Characterizations of max-balanced flows

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Abstract

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Let $G = (V, A)$ be a *graph* with *vertex set* V and *arc set* A. A *flow for* G is an arbitrary real-valued function defined on the arcs A. A flow f is called *max-balanced* if for every cut W, $\emptyset \neq W \subset V$, the maximum flow over arcs leaving W equals the maximum flow over arcs entering W. We describe ten characterizations of max-balanced flows using properties of graph contractions, maximum cycle means, flow maxima, level sets of flows, cycle covers, and minimality with respect to order structure in the set of flows derived from a given flow by reweighting. We also give a linear programming based proof for an existence result of Schneider and Schneider.

1. Introduction

Let $G = (V, A)$ be a strongly connected directed graph, and let f be an arbitrary real-valued function defined on the arcs A (we will refer to f as a *flow* for G). Then f is called *max-balanced* if for every cut W, the maximum flow over arcs leaving W equals the maximum flow over arcs entering W. A max-balanced flow is a max-analogue of a *circulation* in which the summation operators are replaced by maximization operators.

We describe ten characterizations of max-balanced flows. First, we prove some elementary characterizations of max-balanced flows using properties of graph contractions and maximum cycle means. Next, we prove a useful characterization using a notion of the *level sets of f*. We then apply this characterization to prove a result of Schneider and Schneider [20] showing that f is max-balanced if and only if G has an f-cycle cover. Then we define a partial order on the set of flows for G and show that f is max-balanced if and only if it is the least element in the set of all flows derived from f by reweighting. Finally, we prove an analogous result for functions defined on the set of all cuts of G.

Our characterizations of max-balanced flows have equivalent formulations in terms of matrices. Under a standard correspondence between flows and square nonnegative matrices, max-balanced flows correspond to square matrices with the property that the row maxima equal the corresponding column maxima. The operation of reweighting a flow corresponds (via an exponentiation transformation) to diagonal equivalence scaling of a square nonnegative matrix (see [21, Section 8]). In particular, some of the characterizations we obtain produce interesting results for the matrix formulation of the problem.

Max-balanced flows have been studied by Schneider and Schneider [19–21]. See also Hartmann and Schneider [10] for a discussion of max-balanced flows satisfying lower and upper bounds, Rothblum et al. [16] for a discussion of a related algebraic matrix scaling problem, and Young et al. [23] for a discussion of efficient algorithms for max-balancing. Related algebraic generalizations of network flow and linear programming problems have been considered by Hoffman [11], Cunningham-Green [3], Hamacher [6–9], and Zimmerman [24,25]. See also the survey paper by Burkard and Zimmermann [2] and the collection of papers in [1].

We were originally motivated to study max-balanced flows by their connection to matrix scaling problems as described, for example, in [4, 12, 15, 17, 18]. These scaling problems are *sum variants* of the *max-problems* we consider in this paper, and they have numerous applications in economics, finance, statistics, and probability (see [22], and the references therein). As an illustration of this connection, Eaves et al. [4] study the problem of identifying for a given square, nonnegative matrix a diagonal equivalence scaling whose row sums equal its column sums, that is, the l_1 norm of each row must equal the l_1 norm of the corresponding column. They characterize those matrices for which such scalings exist [4, Theorem 2]. Schneider and Schneider in [17, 18] study the problem of identifying for a given flow

a reweighting that is max-balanced, and they show that every flow on a strongly connected graph has such a reweighting [18, Theorem 6]. The problem studied in [19,21] is the l_∞ analogue of the l_1 problem studied in [4]. The l_p analogue for $1 < p < \infty$ can be reduced to the case of $p = 1$, whereas no such reduction seems possible for the case of $p = \infty$. These connections are described in greater detail in [16, Section 8] and [18, Section 2]. In this paper, we continue our study of max-balanced flows initiated in [19-21].

We now summarize our paper. In Section 2 we define the graph-theoretic properties we will use. In Section 3 using the *level sets* of a function, we consider a partial order $\leq$ and a linear pre-order $\leq_L$ on the set of all functions defined on a finite set. In Sections 4 and 5 we give our characterizations of max-balanced flows. In Theorem 6 we prove four elementary characterizations using properties of graph contractions and maximum cycle means. In Theorem 7 we prove that f is max-balanced if and only if its level sets have *isolated strong components*, or, equivalently, if and only if G has an *f-cycle cover*. In Theorem 8 we show that f is max-balanced if and only if it is the least element with respect to the partial order $\leq$ in the set of all flows obtained from f by reweighting. Finally, in Theorem 11 we define the *cut function* induced by a flow f and prove an analogous least-element characterization of max-balanced flows using the cut function.

In Appendix A, we present a linear programming based proof of the existence result of Schneider and Schneider (Theorem 3). Our characterizations in Theorems 8 and 11 can be proved using that result or independently using a simple descent argument. Since the proof of Theorem 3 in Appendix A uses only the characterizations of Theorem 7, there is no circularity in the application of Theorem 3. Consequently, this paper can be read independently of papers [19,21].

2. Notation and preliminaries

Let $G = (V, A)$ be a (directed) graph with *vertex set* V and *arc set* $A \subseteq V \times V$. We use the notation $a = (u, v)$ to denote an arc $a \in A$ directed from vertex u to vertex v. We use the symbols $\subset$ and $\subseteq$ to denote, respectively, strict and weak containment. A subset W of V is called *nontrivial* if $\emptyset \neq W \subset V$. A *cut* of G is a nontrivial subset of the vertices. For a cut W of G, we define the *set of arcs leaving and entering* W, written $\delta^+(W; G)$ and $\delta^-(W; G)$, respectively, by

$$\delta^+(W; G) = \{a = (u, v) \in A \mid u \in W, \text{ and } v \in V \setminus W\},$$

and

$$\delta^-(W; G) = \{a = (u, v) \in A \mid u \in V \setminus W, \text{ and } v \in W\}.$$

When there is no possibility of confusion, we will omit the dependence on G.

A *flow* for the graph G is an arbitrary real-valued function f defined on the arcs A. We will use f_a for $a \in A$ to denote the flow of arc a. For a cut W of G, we say

that the flow f is *max-balanced at W* if

$$\max_{a \in \delta^-(W)} f_a = \max_{a \in \delta^-(W)} f_a. \tag{1}$$

We say that f is *max-balanced* if it is max-balanced at every cut W.

A *potential* for the graph G is a real-valued function p defined on the vertex set V. A potential p is called *trivial* if for some constant α, $p_v = \alpha$ for all $v \in V$; otherwise p is called *nontrivial*. For a flow f and a potential p for G, we define the *p-reweighted flow of f* to be the flow f^p defined by

$$(f^p)_a = p_u + f_a - p_v \quad \text{for } a = (u, v) \in A. \tag{2}$$

When there is no possibility of confusion, we will use f_a^p to denote the flow of f^p on arc a. We note that the operation of reweighting flows via potentials arises throughout network optimization (see, for example [14]).

