 3 to re-prove Theorem 5.10 of [5].

THEOREM 3 (Friedland and Schneider [5, Theorem 5.10]) *Let P be an $n \times n$ nonnegative matrix given in* (5.1). *Let $i,j \in \langle p \rangle$ and let $Q = P\{i,j\}$. Set $\delta = \delta(i,j)$, $s = s(i,j)$, $g = g(i,j)$, and let $M = M(i,j)$. Suppose that i has access to j in $\mathcal{R}(A)$. Then $(Z^{(\delta-1)}(Q))_{i,j} \gg 0$ and if $s > 0$, then*

$$(MP^m)_{i,j} = \frac{m^{\delta-1}s^{m-\delta+1}(Z_s^{(\delta-1)}(Q))_{i,j}}{(\delta-1)!} + o(m^{\delta-1}s^{m-\delta+1}) \tag{5.3}$$

as $m \to \infty$.

Remark Note that the matrices Q, $Z(Q)$, and N used in the statement and proof are block matrices with fewer blocks in their block dimension than P. We shall retain the block row and columns indexing of P. Thus, for example, $Q_{j,j}$ is the upper left hand block of Q and $Q_{i,i}$ is the lower right hand block of Q. In particular, $(Z_s^{(\delta-1)}(Q))_{i,j} = (Z_s^{(\delta-1)}(P))_{i,j}$.

Proof By Theorem 1, $(Z_s^{(\delta-1)}(Q))_{i,j} \gg 0$. Suppose that $s > 0$. We first assume that $s = 1 = \rho(P)$. Let $N = p(Q)$, where $p(z)$ is given in (5.2). By Lemma 1, $(MP^m)_{i,j} = (NQ^m)_{i,j} = (p_m(Q))_{i,j}$, where, since $s = 1$,

$$p_m(z) = z^m p(z) = (z^m + z^{m+1} + \cdots + z^{m+g-1})/g, \qquad m = 0, 1, \ldots .$$

It is well known (cf. Lancaster and Tismenetski [9, p. 314]) that

$$NQ^m = \sum_{\lambda \in \sigma(Q)} \sum_{r=0}^{\nu_\lambda(Q)-1} \frac{p_m^{(r)}(\lambda)}{r!} Z_\lambda^{(r)}(Q). \tag{5.4}$$

Since

$$\frac{p_m^{(r)}(\lambda)}{r!} = \frac{1}{g}\left[\binom{m}{r}\lambda^{m-r} + \binom{m+1}{r}\lambda^{m+1-r} + \cdots + \binom{m+g-1}{r}\lambda^{m-r+g-1}\right] \tag{5.5}$$

it follows that

$$NQ^m = \sum_{\lambda \in \sigma(Q),\ |\lambda|=1} \sum_{r=0}^{\nu_\lambda(Q)-1} \frac{p_m^{(r)}(\lambda)}{r!} Z_\lambda^{(r)}(Q) + o(1). \tag{5.6}$$

Since by Lemma 4(i), we have that $\nu_\lambda(Q) \leq \delta$ when $|\lambda| = 1$, it follows from (5.5), (5.6), and Lemma 4 that

$$NQ^m = \sum_{\lambda \in \sigma(Q),\ |\lambda|=1} \frac{(1 + \lambda + \cdots + \lambda^{g-1}) m^{\delta-1} \lambda^{m-\delta+1} Z_\lambda^{(\delta-1)}(Q)}{g(\delta-1)!} + o(m^{\delta-1}). \tag{5.7}$$

Suppose that $\lambda \in \sigma(Q)$, $|\lambda| = 1$, but $\lambda \neq 1$. Then either $\lambda^g = 1$, in which case $1 + \lambda + \cdots + \lambda^{g-1} = 0$, or $(Z_\lambda^{(\delta-1)}(Q))_{i,j} = 0$ by Lemma 4. We now obtain that

$$(NQ^m)_{i,j} = \frac{m^{d-1}(Z_1^{(\delta-1)}(Q))_{i,j}}{(\delta-1)!} + o(m^{\delta-1}). \tag{5.8}$$

To obtain the general case of $s > 0$ from (5.8), we note that $p_m(sQ) = s^m p_m(Q)$ and that $Z_s^{\delta-1}(sQ) = s^{\delta-1} Z_1^{(\delta-1)}(Q)$. ■

We observe that if $s(i,j) = 0$ for a nonnegative matrix P given by (5.1), then $P\{i,j\}$ is strictly lower triangular and hence nilpotent. Thus $(P^m)_{i,j} = 0$ for sufficiently large m.

