Towers and Cycle Covers for Max-Balanced Graphs

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Abstract

Let $G = (V, A, g)$ be a strongly connected weighted graph. We say that G is *max-balanced* if for every cut W, the maximum weight over arcs leaving W equals the maximum weight over arcs entering W. A subgraph H of G is *max-sufficient* if for every cut W, the maximum weight over arcs of G leaving W is attained at some arc of H. A *tower* $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ is a sequence of arc-sets of G where C_{i+1} is a cycle all of whose weights are maximal in the graph formed by contracting the sets C_1, C_2, ..., C_i to a point. We show that G is max-balanced if and only if G contains a tower. A *cycle cover for* G is a collection of cycles $\mathcal{D} = \{D_a \mid a \in A\}$ such that arc a is the minimum weight arc of D_a. We use the tower construction to show that the existence of a cycle cover characterizes max-balanced graphs. We show that the graph H of a tower is max-sufficient, thereby showing that a max-balanced graph contains a max-sufficient subgraph with at most $2(|V| - 1)$ arcs. Further, we use the tower construction to show that H has a cycle cover with at most $|V|$ cycles.

1 Introduction

In this paper we study max-balanced weighted directed graphs, which were introduced in [4] and [5]. We define three concepts for such graphs G, namely a max-sufficient subgraph for G, a tower for G, and a cycle cover for G. We study connections between these concepts, and we prove characterizations of max-balanced graphs associated with them. A summary of our results is found in the abstract above, and we give further details in this introduction after some definitions and an explanation of the relation of our results to previous work. Further results on max-balanced graphs are contained in [3].

Let (V, A) be a (directed) graph with vertex set V and arc set A. For $a \in A$, we will use the notation $a \sim (u, v)$ to denote the arc a from vertex u to vertex v, and refer to the vertices u and v as the *endpoints of* a. Note, that a graph (V, A) may contain parallel arcs (i.e., two arcs a and a' of the form $a \sim (u, v)$ and $a' \sim (u, v)$). We will assume, however, that (V, A) does not contain loops (i.e., an arc a of the form $a \sim (v, v)$).

A *weight function for* (V, A) is a real-valued function g defined on the arcs A. We will use the notation g_a to denote the *weight of* a. A *weighted graph* is a triple $G = (V, A, g)$ where (V, A) is a graph, and g is weight function for (V, A). A *cut for* G is a *nontrivial* subset W of V (i.e., $\emptyset \subset W \subset V$). (We will use the symbols $\subset$ and

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max-balanced if and only if it contains a tower. A tower is built up from arcs sets $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ where C_{i+1} is a cycle in the graph formed by contracting the cycles C_1, C_2, ..., C_i to a point. We show that the subgraph

$$H = C_1 \cup C_2 \cup \cdots \cup C_r,$$

which we call the *graph of* $\mathcal{T}$, is max-sufficient for G. We show further that $r < |V|$ and that H contains at most $2(|V| - 1)$ arcs.

In Section 4, we define a *cycle cover for* G *associated with a subgraph* H and in Theorem 6 we apply the tower construction to generate a cycle cover for G associated with the graph H of the tower. Further, we show in Corollary 7 that H has a cycle cover containing fewer than $|V|$ cycles. Finally, in Corollary 8 we show that a weighted graph is max-balanced if and only if it has a cycle cover. This result is an analogue of a cycle decomposition for a circulation in a graph.

In this paper, we use the framework for max-balanced graphs described in [4]. In particular, we use the definition of contraction from [4] (rather than [3] or [5]). The definition used here is natural for describing our tower construction since it allows us to identify the arcs of a contracted graph with arcs in the original graph. The results of [3] and [5] apply (with trivial modifications) to the setting of this paper. Similarly, the results in this paper apply to the setting of [3] and [5]. We consider only strongly connected graphs, although all of our results extend with minor modifications to graphs that are the disjoint union of strongly connected graphs.

2 The Operation of Contraction

Let $G = (V, A, g)$ be a strongly connected weighted graph, and let Π be a partition of V. We define the *contraction of* G *with respect to* Π, written G/Π, to be the weighted graph (Π, A', g') such that there exists an embedding $\phi\colon A' \mapsto A$ satisfying the following conditions:

(i) If $\phi(a') = a$, where $a' \sim (I, J) \in A'$ and $a \sim (u, v) \in A$, then $u \in I$ and $v \in J$;

(ii) If $a' \in A'$, then $g'_{a'} = g_{\phi(a')}$.