Our formulation assumes that the underlying graph G has no *parallel arcs* (i.e., two arcs directed from u to v for some pair of vertices u and v) and no loops (i.e., arcs of the form $a = (v, v)$ for some vertex v). We note, however, that all of our results extend easily to handle the more general situation. Also, we will assume throughout that $V \neq \emptyset$ and $A \neq \emptyset$. For a finite set S, we will use the notation $|S|$ to denote the number of elements of S. For example, $|V|$ and $|A|$ denote, respectively, the number of vertices and arcs of G. We will use $\mathbb{R}^S$ to denote the set of all real-valued functions with domain S.

Let u and v be vertices of G. A *(directed) path from u to v* is a sequence of the form $P = (v_0, a_1, v_1, \ldots, a_k, v_k)$, such that $v_0 = u$, $v_k = v$, and $a_i = (v_{i-1}, v_i)$ for $i = 1, 2, \ldots, k$. The path P is said to *start* and *end* at the vertices u and v, respectively. We will identify a path containing at least one arc with its underlying arc set. In particular, the *length* of a path P, written $|P|$, is defined to be the number of arcs in the sequence P. (Note that the sequence (v) is a path starting and ending at v of length 0.) A *(simple) cycle* is a path containing an arc that starts and ends at the same vertex and contains no other repeated vertices. Vertices $u, v \in V$ (which need not be distinct) are called *connected* if there exists a path from u to v and a path from v to u.

Let $G = (V, A)$ be a graph, and let W be a subset of the vertices V. We define the *subgraph of G induced by W*, written $G(W)$, to be the graph (W, E) where E is the set of all arcs $a = (u, v) \in A$ such that $u, v \in W$. The relation connectedness is an equivalence relation on V, which therefore induces a partition $\{V_1, V_2, \ldots, V_m\}$ of the vertices. The resulting induced graphs $G(V_1), G(V_2), \ldots, G(V_m)$ are the *strong components* of G. The graph G is called *strongly connected* if it has exactly one strong component. Also, we say that G has *isolated strong components* if every arc a of A is contained in a strong component of G.

In the following lemma, we state (without proof) two elementary characterizations of graphs with isolated strong components.

Lemma 1. *Let $G = (V, A)$ be a graph. Then the following are equivalent*:
(i) *The graph G has isolated strong components*;
(ii) *for every cut W of G, $\delta^+(W)$ is nonempty if and only if $\delta^-(W)$ is nonempty*;
(iii) *every arc of A lies on a cycle.*

Let $G = (V, A)$ be a graph. For notational convenience, we will identify a nonempty subset $E \subseteq A$ with the graph (V, E). In particular, we will say that E has *isolated strong components* if the graph (V, E) has isolated strong components. Similarly, we will refer to the strong components of (V, E) as the strong components of the set E.

Let $G = (V, A)$ be a graph, and let f be a flow for G. For a nonempty subset E of A, we define the *flow of E*, written $f(E)$, by

$$f(E) = \sum_{a \in E} f_a,$$

and the *mean flow of E*, written $\bar{f}(E)$, by

$$\bar{f}(E) = \frac{1}{|E|} \sum_{a \in E} f_a.$$

In particular, we apply these definitions to a cycle C by applying them to the set of arcs of C. We define the *maximum cycle mean* of f, written $\mathrm{mcm}(f)$, by

$$\mathrm{mcm}(f) = \max\{\bar{f}(C) \mid C \text{ is a cycle of } G\}.$$

A cycle C of G is a *maximum mean cycle* for G if

$$\bar{f}(C) = \mathrm{mcm}(f).$$

We observe that for each potential p for G, we have $f(C) = f^p(C)$ for every cycle C for G, implying that $\mathrm{mcm}(f) = \mathrm{mcm}(f^p)$. Next, we characterize potentials p of strongly connected graphs for which $f = f^p$.

Lemma 2. *Let G be a strongly connected graph, and let f and p be, respectively, a flow and a potential for G. Then $f = f^p$ if and only if p is trivial.*

Proof. Clearly, if p is trivial, the $f = f^p$. Conversely, if p is nontrivial, then define

$$W = \left\{ w \in V \mid p_w = \max_{v \in V} p_v \right\}.$$

Since G is strongly connected and $\emptyset \neq W \subset V$, it follows that $\delta^+(W) \neq \emptyset$ and for any $a = (u, v) \in \delta^+(W)$, we have

$$f_a^p = p_u + f_a - p_v > f_a,$$

so $f = f^p$ does not hold. $\square$

The following result was proved constructively by Schneider and Schneider in [19,21]. In Appendix A, we present a proof using linear programming duality

Theorem 3. *Let $G = (V, A)$ be a strongly connected graph, and let f be a flow for G. Then there exists a potential p (unique up to a trivial potential) for G such that f^p is max-balanced.*

Let $G = (V, A)$ be a graph, and let f be a flow for G. A set of cycles $\mathscr{C}$ of G is called and *f-cycle cover for G* if there exists a map from A onto $\mathscr{C}$, where we denote the image of $a \in A$ by C_a, such that for all $a \in A$

(i) $a \in C_a$, and

(ii) $f_a \le f_b$ for every $b \in C_a$.

It follows directly that G has an f-cycle cover if and only if every arc is contained in some cycle of G for which it is the arc with minimum flow. Cycle covers were studied in [20], where it was shown that f is max-balanced if and only if there exists an f-cycle cover for G. We provide an alternative proof of this result in Theorem 7. This is an instance of a more general decomposition theory for matroid flows (see [9, Theorem 2.26]).

Next, we define the operation of contraction of a graph with respect to a partition of the vertices. Let $G = (V, A)$ be a graph, and let Π be a partition of the vertex set V. We define the *contr   ion of G with respect to Π*, written G/Π, to be the graph (Π, A') where

$$A' = \{(I, J) \in \quad \Pi \mid \exists (u, v) \in A \text{ with } u \in I \text{ and } v \in J\}. \tag{3}$$

It is easy to see that the operation of contraction preserves strong connectivity.

For a flow f for G, we define the *contraction of f with respect to Π*, written f/Π, to be the flow for G/Π such that for $a' = (I, J) \in A'$,

$$(f/\Pi)_{a'} = \max\{f_a \mid a = (u, v) \in A, u \in I, \text{ and } v \in J\}. \tag{4}$$

That is, G/Π is derived from G by identifying all vertices of V contained in the same element of Π, deleting arcs between identified vertices, and identifying parallel arcs. The flow f/Π is defined by *max-projecting f* onto the arc set A'. Of course, the definition of A' ensures that the maximum in (4) is taken over a nonempty set.

In the important special case where $\Pi = \{W, V \setminus W\}$ for some cut W, we write G/W and f/W for G/Π and f/Π, respectively.

3. Order relations on sets of functions

In this section we define the partial order used in Theorems 8 and 11 to characterize max-balanced flows. Further, we discuss the relation between our partial order and the usual lexicographic order.

Let S be any finite set. For $f \in \mathbb{R}^S$ and $\alpha \in \mathbb{R}$ we define the *α-level set of f*, written

$\text{lev}_\alpha(f)$, by

$$\text{lev}_\alpha(f) = \{s \in S \mid f_s \geq \alpha\}.$$

We define the *maximum of* f, written $\max(f)$, by

$$\max(f) = \max_{s \in S} f_s.$$

In our applications, the set S will be the set of arcs or the set of cuts of a graph.