6. AN OPEN PROBLEM

We wish to close this paper with a question. For $\epsilon > 0$ define the *trailing sums* of J given in (3.7) by

$$J^{(k)}(\epsilon) = \frac{Z^{(k)}}{\epsilon^{k+1}} + \cdots + \frac{Z^{(\nu(A)-1)}}{\epsilon^{\nu(A)}}, \qquad k = 0, \ldots, \nu(A) - 1.$$

Note that in this notation J of (3.7) is now $J^{(0)}(\epsilon)$. We can prove the following observation: *Let* $0 \leq k \leq \nu(A) - 1$. *Then* $Z^{(k)} \geq 0$ *if and only if* $J^{(k)}(\epsilon) \geq 0$ **for all** $\epsilon > 0$. For the case $k = 0$ this provides us with an alternative characterization for the nonnegativity of the eigenprojection $Z_A = Z^{(0)}$ to that which we developed in Section 4. The difference between the two characterizations is that the characterization of Section 4 gives us an idea of what kind of a nonnegative bases must the Perron eigenspace $E(A)$ possess in order for $Z_A \geq 0$, whereas the characterization mentioned above is one that relies entirely on asymptotic expansions. We therefore raise here the question of characterizing the nonnegativeness of $Z^{(k)}$, $k = 1, \ldots, \nu(A) - 1$, more in terms of properties which certain bases for $E(A)$ must have.

ACKNOWLEDGMENTS

The authors would like to thank Professor Ludwig Elsner for reading the first written version of this manuscript and for making various suggestions to improve its presentation. We also thank the referee for his/hers helpful comments.

References

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
[2] C. D. H. Cooper, On the maximum eigenvalue of a reducible nonnegative matrix, *Math. Z.* **131** (1973), 213–217.
[3] R. E. Hartwig, M. Neumann and N. J. Rose, An algebraic-analytic approach to nonnegative basis, *Lin. Alg. Appl.* **133** (1990), 77–88.
[4] S. Friedland and D. Hershkowitz, The rank of powers of matrices in a block triangular form, *Lin. Alg. Appl.* **107** (1988), 17–22.
[5] S. Friedland and H. Schneider, The growth of powers of a nonnegative matrix, *SIAM J. Algebraic and Discrete Methods* **2** (1980), 25–34.
[6] D. Hershkowitz and H. Schneider, Applications of combinatorial bases and derived Jordan sets in the study of the equality of the height and level characteristics of an M-matrix, *Lin. Multilin. Alg.* **29** (1991), 21–42.
[7] D. Hershkowitz, U. Rothblum and H. Schneider, The combinatorial structure of the generalized nullspace of a block triangular matrix, *Lin. Alg. Appl.* **116** (1989), 9–26.
[8] T. Kato, *Perturbation Theory for Linear Operators*, 2nd Edition. Springer-Verlag, New York, 1984.
[9] P. Lancaster and M. Tismentsky, *The Theory of Matrices*, 2nd Edition. Academic Press, New York, 1985.
[10] D. Richman and H. Schneider, On the singular graph and Weyr characteristic of an M-matrix, *Aequations Math.* **17** (1978), 208–234.
[11] U. G. Rothblum, Algebraic eigenspaces for nonnegative matrices, *Lin. Alg. Appl.* **12** (1975), 281–292.
[12] H. H. Schaeffer, *Topological Vector Spaces*. Springer-Verlag, New York, 1980.
[13] H. Schneider, The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and related properties: A survey, *Lin. Alg. Appl.* **84** (1986), 161–189.