Contraction with respect to a partition Π can be described intuitively as follows: Given an element W of Π, add a new vertex v_W to the graph $G - W$ (i.e., the graph formed by deleting W and all arcs entering or leaving W) and join to v_W an arc $a' \sim (u, v_W)$ for each arc $a \sim (u, v) \in A$ with $v \in W$ and an arc $a' \sim (v_W, u)$ for each arc $a \sim (v, u)$ with $v \in W$. (See Fig 1.) Set the weight of each arc of the resulting graph to the weight of the corresponding arc of G. We will refer to this operation as *contracting the set* W *to a point*. The graph G/Π is formed by contracting each element of Π to a point.

Let I and J be distinct elements of the partition Π. Note that the contracted graph G/Π contains an arc with endpoints I and J for each arc $a \sim (u, v)$ for which $u \in I$ and $v \in J$. In our definition of contraction, we do *not* identify resulting parallel arcs, and therefore G/Π will, in general, contain parallel arcs. For the tower construction described in this paper, this is the natural definition of contraction since it allows us to identify the arcs of G/Π with arcs of the graph G (under the mapping ϕ).

In summary an arc a' in G/Π corresponds in a natural way to a unique arc a of the original graph G. Since we identify the arcs a and a', we will refer to *a as an arc of G/Π*. It is natural and intuitive to think of the arcs of G/Π as those arcs of G which are not deleted by the contraction operation. In particular, we will identify a cycle C of G/Π with the set of arcs $\phi(C)$ of G and thus refer to the set $C \subset A$ as a cycle of G/Π. It is easy to see that a cycle of G/Π is a disjoint union of paths in G between elements of Π. See Fig 1.

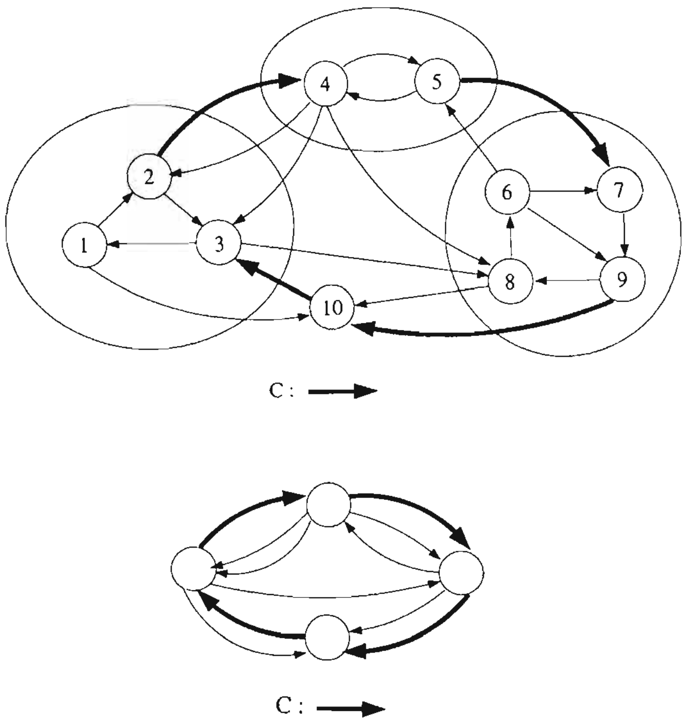

Figure 1: G/Π, where $\Pi = \left\{ \{1,2,3\}, \{4,5\}, \{6,7,8,9\}, \{10\} \right\}$

We will use the next two lemmas to prove some of our results.