Next, we consider two relations on $\mathbb{R}^S$ and one on $\mathbb{R}^{|S|}$ that will be useful for our development. First, we define the relation $<$ on $\mathbb{R}^S$ as follows: For $f, g \in \mathbb{R}^S$, we define $f < g$ if there exists a constant β such that

$$\text{lev}_\alpha(f) = \text{lev}_\alpha(g) \quad \text{for all } \alpha > \beta,$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

$$\text{lev}_\beta(f) \subset \text{lev}_\beta(g).$$

(Recall, $\subset$ denotes strict containment.) Further, we define $f \leq g$ if either $f < g$ or $f = g$. It is easy to see that if $f < g$, then the constant β in (5) is unique. We will denote this constant by $\beta(f, g)$. We define $\beta(f, f) = -\infty$.

We observe that $f \leq f$, and that $f = g$ if and only if $f \leq g$ and $g \leq f$. Further, if $f \leq g$, and $g \leq h$, then $f \leq h$ and $\beta(f, h) = \max\{\beta(f, g), \beta(g, h)\}$. Therefore $\leq$ is a partial order.

In the following lemma, we state a useful property of relation $\leq$ that is needed in the proof of Lemma 10.

Lemma 4. *Let S be a finite set, and let $<$ be defined by* (5). *If $f < g$ and $s \in S$ satisfies $f_s \geq \beta(f, g)$, then $f_s = g_s$.*

Proof. If not, then set $\alpha = \max\{f_s, g_s\}$ and use the definition (5) to derive a contradiction. $\square$

Next, we define the relation $<_L$ on $\mathbb{R}^S$ as follows: For $f, g \in \mathbb{R}^S$ we define $f <_L g$ if there exists a constant β such that

$$|\text{lev}_\alpha(f)| = |\text{lev}_\alpha(g)| \quad \text{for all } \alpha > \beta,$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (6)

$$|\text{lev}_\beta(f)| < |\text{lev}_\beta(g)|.$$

As before, if $f <_L g$, then the constant β in (6) is unique. Further, we define $f <_L g$ if either $f <_L g$ or

$$|\text{lev}_\alpha(f)| = |\text{lev}_\alpha(g)| \quad \text{for all } \alpha \in \mathbb{R}.$$

We define $f \sim_L g$, if $f \leq_L g$ and $g \leq_L f$. (Note that $f \sim_L g$ does not imply that $f = g$.)

It is easy to see that $\leq_L$ is a *linear pre-order*. That is, for $f, g, h \in \mathbb{R}^S$,

(i) $f \leq_L f$,

(ii) if $f \leq_L g$ and $g \leq_L h$, then $f \leq_L h$,

(iii) either $f \leq_L g$ or $g \leq_L f$, and

(iv) we cannot have both $f <_L g$ and $g <_L f$ simultaneously.

Finally, the (usual) *lexicographic order* on $\mathbb{R}^{|S|}$, writen $\leq_{lex}$, is defined as follows: For $x, y \in \mathbb{R}^{|S|}$, $x \leq_{lex} y$ if either $x = y$ or for some positive integer $k \leq n$, $x_i = y_i$ for $i = 1, 2, \ldots, k-1$ and $x_k ._{,k}$. For $f \in \mathbb{R}^S$ we define the *rank vector of f*, written f^{ord}, to be the vector in $\mathbb{R}^{|S|}$ such that there exists an ordering $(s_1, s_2, \ldots, s_{|S|})$ of the elem~r ~ c.: S satisfyin.~

$$f_i^{ord} = f_{s_i}, \quad \text{for } i = 1, 2, \ldots, |S|,$$

and

$$f_1^{ord} \geq f_2^{ord} \geq \cdots \geq f_{|S|}^{ord}.$$

That is, we choose a fixed ordering of the elements of S such that the values of the function f are ordered by decreasing size. Note that $f_1^{ord} = \max(f)$.

In the following lemma, we summarize the relationships between $\leq$, $\leq_L$, and $\leq_{lex}$. The implications follow directly from the respective definitions and from Lemma 4; the details are omitted.

Lemma 5. *Let S be a finite set, and let $\leq$, $\leq_{lex}$ and $\leq_L$ be the orders defined above. Then for $f, g \in \mathbb{R}^S$,*

  (i) *$f <_L g$ if and only if $f^{ord} <_{lex} g^{ord}$,*

  (ii) *$f \sim_L g$ if and only if for some permutation σ of S, $g_s = f_{\sigma(s)}$ for all $s \in S$, and*

  (iii) *if $f < g$, then $f <_L g$.*

Part (i) states that the linear pre-order $<_L$ can be derived by applying the usual lexicographic order to the rank vectors. It follows from part (ii) that the order $\leq_L$ is defined by identifying certain incomparable elements under the order $\leq$. Specifically, the set of elements equivalent to f are precisely those elements that can be derived from f by permuting the function values. Two such elements, if distinct, must be incomparable under the partial order $\leq$. Equivalently, $f \sim_L g$ if the ranges of f and g, including multiplicities, coincide. Part (iii) states that the linear pre-order $<_L$ is compatible with the partial order $<$.

4. Arc characterizations

In this section we prove eight necessary and sufficient conditions for a flow f to be max-balanced using contractions, level sets, cycle covers and the relations $<$ and $<_L$.

Theorem 6. *Let $G = (V, A)$ be a strongly connected graph, and let f be a flow for G. Then the following are equivalent.*

  (i) *f is max-balanced;*

(ii) f/Π *is max-balanced for each partition* Π *of* V;

(iii) f/W *is max-balanced for each cut* W *of* G;

(iv) $\mathrm{mcm}(f/W) = \max(f/W)$ *for each cut* W *of* G; *and*

(v) $\mathrm{mcm}(f/\Pi) = \max(f/\Pi)$ *for each partition* Π *of* V.

Proof. The order of the proof is as follows: First, we show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Then we show that (i) $\Rightarrow$ (v) and that (v) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (ii): Let Π be a partition of V, and let $G' = G/\Pi$ and $f' \quad f/\Pi$. For a cut W' of G', we define the corresponding cut W of G by

$$W = \{v \in V \mid v \in I \text{ for some } I \in W'\}.$$

Then it follows directly from the definition of f' in (4) that

$$\max_{a \in \delta^+(W';G')} f'_a = \max_{a \in \delta^+(W;G)} f_a,$$

and

$$\max_{a \in \delta^-(W';G')} f'_a = \max_{a \in \delta^-(W;G)} f_a.$$

Therefore, if f is max-balanced, then so is f'.

(ii) $\Rightarrow$ (iii): This implication is trivial.

(iii) $\Rightarrow$ (iv): This implication is obvious since G/W contains two vertices and two arcs.

(iv) $\Rightarrow$ (i): Suppose that f is not max-balanced at some cut W. Then the two arcs of G/W form a cycle for which f/W differs. It follows that $\mathrm{mcm}(f/W) \neq \max(f/W)$.

(i) $\Rightarrow$ (v): First, we show that $\mathrm{mcm}(f) = \max(f)$ whenever f is max-balanced. Let $a = (u, v)$ be an arc of G satisfying $f_a = \max(f)$. Since f is max-balanced at the singleton cut $\{v\}$, it follows that there exists an arc leaving v with flow $\max(f)$. Continuing in this fashion, we can construct a cycle C all of whose arcs have flow $\max(f)$. Clearly C is a maximum mean cycle for G, and, therefore, $\mathrm{mcm}(f) = \max(f)$. The implication (i) $\Rightarrow$ (v) now follows for an arbitrary partition Π, since f/Π is max-balanced whenever f is.

(v) $\Rightarrow$ (iv): This implication is trivial. $\quad \square$

Theorem 7. *Let* $G = (V, A)$ *be a strongly connected graph, and let* f *be a flow for* G. *Then the following are equivalent.*

  (i) f *is max-balanced*;

 (ii) *every level set of* f *has isolated strong components*;

(iii) *there exists an* f-*cycle cover for* G.

Proof. (i) $\Rightarrow$ (ii): Suppose that for some real number α, $\mathrm{lev}_\alpha(f)$ does not have isolated strong components. Then by Lemma 1 there exists a cut W such that $\mathrm{lev}_\alpha(f) \cap \delta^+(W) \neq \emptyset$ and $\mathrm{lev}_\alpha(f) \cap \delta^-(W) = \emptyset$. That is,

$$f_a \geq \alpha \quad \text{for some } a \in \delta^+(W),$$

and

$$f_a < \alpha \quad \text{for some } a \in \delta^-(W).$$

Therefore f is not max-balanced at W.

(ii) $\Rightarrow$ (iii): Suppose (ii) holds. For each $a \in A$ consider $\text{lev}_\alpha(f)$ where $\alpha = f_a$. Since $\text{lev}_\alpha(f)$ has isolated strong components, it follows from Lemma 1 that a is contained in some cycle C_a of G such that $C_a \subseteq \text{lev}_\alpha(f)$. Now let $\mathscr{C} = \{C_a \mid a \in A\}$. Then it follows directly from the definition of a level set that $\mathscr{C}$ is an f-cycle cover for G.

(iii) $\Rightarrow$ (i): Let $\mathscr{C}$ be an f-cycle cover for G, and let W be a cut for G. Then for each $a \in \delta^+(W)$ there exists a cycle $C \in \mathscr{C}$ such that $f_a \le f_b$ for all $b \in C$. Since C must also intersect $\delta^-(W)$, it follows that there exists an arc $c \in \delta^-(W)$ such that $f_a \le f_c$. Thus, we have shown that

$$\max_{a \in \delta^+(W)} f_a \le \max_{a \in \delta^-(W)} f_a. \tag{7}$$

A similar argument shows that the reverse inequality in (7) is also satisfied. This proves that f is max-balanced. $\square$

For a cycle C of G, we define the *characteristic function of C* to be the function $\chi^C \in \{\mathbb{R} \cup \{-\infty\}\}^A$ defined by

$$\chi_a^C = \begin{cases} 0, & \text{if } a \in C, \\ -\infty, & \text{if } a \in A \setminus C. \end{cases}$$

We observe that a set of cycles $\mathscr{C}$ is an f-cycle cover for G if and only if there exist real numbers α_C for $C \in \mathscr{C}$ such that

$$f_a = \max_{C \in \mathscr{C}} \{\alpha_C + \chi_a^C\} \quad \text{for all } a \in A. \tag{8}$$

To see this, note that if $\mathscr{C}$ is a set of cycles for G and $a \in A$, then it follows from the definition of χ_C that

$$\max_{C \in \mathscr{C}} \{\alpha_C + \chi_a^C\} = \max\{\alpha_C \mid a \in C \text{ and } C \in \mathscr{C}\}, \tag{9}$$

and therefore it suffices to show that $\mathscr{C}$ is an f-cycle cover if and only if

$$f_a = \max\{\alpha_C \mid a \in C \text{ and } C \in \mathscr{C}\}. \tag{10}$$

Now, suppose that $\mathscr{C}$ is an f-cycle cover for G; define

$$\alpha_C = \min_{a \in C} f_a.$$

Then $\alpha_C \le f_a$ for all $a \in C$ and $C \in \mathscr{C}$, and since $f_a = \alpha_{C_a}$, (10) follows.

Conversely, suppose that for some set of cycles $\mathscr{C}$ for G,

$$f_a = \max\{\alpha_C \mid a \in C \text{ and } C \in \mathscr{C}\}. \tag{11}$$

Then it follows that $f_b \geq \alpha_C$ whenever $b \in C$ and $C \in \mathscr{C}$. For $a \in A$, let C_a be any cycle at which the maximum in (11) is attained. (There must be such a cycle since $f_a > -\infty$.) Then $a \in C_a$ and $\alpha_C = f_a \leq f_b$ whenever $b \in C_a$, and it follows that $\mathscr{C}$ is an f-cycle cover for G.

In summary, there exists an f-cycle cover for G if and only if f is in the span of the cycles of G with respect to the algebra in which multiplication is replaced by summation and summation is replaced by maximization. Thus, characterization (iii) of Theorem 7 is an analogue of the well-known result that a circulation can be decomposed into the sum of flows around cycles (see [20] for further discussions). Similar cycle decompositions are described in [9].

The following theorem shows that a max-balanced flow f is characterized via minimization with respect to the partial order $<$ and the linear pre-order $<_L$ in the set of all flows f^p derived from f by reweighting.

Theorem 8. *Let $G = (V, A)$ be a strongly connected graph, and let f be a flow for G. Then the following are equivalent.*

(i) *f is max-balanced;*

(ii) *$f < f^p$ for each nontrivial potential p for G;*

(iii) *$f <_L f^p$ for each nontrivial potential p for G.*

Proof. (i) $\Rightarrow$ (ii): Suppose that f is max-balanced, and let p be a nontrivial potential for G. We define

$$\beta = \sup\{\alpha \mid \mathrm{lev}_\alpha(f) \neq \mathrm{lev}_\alpha(f^p)\}.$$

Then $-\infty < \beta < \infty$ since $\mathrm{lev}_\alpha(f) = \mathrm{lev}_\alpha(f^p) = \emptyset$ for α large and since $f \neq f^p$ by Lemma 2. Also, we must have $\mathrm{lev}_\beta(f) \neq \mathrm{lev}_\beta(f^p)$ since the set A is finite.

We must show that $\mathrm{lev}_\beta(f) \subset \mathrm{lev}_\beta(f^p)$. If not, then since $\mathrm{lev}_\beta(f) \neq \mathrm{lev}_\beta(f^p)$ there is some arc a such that $a \in \mathrm{lev}_\beta(f)$ and $a \notin \mathrm{lev}_\beta(f^p)$; that is, $f_a \geq \beta > f_a^p$. Since by Theorem 7, $\mathrm{lev}_\beta(f)$ has isolated strong components, it follows from Lemma 1 that a lies on a cycle C contained in $\mathrm{lev}_\beta(f)$. Since $f^p(C) = f(C)$ there is some arc b of C with $f_b^p > f_b$. Since $b \in \mathrm{lev}_\beta(f)$ we must have $f_b \geq \beta$. Therefore $\beta' = f_b^p > f_b \geq \beta$. We conclude that $\mathrm{lev}_{\beta'}(f) \neq \mathrm{lev}_{\beta'}(f^p)$ contradicting the maximality of β and thereby proving the desired implication.

(ii) $\Rightarrow$ (iii): This implication follows directly from Lemma 5, part (iii).