Lemma 1 *Let $G = (V, A, g)$ be a strongly connected weighted graph. Then G is max-balanced if and only if G/Π is max-balanced for every partition Π of V.*

Proof. Let G be max-balanced, and let W' be a cut for G/Π. Define W (a cut for G) by

$$W = \left\{ v \in V \mid v \in I \text{ for some } I \in W' \right\}.$$

Since G is max-balanced at W, it follows directly from the definition of contraction that G/Π is max-balanced at W'. The converse follows by letting Π be the discrete partition of V. ∎

Lemma 2 *Let $G = (V, A, g)$ be max-balanced, and let $b \in A$ with $g_b = \max(G)$. Then b is contained in a cycle C for G such that $g_a = \max(G)$ for all $a \in C$.*

Proof. Suppose $g_b = \max(G)$, $b \sim (u, v)$. It suffices to show that there exists a path P from v to u all of whose arcs have weight $\max(G)$. Let W be the set of vertices w such that there exists such a path from v to w. If $u \notin W$, then since $b \sim (u, v) \in \delta^-(W)$ it follows directly from the definition of W that

$$\delta^+(W) < \max(G) = \delta^-(W),$$

which violates the definition of max-balanced graphs. Therefore $u \in W$ and $b \sim (u, v)$ must lie on a cycle all of whose arcs have weight $\max(G)$. ∎

3 Towers for G

Let $G = (V, A, g)$ be a weighted graph. We wish to define a construction that we will call a *tower for G*. We give an algorithm for computing a tower and show that G is max-balanced if and only if G contains a tower.

Let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a sequence of subsets of A. Let $H_0 = (V, \emptyset)$, and define the subgraphs

$$H_{i+1} = H_i \cup C_{i+1} \quad \text{for } i = 0, 1, \ldots, r-1. \tag{4}$$

For $i = 0, 1, \ldots, r$, let Π_i be the partition of V induced by the strong components of H_i. Then the sequence $\mathcal{T}$ is called a *tower for G* if

(i) C_{i+1} is a cycle of the contracted graph G/Π_i for $i = 0, 1, \ldots, r-1$,

(ii) $g_a = \max(G/\Pi_i)$ for $a \in C_{i+1}$ and $i = 0, 1, \ldots, r-1$, and

(iii) $|\Pi_r| = 1$.

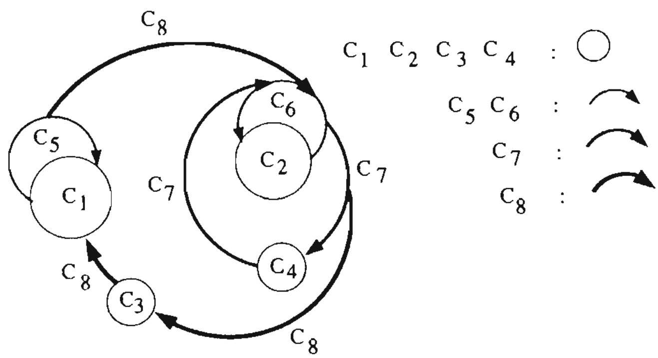

Figure 2: A Tower for G

Note that since each subgraph H_i is spanning, condition (iii) is equivalent to requiring that H_r is strongly connected. Note that the arc sets $(C_1, C_2, \ldots, C_r)$ are pairwise disjoint since the arcs of $C_1, C_2, \ldots, C_i$ are deleted when H_i is contracted to form

G/Π_i. We will call the subgraph

$$H_r = C_1 \cup C_2 \cup \cdots \cup C_r$$

the *graph of the tower* $\mathcal{T}$. See Fig. 2 for an example of a tower for G.

Theorem 3 *Let $G = (V, A, g)$ be a strongly connected weighted graph. Let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a tower for G, and let $H = (V, E)$ be the graph of $\mathcal{T}$. Then the following are true:*

(i) H is max-sufficient for G;

(ii) G is max-balanced;

(iii) $r \leq |V| - 1$;

(iv) $|E| \leq 2(|V| - 1)$.

Proof. (i) and (ii): Let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a tower for G, and let W be any cut for G. Let j be the largest integer such that the partition Π_j is finer than the two element partition $\{W, V \backslash W\}$. Note that $0 \leq j < r$ since Π_0 and Π_r are, respectively, the discrete and the indiscrete partitions of V. Now define the cut W' for G/Π_j by