(iii) $\Rightarrow$ (i): Suppose that f is not max-balanced. We will show then there exists a nontrivial potential p such that $f^p <_L f$, implying that (iii) cannot hold. Let W be a cut for G such that

$$c^+ = \max_{a \in \delta^+(W)} f_a \neq \max_{a \in \delta^-(W)} f_a = c^-.$$

By possibly exchanging W with $V \setminus W$, we may assume that $c^+ > c^-$. Let $\varepsilon = 1/2(c^+ - c^-)$, and define the potential p by

$$p_v = \begin{cases} \varepsilon, & \text{if } v \in V \setminus W, \\ 0, & \text{if } v \in W. \end{cases}$$

Let $E = A \setminus [\delta^+(W) \cup \delta^-(W)]$. Then

$$f_a^p = \begin{cases} f_a - \varepsilon, & \text{if } a \in \delta^+(W), \\ f_a + \varepsilon, & \text{if } a \in \delta^-(W), \\ f_a, & \text{if } a \in E. \end{cases} \tag{12}$$

It follows directly from (12) that

$$f_a^p \leq c^+ - \varepsilon \quad \text{for } a \in \delta^+(W) \cup \delta^-(W).$$

Let $\beta = c^+$. Since $f_a = f_a^p$ for $a \in E$, we conclude that

$$\text{lev}_\alpha(f^p) = \text{lev}_\alpha(f) \quad \text{for } \alpha > \beta,$$

and

$$\text{lev}_\beta(f^p) = \text{lev}_\beta(f^p) \cap E = \text{lev}_\beta(f) \cap E \subset \text{lev}_\beta(f).$$

Therefore $f^p < f$, and by Lemma 5, part (iii), we have $f^p <_L f$. $\quad \square$

We note that once the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) of Theorem 8 are established, the implication (iii) $\Rightarrow$ (i) can be derived by a quick argument using Theorem 3. Suppose that f is not max-balanced. Then by Theorem 3 there exists a potential p (which must be nontrivial) such that f^p is max-balanced. Then by the implication (i) $\Rightarrow$ (iii), we have that $f^p <_L (f^p)^{-p} = f$. Thus, we do not have $f <_L f^q$ for all nontrivial potentials q. We observe that it follows from Theorem 8 that if f is max-balanced, then in the set of flows derived from f by reweighting, f is the (unique) least element with respect to the partial order $\leq$, and f is a least element with respect to the linear pre-order $\leq_L$.

For the case of strongly connected graphs, Theorem 8 sharpens the following theorem of Engel and Schneider.

Theorem 9 [5, Theorem 7.5]. *Let $G = (V, A)$ be a graph containing a cycle, and let f be a flow for G. Then*

$$\text{mcm}(f) = \min_{p \in \mathbb{R}^V} \left\{ \max_{\substack{a \in A \\ a = (u,v)}} (p_u + f_a - p_v) \right\}. \tag{13}$$

Let q be a potential for which f^q is max-balanced. Since $f < g$ whenever $\max(f) < \max(g)$, there exists no $p \in \mathbb{R}^V$ such that $\max(f^p) < \max(f^q)$, and it follows that the minimum in (13) must be attained at q. Theorem 8 asserts a much stronger minimal property of the function f^q, since it must be minimal with respect to the order $\leq$. In Appendix B, we give a proof of Theorem 9 using linear programming duality.

5. Cut characterizations

In this section we present three additional characterizations of max-balanced flows based on lexicographic order properties of functions defined on cuts. First, we need some definitions.

Let $G = (V, A)$ be a strongly connected graph, and let f be a flow for G. We will use Cuts(G) to denote the set of all cuts of G. (Note, Cuts(G) is just the set of all nontrivial subsets of V.) We defined the *cut function induced by f*, written $\mathcal{F}$, to be the real-valued function defined on Cuts(G) such that

$$\mathcal{F}(W) = \max_{a \in \delta^+(W)} j_a \quad \text{for } W \in \text{Cuts}(G).$$

That is, $\mathcal{F}(W)$ is the maximum flow over all arcs leaving W. It follows from Lemma 1 that the set $\delta^+(W)$ is nonempty for every cut W whenever G is strongly connected. For a potential p, we will use $\mathcal{F}^p$ to denote the cut function induced by f^p.

Note that the definitions of $\text{lev}_\alpha(\mathcal{F})$, $\max(\mathcal{F})$, and $\mathcal{F}^{\text{ord}}$ are given in Section 3 for the case of $S = \text{Cuts}(G)$. It is easy to see that

$$W \in \text{lev}_\alpha(\mathcal{F}) \quad \Leftrightarrow \quad \delta^+(W) \cap \text{lev}_\alpha(f) \neq \emptyset. \tag{14}$$

We will use the following lemma in our next characterization of max-balanced flows.

Lemma 10. *Let $G = (V, A)$ be a strongly connected graph. Let f and p be, respectively, a flow and a potential for G such that $f < f^p$, and let $\beta = \beta(f, f^p)$. Then the following are true:*

(i) $\text{lev}_\beta(f)$ *and* $\text{lev}_\beta(f^p)$ *have the same strong components;*

(ii) *if $b = (u, v) \in \text{lev}_\beta(f^p) \setminus \text{lev}_\beta(f)$, then u and v are contained in distinct strong components of $\text{lev}_\beta(f)$, or equivalently of $\text{lev}_\beta(f^p)$.*

Proof. (i) Suppose that the strong components of $\text{lev}_\beta(f)$ and $\text{lev}_\beta(f^p)$ do not coincide. Since $\text{lev}_\beta(f) \subset \text{lev}_\beta(f^p)$, it follows directly that there exists an arc $b = (u, v) \in \text{lev}_\beta(f^p) \setminus \text{lev}_\beta(f)$ such that vertices u and v are contained in the same strong component of $\text{lev}_\beta(f^p)$ and in distinct strong components of $\text{lev}_\beta(f)$. Since b is contained in a strong component of $\text{lev}_\beta(f^p)$, there exists a cycle C such that $b \in C \subseteq \text{lev}_\beta(f^p)$ (see Lemma 1).

Since $f(C) = f^p(C)$, we have

$$0 = \sum_{a \in C} (f_a^p - f_a). \tag{15}$$

For each $a \in C$, if $a \in \text{lev}_\beta(f)$, then it follows from Lemma 4 that $f_a = f_a^p$, whereas if $a \notin \text{lev}_\beta(f)$, then $f_a^p \geq \beta > f_a$. Since $b \notin \text{lev}_\beta(f)$, it follows that the summation in (15) must be strictly positive. This contradiction completes the proof of part (i).

(ii) If $b = (u, v) \in \text{lev}_\beta(f^p) \setminus \text{lev}_\beta(f)$ is contained in a strong component of $\text{lev}_\beta(f)$, then there exists a path $P \subseteq \text{lev}_\beta(f)$ from v to u. Now we can apply the argument used in part (i) to the cycle $C = P \cup \{b\}$ to derive a contradiction. □

Next, we state and prove a result for cut functions that is analogous to Theorem 8 for flows.