$$W' = \left\{ I \in \Pi_j \mid I \subseteq W \right\}. \tag{5}$$

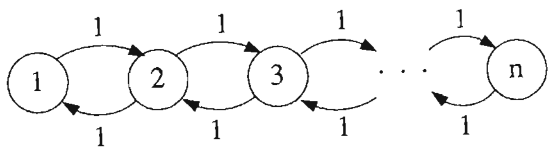

Figure 3: $r = |V| - 1$ and $|E| = 2(|V| - 1)$

It follows from the definition of j that C_{j+1} must intersect both $\delta^+(W'; G/\Pi_j)$ and $\delta^-(W'; G/\Pi_j)$. Since the endpoints of each arc of $\delta^+(W; G)$ lie in distinct elements of the partition Π_j, it follows that $\delta^+(W; G)$ and $\delta^+(W'; G/\Pi_j)$ coincide. Because $g_a = \max(G/\Pi_j)$ for each $a \in C_{j+1}$, it follows from the definition of contraction that

$$\max_{a \in \delta^+(W;G)} g_a = \max_{a \in \delta^-(W;G)} g_a = \max(G/\Pi_j), \tag{6}$$

and furthermore both maxima in (6) must be attained at some arc of C_{j+1}. This proves that H is max-sufficient for G, and that G is max-balanced.

(iii): Since each cycle in a tower must have length at least 2 (recall, G and hence G/Π_i contains no loops), we must have $|\Pi_{i+1}| < |\Pi_i|$, and therefore a tower can have length at most $r \leq |V| - 1$

(iv): Since the vertices of C_{i+1} (which are distinct) are identified to form Π_{i+1}, we must have

$$|\Pi_i| = |\Pi_{i+1}| + |C_{i+1}| - 1, \quad \text{for } i = 0, 1, \ldots, r-1.$$

Since $|\Pi_0| = |V|$ and $|\Pi_r| = 1$, we have

$$|E| = \sum_{i=1}^{r} |C_i| = |V| - 1 + r \leq 2(|V| - 1).$$

This completes the proof. ∎

The max-balanced graph given in Fig. 3 shows that the bounds in parts (iii) and (iv) of Theorem 3 are, in general, the best possible.

It follows from Part (ii) of Theorem 3 that the existence of a tower is a sufficient condition for a weighted graph to be max-balanced. This condition is also necessary, and the following algorithm shows how to compute a tower for a given max-balanced graph.

The Tower Algorithm

Input: A strongly connected max-balanced graph $G = (V, A, g)$.

Output: A tower $(C_1, C_2, \ldots, C_r)$ for G.

Step 0: Set $H_0 = (V, \emptyset)$ and $i = 0$.

Step 1: If H_i is strongly connected, set $r = i$, return the sequence $(C_1, C_2, \ldots, C_r)$, and **STOP**.

Step 2: Let Π_i be the partition of V induced by the strong components of H_i, and let C_{i+1} be a cycle of G/Π_i satisfying

$$g_a = \max(G/\Pi_i) \quad \text{for } a \in C_{i+1}. \tag{7}$$

Step 3: Let $H_{i+1} = H_i \cup C_{i+1}$; set $i = i + 1$ and return to Step (1).

It follows from Lemma 1 that the graph G/Π_i in Step 2 is max-balanced, and therefore by Lemma 2 it contains a cycle C_{i+1} satisfying (7). It follows directly from Steps 1 and 2 that the output satisfies conditions (ii) and (iii) in the definition of a tower. Since $|\Pi_{i+1}| < |\Pi_i|$, for all i in Step 3, we have the following result:

Theorem 4 *Let $G = (V, A, g)$ be a strongly connected weighted max-balanced graph. Then the tower algorithm terminates in at most $|V| - 1$ iterations with a tower $(C_1, C_2, \ldots, C_r)$ for G.*

As a consequence of Theorems 3 and 4, we have the following characterization of max-balanced graphs.

Theorem 5 *Let $G = (V, A, g)$ be a strongly connected weighted graph. Then G is max-balanced if and only if G contains a tower.*

4 Cycle Covers for G

In this section, we define the notion of a cycle cover for G. We show that G is max-balanced if and only if G has a cycle cover (see [3] for an alternative proof of this result) and use a tower for G to construct a cycle cover. As a consequence, we derive for max-balanced graphs a result that is an analogue of a cycle decomposition for a circulation in a graph.