Theorem 11. *Let $G = (V, A)$ be a strongly connected graph, and let f be a flow from G. Then the following are equivalent*:

    (i) *f is max-balanced*;

    (ii) *$\mathscr{F} < \mathscr{F}^p$ for each nontrivial potential p; and*

    (iii) *$\mathscr{F} <_L \mathscr{F}^p$ for each nontrivial potential p.*

Proof. (i)$\Rightarrow$(ii): Let f be max-balanced, and let p be a nontrivial potential for G. Then it follows from Theorem 8 that $f < f^p$; let $\beta = \beta(f, f^p)$. It follows directly from (14) and the definition of β in (5) that

$$\mathrm{lev}_\alpha(\mathscr{F}) = \mathrm{lev}_\alpha(\mathscr{F}^p) \quad \text{for all } \alpha > \beta,$$

and   (16)

$$\mathrm{lev}_\beta(\mathscr{F}) \subseteq \mathrm{lev}_\beta(\mathscr{F}^p).$$

We need to show that the inclusion in (16) is strict.

It follows from Lemma 10 that any $a \in \mathrm{lev}_\beta(f^p) \setminus \mathrm{lev}_\beta(f)$ must be directed between strong components of $\mathrm{lev}_\beta(f)$. Since $\mathrm{lev}_\beta(f)$ has isolated strong components, it follows that there exists a cut W such that

$$\delta^+(W) \cap \mathrm{lev}_\beta(f) = \emptyset,$$

and

$$\delta^+(W) \cap \mathrm{lev}_\beta(f^p) \neq \emptyset.$$

Clearly, $W \in \mathrm{lev}_\beta(\mathscr{F}^p)$ and $W \notin \mathrm{lev}_\beta(\mathscr{F})$.

(ii)$\Rightarrow$(iii): This follows directly from Lemma 5, part (iii).

(iii)$\Rightarrow$(i): Suppose that f is not max-balanced. We will show that there exists a potential p such that $\mathscr{F}^p <_L \mathscr{F}$. The proof of the implication (iii)$\Rightarrow$(i) in Theorem 8 shows the existence of a potential p for which $f^p < f$. Thus, it follows from (14) that for $\beta = \beta(f^p, f)$, we have

$$\mathrm{lev}_\alpha(\mathscr{F}^p) = \mathrm{lev}_\alpha(\mathscr{F}) \quad \text{for all } \alpha > \beta,$$

and   (17)

$$\mathrm{lev}_\beta(\mathscr{F}^p) \subseteq \mathrm{lev}_\beta(\mathscr{F}).$$

Further, in the proof of the implication (iii)$\Rightarrow$(i) in Theorem 8 we identified a cut W for which $\delta^+(W) \cap \mathrm{lev}_\beta(f) \neq \emptyset$ and $\delta^+(W) \cap \mathrm{lev}_\beta(f^p) = \emptyset$. Therefore, the inclusion in (17) is strict, and we have $\mathscr{F}^p < \mathscr{F}$. It follows from Lemma 5, part (iii) that $\mathscr{F}^p <_L \mathscr{F}$. $\square$

We observe that the remark following the proof of Theorem 8 can be used to produce a simple proof of the implication (iii)$\Rightarrow$(i) in Theorem 11 using Theorem 3.

Appendix A

In this appendix we give a linear programming based proof for the result of Schneider and Schneider [21] which asserts that each flow on a strongly connected graph can be potential reweighted to obtain a max-balanced flow.

Let $G = (V, A)$ be a graph, and let f be a flow for G. The following linear programming will be key to an iterative construction of potentials that will be used to produce a max-balanced potential reweighted flow of f. Let p be a potential for G and let λ be a real number where $\text{lev}_\lambda(f^p) \neq A$ and consider the linear program

Program(p, λ).

$$\begin{aligned}
\min \quad & y, \\
\text{subject to} \quad & x_i + f_{ij} - x_j \leq y && \text{for } (i,j) \in A \setminus \text{lev}_\lambda(f^p), \\
& x_i + f_{ij} - x_j = (f^p)_{i,j} && \text{for } (i,j) \in \text{lev}_\lambda(f^p), \\
& x \in \mathbb{R}^V, \, y \in \mathbb{R}.
\end{aligned}$$

The following two lemmas show that *Program(p, λ)* has an optimal solution and establishes useful properties of optimal solutions of that program.

Lemma A.1. *Let G be a strongly connected graph, f a flow for $\mathcal{G}$, p a potential for G and λ a real number with $\text{lev}_\lambda(f^p) \neq A$. Then Program$(p, \lambda)$ has an optimal solution and each optimal solution (q, μ) of that program has*

$$\mu < \lambda, \tag{A1}$$

$$\{(f^q)_a \mid a \in \text{lev}_\mu(f^q)\} = \{(f^p)_a \mid a \in \text{lev}_\lambda(f^p)\} \cup \{\mu\}, \tag{A2}$$

$$\text{lev}_\alpha(f^q) = \text{lev}_\alpha(f^p) \quad \text{for all } \alpha \geq \lambda, \tag{A3}$$

$$\text{lev}_\mu(f^q) = \text{lev}_\lambda(f^p) \cup \{a \in A \mid (f^q)_a = \mu\} \supset \text{lev}_\lambda(f^p). \tag{A4}$$

Further, all optimal solutions (q, u) of Program(η, λ) have a common μ.

Proof. We first demonstrate that *Program(p, λ)* is feasible. Evidently, $p_i + f_{ij} - p_j < \lambda$ for all $(i, j) \in A \setminus \text{lev}_\lambda(f^p)$; hence, for some positive ε, $(p, \lambda - \varepsilon)$ is feasible for *Program(p, λ)*. We next show that *Program(p, λ)* has a bounded objective. Let (x, y) be a feasible solution of *Program(p, λ)* and let a^* be an arc at which $\max\{(f^x)_a \mid a \in A \setminus \text{lev}_\lambda(f^p)\}$ is attained. As G is strongly connected, a^* lies on some cycle, say C. Let $r \equiv |C \cap \text{lev}_\lambda(f^p)|$ and $s \equiv |C \cap [A \setminus \text{lev}_\lambda(f^p)]|$. Then $s \geq 1$, $r + s \leq |V|$ and

$$\bar{f}(C) = \bar{f}^x(C) \leq (r + s)^{-1} [s(f^x)_{a^*} + r \max(f^p)].$$

Therefore,

$$y \geq (f^x)_{a^*} \geq [(r + s)/s] \bar{f}(C) - (r/s) \max(f^p)$$

$$\geq -|V| [|\bar{f}(C)| + |\max(f^p)|].$$

As the set of cycles is finite, $\tilde{f}(C)$ is bounded from below and we conclude that *Program*(p,λ) has a bounded objective and therefore must have an optimal solution.

Next let (q,μ) be an optimal solution of *Program*(p,λ). As we have seen that $(p,\lambda-\varepsilon)$ is feasible for some $\varepsilon>0$, we have that $\mu\le\lambda-\varepsilon<\lambda$, proving (A1). Also, we have from the feasibility and optimality of (q,μ) for *Program*(p,λ) that

$$(f^q)_{ij}=(f^p)_{ij}\ge\lambda>\mu\quad\text{for all }(i,j)\in\text{lev}_\lambda(f^p),\tag{A5}$$

and

$$\max\{(f^q)_{ij}\mid(ij)\in A\setminus\text{lev}_\lambda(f^p)\}=\mu<\lambda.\tag{A6}$$

(Note that the assumption that $\text{lev}_\lambda(f^p)\ne A$ assures that the max in (A6) is well defined.) Now, (A5) and (A6) combine to show (A2), (A3) and (A4). Finally the fact that all optimal solutions (q,μ) of *Program*(p,λ) share the same μ is straightforward. □

Lemma A.2. *Let G be a strongly connected graph, f a flow for G, p a potential for G and λ a real number where $\text{lev}_\lambda(f^p)\ne A$ and where $\text{lev}_\lambda(f^p)$ has isolated strong components. Also, let (q,μ) be an optimal solution of Program(p,λ) which minimizes the number of arcs (i,j) with $q_i+f_{ij}-q_j=\mu$ among all optimal solutions of Program(p,λ). Then $\text{lev}_\mu(f^q)$ has isolated strong components.*

Proof. Assume that $\text{lev}_\mu(f^q)$ does not have isolated strong components. Then, by Lemma 1, there exists an arc $(u,v)\in\text{lev}_\mu(f^q)$ which does not lie on a cycle all of whose arcs are in $\text{lev}_\mu(f^q)$. Let $W=\{i\in V\mid\text{ there exists a path from }v\text{ to }i\text{ with edges in }\text{lev}_\mu(f^q)\}$. Then $v\in W$, $u\in V\setminus W$ and for each $i\in W$ and $j\in V\setminus W$ there is no path i to j with arcs in $\text{lev}_\mu(f^q)$. In particular, as Lemma A.1 implies that $\text{lev}_\mu(f^q)\supset\text{lev}_\lambda(f^p)$, we have that no such path exists with arcs in $\text{lev}_\lambda(f^p)$. But, as $\text{lev}_\lambda(f^p)$ has isolated strong components, we conclude that there exists no path from j to i with arcs in $\text{lev}_\lambda(f^p)$. So, if $i\in W$ and $j\in V\setminus W$, then $(i,j)\notin\text{lev}_\mu(f^q)\supset\text{lev}_\lambda(f^p)$ and $(j,i)\notin\text{lev}_\lambda(f^p)$. In particular, $(u,v)\notin\text{lev}_\lambda(f^p)$.

For $(i,j)\in A\setminus\text{lev}_\mu(f^q)$, we have $(f^q)_{ij}<\mu$. Hence, for some $\varepsilon>0$,

$$(f^q)_{ij}+\varepsilon<\mu\quad\text{for all }(i,j)\in A\setminus\text{lev}_\mu(f^q).\tag{A7}$$

Consider the vector $\varepsilon^W\in\mathbb{R}^V$ defined by

$$(\varepsilon^W)_i=\begin{cases}\varepsilon,&\text{if }i\in W,\\0,&\text{if }i\in V\setminus W.\end{cases}$$

Then

$$(\varepsilon^W)_i-(\varepsilon^W)_j\begin{cases}=0,&\text{if }(i,j)\in\text{lev}_\lambda(f^p),\\\le0,&\text{if }(i,j)\in\text{lev}_\mu(f^q),\text{ and}\\\le\varepsilon,&\text{if }(i,j)\in A\setminus\text{lev}_\mu(f^q),\end{cases}$$

hence

$$(f^{q+\varepsilon^{W}})_{ij} \begin{cases} = (f^{q})_{ij} = (f^{p})_{ij}, & \text{if } (i,j) \in \text{lev}_{\lambda}(f^{p}), \\ \leq (f^{q})_{ij} \leq \mu, & \text{if } (i,j) \in \text{lev}_{\mu}(f^{q}) \setminus \text{lev}_{\lambda}(f^{p}), \text{ and} \\ \leq (f^{q})_{ij} + \varepsilon < \mu, & \text{if } (i,j) \in A \setminus \text{lev}_{\mu}(f^{q}). \end{cases} \quad \text{(A8)}$$

So, $(q + \varepsilon^{W}, \mu)$ is feasible for *Program*(p, λ).

We next argue that

$$\{(i,j) \in A \mid (f^{q+\varepsilon^{W}})_{ij} = \mu\} \subseteq \{(i,j) \in A \mid (f^{q})_{ij} = \mu\}. \quad \text{(A9)}$$

To verify this inclusion, let $(i,j) \in A$ satisfy $(f^{q+\varepsilon^{W}})_{ij} = \mu$. Then, by (A8), either $\mu = (f^{q+\varepsilon^{W}})_{ij} = (f^{q})_{ij}$ or $\mu = (f^{q+\varepsilon^{W}})_{ij} \leq (f^{q})_{ij} = \mu$. In either case we conclude that $(f^{q})_{ij} = \mu$, thereby establishing (A9).

We next show that the inclusion in (A9) is strict by showing that $(f^{q})_{uv} = \mu$ while $(f^{q+\varepsilon^{W}})_{uv} < \mu$. First, as $(u,v) \in A \setminus \text{lev}_{\lambda}(f^{p})$, the feasibility of (q, μ) for *Program*(p, λ) assures that $(f^{q})_{uv} \leq \mu$; hence the assumption that $(u,v) \in \text{lev}_{\mu}(f^{q})$ implies that $(f^{q})_{uv} = \mu$. Further, we have that

$$(f^{q+\varepsilon^{W}})_{uv} = (f^{q})_{uv} + (\varepsilon^{W})_{u} - (\varepsilon^{W})_{v} = \mu - \varepsilon < \mu.$$

Thus, strict inclusion does hold in (A9). This fact contradicts the minimality property of (q, μ) and thereby completes our proof. $\square$

The following example shows that the minimality requirement of the solution of *Program*(p, λ) cannot be dropped. Consider the graph and the flow represented by Fig. 1 and let $(p, \lambda) = (0, 5) \in \mathbb{R}^{4} \times \mathbb{R}$. Then $\text{lev}_{\lambda}(f^{p}) = \phi$ has isolated strong components and the pair $(q, \mu) = (0, 4) \in \mathbb{R}^{4} \times \mathbb{R}$ is an optimal solution of *Program*(p, λ), but $\text{lev}_{\mu}(f^{q}) = \text{lev}_{4}(f)$ does not have isolated strong components.

We note that the minimality property of the optimal solutions of *Program*(p, λ) assumed in Lemma A.2 can be weakened by assuming that there is no optimal solution (x, μ) of *Program*(p, λ) for which

$$\{(i,j) \in A \mid q_{i} + f_{ij} - x_{j} = \mu\} \subset \{(i,j) \in A \mid q_{i} + f_{ij} - q_{j} = \mu\}.$$

Our current proof of Lemma A.2 can be used directly to establish this stronger conclusion.

We also observe that our proof of Lemma A.2 can actually be used to construct an optimal solution (q, μ) of *Program*(p, λ) for which $\text{lev}_{\mu}(f^{q})$ has isolated strong components. This can be accomplished by first computing any optimal solution (q', μ) of *Program*(p, λ), e.g., by applying the simplex method. The construction described in the proof of Lemma A.2 can then be used to eliminate, by applying further reweighting, arcs of the $\text{lev}_{\mu}(f^{q'})$ which are not contained in a cycle all of whose arcs are in that level set. This can be done without adding any new arcs to the μ-level set of the reweighted flow. The repeated use of the procedure will result in a potential q, where (q, μ) is optimal for *Program*(p, λ) and where every arc of

$\text{lev}_\mu(f^q)$ is contained in a cycle of $\text{lev}_\mu(f^q)$. By Lemma 1 we are then guaranteed that $\text{lev}_\mu(f^q)$ will then have isolated strong components.