Let $G = (V, A, g)$ be max-balanced, and let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a tower for G. For $i = 0, 1, \ldots, r$, let H_i be defined by (4) and let Π_i be the partition of V determined by the strong components of H_i. Define λ_i by

$$\lambda_i = \max(G/\Pi_{i-1}) = g_a \quad \text{for all } a \in C_i. \tag{8}$$

It follows directly from the definition of C_i that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r. \tag{9}$$

Each set C_i (since it is a cycle of G/Π_{i-1}) is a disjoint union of paths in G between the strong components of H_{i-1}. Thus each C_i can be extended to a cycle C_i' of G by traversing arcs in $C_1 \cup C_2 \cup \cdots \cup C_{i-1}$. It follows from (9) that the resulting cycle C_i' satisfies

$$\begin{aligned} \lambda_i &\leq g_a \quad \text{for } a \in C_i', \quad \text{and} \\ g_a &= \lambda_i \quad \text{for } a \in C_i. \end{aligned} \tag{10}$$

Let $G = (V, A, g)$ be a weighted graph, and let H be a subgraph for G. A *cycle cover for G associated with H* is a collection $\mathcal{D} = \{D_a \mid a \in A\}$ of (not necessarily distinct) cycles of G such that for each $b \in A$

(i) $b \in D_b$,

(ii) $D_b \backslash \{b\}$ is contained in H, and

(iii) $g_b \leq g_a$ for $a \in D_b$.

If H equals G, then we will refer to $\mathcal{D}$ as a *cycle cover for G*.

We make three observations on the relation between towers and cycle covers.

1. Equivalently, we may define a cycle cover for $G = (V, A, g)$ as a sequence of cycles $(D_1, D_2, \ldots, D_s)$ for G such that there exist numbers $(\mu_1, \mu_2, \ldots, \mu_s)$ so that for all $a \in A$
$$g_a = \max_{\{i \mid a \in D_i\}} \mu_i. \tag{11}$$

2. If $\mathcal{D}$ is a cycle cover for G associated with H and $H \subseteq H' \subseteq G$, then $\mathcal{D}$ is a cycle cover for G associated with H'.

3. Let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a tower for G, and let $G' = C_1 \cup C_2 \cup \cdots \cup C_r$ be the graph of $\mathcal{T}$. Let $\mathcal{D} = \{C_i' \mid i = 1, 2, \ldots, r\}$ be the extended cycles constructed above. Since C_i' is contained in $C_1 \cup C_2 \cup \cdots \cup C_i$, it follows directly from (10) that $g_a = \max_{\{i \mid a \in C_i'\}} \lambda_i$. Therefore, it follows from Observation 1 that $\mathcal{D}$ is a cycle cover for G'.

Theorem 6 *Let G be a max-balanced graph, and let $\mathcal{T} = (C_1, C_2, \ldots, C_r)$ be a tower for G. Then there exists a cycle cover for G associated with the graph of $\mathcal{T}$.*

Proof. Let $H = (V, E)$ be the graph of the tower $\mathcal{T}$. We will define a collection of cycles $\mathcal{D} = \{D_b \mid b \in A\}$ for G (not all distinct) and show that $\mathcal{D}$ is a cycle cover for G associated with H. First, let $b \in E$ be an arc of $\mathcal{T}$ contained in cycle C_j. It follows from Observation 3 that $D_b = C_j'$ (where C_j' is the extension of C_j described above) satisfies the required conditions.

Next, let b be an arc not contained in H, and let Π_i, $i = 0, 1, \ldots, r$, be the partitions determined by H_i defined in (4). It is intuitively obvious that the graph G/Π_i is formed by sequentially contracting the cycles $C_1, C_2, \ldots, C_i$ (see [4] for a careful proof of this). The arcs of G/Π_{i-1} that are deleted by contracting C_i to a point are precisely those arcs with both endpoints in C_i. Let j be the integer such that b is deleted when C_j is contracted to a point. It follows that b has both endpoints in the extended cycle C_j'. Thus we can form D_b by concatenating b and the path in C_j' between the endpoints of b in the direction of b. This proves that $D_b \backslash \{b\}$ is contained in H.

Since b is an arc of G/Π_{j-1} it follows that $g_b \leq \lambda_j$. Further, since D_b is contained in C_j', it follows directly from (10) that $g_b \leq g_a$ for $a \in D_b$. This proves that $\mathcal{D}$ is a cycle cover for G associated with H. ▮

We remark that there is an analogy between the graph H of a tower for a max-balanced graph and a spanning tree for an undirected graph in the sense that every arc b not in H is contained in a cycle all of whose arcs except b are contained in H.

We have the following corollary of Theorem 3.

Corollary 7 *Let $G = (V, A, g)$ be a strongly connected max-balanced graph. Then there exists a max-sufficient subgraph H for G that has a cycle cover containing fewer that $|V|$ cycles.*

Proof. Let $\mathcal{T}$ be a tower for G, and let H be the graph of $\mathcal{T}$. It follows from Theorem 3 that H is max-sufficient for G, and by Observation 3 the set $\mathcal{D} = \{C_i' \mid i = 1, 2, \ldots, r\}$ is a cycle cover for H. Furthermore, $r < |V|$ by part (iii) of Theorem 3. ▮

As a consequence of Theorems 3 and 6 we have the following characterization of max-balanced graphs.

Corollary 8 *Let $G = (V, A, g)$ be a strongly connected weighted graph. Then G is max-balanced if and only if G has a cycle cover.*

Proof. Let G be max-balanced. Then by Theorem 3, G has a tower $\mathcal{T}$. It follows from Theorem 6 that G has a cycle cover associated with the graph of $\mathcal{T}$, which by Observation 2 is a cycle cover for G (associated with G).

Conversely, let $\mathcal{D}$ be a cycle cover for G, and let W be a cut for G. Then for each $b \in \delta^+(W)$ there exists a cycle $D \in \mathcal{D}$ such that $g_b \leq g_a$ for $a \in D$. Since D must also intersect $\delta^-(W)$ there exists some arc $c \in \delta^-(W)$ such that $g_b \leq g_c$. Thus we have show that

$$\max_{a \in \delta^+(W)} g_a \leq \max_{a \in \delta^-(W)} g_a. \tag{12}$$

A similar argument shows that the reverse inequality in (12) is also satisfied. This proves that G is max-balanced. ▮

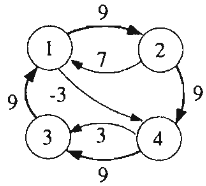

Figure 4: A Max-Balanced Graph

Corollary 8 is an analogue of a cycle decomposition for a circulation in a graph. Specifically, it is easy to prove that a weight function g is a circulation for a graph (V, A) if and only if there exist cycles $D_1, D_2, \ldots, D_s$ and positive numbers μ_1, μ_2, $\ldots$, μ_s such that

$$g_a = \sum_{\{i \mid a \in D_i\}} \mu_i \quad \text{for } a \in A.$$

Given a circulation g it is easy to construct such a cycle decomposition: Let D_1 be a cycle for (V, A) such that $g_a > 0$ for $a \in D_1$. Let $\mu = \max\{g_a \mid a \in D_1\}$, and subtract μ from each weight g_a, $a \in D_1$ and repeat this operation on the resulting circulation. Continuing in this fashion, we can easily construct the desired cycle decomposition. The tower algorithm is, in a sense, an analogue of this algorithm for decomposing circulations.

5 Examples

We conclude the paper by providing examples of max-balanced graphs. We proved in Theorem 5 that G is max-balanced if and only if G contains a tower. Thus every max-balanced has the structure of a tower together with appropriately weighted chords. Every max-balanced graph contains a cycle all of whose weights are maximal. If a weighted graph G contains such a cycle that is also Hamiltonian, then G is max-balanced. See Fig. 4.

More complicated examples can be built by contracting the maximal cycle to a point and repeating this construction. See Fig. (5).

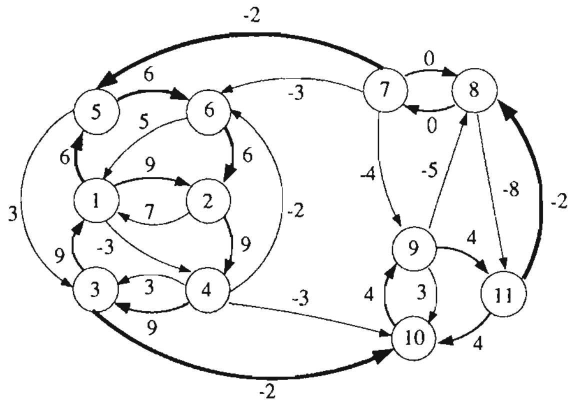

Figure 5: A Max-Balanced Graph

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