We are now ready to prove the existence of max-balanced potential reweighting of every flow on a strongly connected graph.

Theorem A.3 (Schneider and Schneider [21]). *Let $G=(V,A)$ be a strongly connected graph, and let f be a flow for G. Then there exists a potential q for which f^q is max-balanced. Further, q is unique up to the addition of a trivial potential.*

Proof. Let p^0 be the zero potential and let $\lambda^0 > \max(f)$. Then $\text{lev}_{\lambda^0}(f^{p^0}) = \emptyset \neq A$ has isolated strong components. Iteratively, given $r = 1, 2, \ldots$ and a pair (p^r, λ^r) where $\text{lev}_{\lambda^r}(f^{p^r}) \neq A$ has isolated strong components, we construct a pair (p^{r+1}, λ^{r+1}) which is an optimal solution of $Program(p^r, \lambda^r)$ and minimizes the number of arcs (i,j) with $(p^{r+1})_i + f_{ij} - (p^{r+1})_j = \lambda^{r+1}$ among all optimal solutions of $Program(p^r, \lambda^r)$. We observe that by Lemma A.1, $\text{lev}_{\lambda^r}(f^{p^r})$ is strictly increasing. Hence, the procedure will terminate at some stage, say k, with $\text{lev}_{\lambda^k}(f^{p^k}) = A$. Let $q = p^k$ and $\mu = \lambda^k$. We will show that f^q is max-balanced by showing that its level sets have isolated strong components, see Theorem 7.

Lemma A.1 implies that for $r = 1, \ldots, k-1$,

$$\{(f^{p^{r-1}})_a \mid a \in \text{lev}_{\lambda^{r-1}}(f^{p^{r-1}})\} = \{(f^{p^r})_a \mid a \in \text{lev}_{\lambda^r}(f^{p^r})\} \cup \{\lambda^{r+1}\}.$$

As $\text{lev}_{\lambda^0}(f^{p^0}) = \emptyset$, we get from iterating the above equation that

$$\{(f^q)_a \mid a \in A\} = \{(f^q)_a \mid a \in \text{lev}_\mu(f^q)\} = \{\lambda^1, \lambda^2, \ldots, \lambda^k\}.$$

Thus it suffices to show that for $r = 1, \ldots, k$, $\text{lev}_{\lambda^r}(f^q)$ has isolated strong components. Now, by Lemma A.1, λ^r is increasing. Hence, for $r = 1, \ldots, k$ and $j = r, \ldots, k-1$, $\lambda^r \geq \lambda^j > \lambda^{j+1}$ and therefore by (A3) of Lemma A.1, $\text{lev}_{\lambda^r}(f^{p^{j+1}}) = \text{lev}_{\lambda^r}(f^{p^j})$. On iterating the last equation we get that for $r = 1, \ldots, k$, $\text{lev}_{\lambda^r}(f^q) = \text{lev}_{\lambda^r}(f^{p^r})$. But, by Lemma A.2, our construction assures that for each $r = 1, \ldots, k$, $\text{lev}_{\lambda^r}(f^{p^r})$ has isolated strong components and therefore so does $\text{lev}_{\lambda^r}(f^q)$, as asserted.

We finally show the uniqueness up to addition of a trivial potential of a potential p for which f^p is max-balanced. Suppose p and q are potentials for which both f^p and f^q are max-balanced, where $p - q$ is nontrivial. Then, by Theorem 8, $f^p <_L (f^p)^{q-p} = f^q$ and $f^q <_L (f^q)^{p-q} = f^p$, a contradiction which proves that $p - q$ is a trivial potential. $\square$
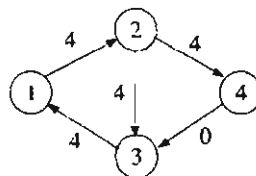

Fig. 1.

The remark following Lemma A.2 suggests that repeated applications of the simplex method and some extra computation can be used to calculate a max-balanced potential reweighted flow whose existence is asserted in Theorem A.3.

Appendix B

In this appendix, we present a proof of Theorem 9 using standard linear programming duality (see [13] and references therein).

Theorem 9 [5, Theorem 7.5]. *Let $G = (V, A)$ be a graph containing a cycle, and let f be a flow for G. Then*

$$\text{mcm}(f) = \min_{p \in \mathbb{R}^i} \left\{ \max_{\substack{a \in A \\ a = (u, v)}} (p_u + f_a - p_v) \right\}. \tag{B1}$$

Proof. The minimum of the right-hand side in (B1) is equal to the optimal value of the linear program

$$\begin{aligned}
&\min_{(p, \lambda)} && \lambda, \\
&\text{subject to} && \lambda \geq p_u + f_a - p_v \text{ for } a = (u, v) \in A.
\end{aligned} \tag{B2}$$

The dual of (B2) is:

$$\begin{aligned}
&\max_{x} && \sum_{a \in A} f_a x_a, \\
&\text{subject to} && \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_t = 0 \text{ for } a \in A, \\
& && \sum_{a \in A} x_a = 1, \\
& && x \geq 0.
\end{aligned} \tag{B3}$$

A nonnegative solution satisfying the first constraints in (B3) is called a *circulation*.

It follows from elementary network flow theory that a circulation in a graph can be decomposed into a sum of nonnegative circulations around cycles and conversely that such a sum must also be a circulation. Let Γ be the set of all cycles for G. Then for any circulation x, there exist nonnegative weights $y(C)$ for $C \in \Gamma$ such that

$$x_a = \sum_{\substack{C \in \Gamma \\ a \in C}} y(C).$$

For such x, we have

$$\sum_{a \in A} f_a x_a = \sum_{a \in A} \sum_{\substack{C \in \Gamma \\ a \in C}} f_a y(C) = \sum_{C \in \Gamma} \sum_{a \in C} f_a y(C) = \sum_{C \in \Gamma} f(C) y(C).$$

and

$$\sum_{a \in A} x_a = \sum_{C \in \Gamma} |C| y(C).$$

Thus, (B3) is equivalent to:

$$\max_{y} \quad \sum_{C \in \Gamma} f(C) y(C),$$

$$\text{subject to} \quad \sum_{C \in \Gamma} |C| y(C) = 1, \tag{B4}$$

$$y(C) \geq 0 \text{ for } C \in \Gamma.$$

Making the substitution $z(C) = |C| y(C)$, (B4) reduces to

$$\max_{z} \quad \sum_{C \in \Gamma} \bar{f}(C) z(C),$$

$$\text{subject to} \quad \sum_{C \in \Gamma} z(C) = 1, \tag{B5}$$

$$z(C) \geq 0 \text{ for } C \in \Gamma.$$

Now it is obvious that C^* is a maximum mean cycle if and only if

$$z(C) = \begin{cases} 1, & \text{if } C = C^*, \\ 0, & \text{otherwise} \end{cases}$$

is an optimal solution for (B5). □

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