Characterizations and Classifications of *M*-Matrices Using Generalized Nullspaces

Daniel Hershkowitz*
Department of Mathematics
Technion — Israel Institute of Technology
Haifa 32000 Israel

Uriel G. Rothblum†
Faculty of Industrial Engineering and Management
Technion — Israel Institute of Technology
Haifa 32000 Israel

and

Hans Schneider‡*
Mathematics Department
University of Wisconsin — Madison
Madison, Wisconsin 53706

Submitted by Richard A. Brualdi

ABSTRACT

Several characterizations of the class of M-matrices as a subclass of the class of Z-matrices are given. These characterizations involve alternating sequences, decompositions, and splittings, and all are related to generalized nullspaces.

1. INTRODUCTION

In this paper we give several new characterizations for a Z-matrix to be an M-matrix. All of our characterizations are related to the generalized nullspace of the matrix.

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Let A be a Z-matrix. In Section 3 we introduce alternating sequences for A and we show that a Z-matrix A is an M-matrix if and only if the length of every alternating sequence is finite. Moreover, it is shown that the index of an M-matrix equals the maximal length of an alternating sequence. Related results appear in [6] and [7]. In Section 4 we show that every vector x such that $Ax \geqslant 0$ has a decomposition of a certain type if and only if A is an M-matrix. In Section 5 we show that if the index of A is less than or equal to 1, then A is an M-matrix if and only if there exists a weakly regular splitting of A, $A = M - N$, such that the matrix $I - M^{-1}N$ is an M-matrix with the same index as A. We introduce a more general class of splittings, called Z-splittings, for which a similar result holds.

Some of our results improve results found in [1].

This paper is the third in a sequence of related papers. The first paper in the sequence is [2] and the second paper is [3].

2. NOTATION AND DEFINITIONS

This section contains most of the definitions and notation used in this paper. In the main we follow the definitions and notation used in [9].

Let A be a square matrix with entries in some field. As is well known (see [9] for further details), after performing an identical permutation on the rows and the columns of A, we may assume that A is in Frobenius normal form, namely a block (lower) triangular form where the diagonal blocks are square irreducible matrices.

NOTATION 2.1. For a positive integer n we denote $\langle n \rangle = \{1, \dots, n\}$.

CONVENTION 2.2. We shall always assume that A is an $n \times n$ matrix in Frobenius normal form (A_{ij}), where the number of diagonal blocks is p. Also every vector b with n entries will be assumed to be partitioned into p vector components b_i conformably with A.

NOTATION 2.3. Let b be a vector with n entries. We denote

$$\operatorname{supp}(b) = \{ i \in \langle p \rangle : b_i \neq 0 \}.$$

DEFINITION 2.4. The *reduced graph of* A is defined to be the graph $R(A)$ with vertices $1, \dots, p$ and where (i, j) is an arc if and only if $A_{ij} \neq 0$.

Definition 2.5. Let i and j be vertices in $R(A)$. We say that j *accesses* i if $i = j$ or there is a path in $R(A)$ from j to i. In this case we write $i =< j$. We write $i -< j$ for $i =< j$ but $i \neq j$. We write $i \neq< j$ $[i \not-< j]$ if $i =< j$ $[i -< j]$ is false.

Definition 2.6. Let W be a set of vertices of $R(A)$, and let i be a vertex of $R(A)$. We say that i *accesses* W $(W =< i)$ if i accesses (at least) one element of W. We say that W *accesses* i $(i =< W)$ if i is accessed by (at least) one element of W.

Notation 2.7. Let W be a set of vertices of $R(A)$. We denote

$$\text{below}(W) = \{\text{vertices } i \text{ of } R(A) : W =< i\},$$

$$\text{above}(W) = \{\text{vertices } i \text{ of } R(A) : i =< W\},$$

$$\text{top}(W) = \{i \in W : j \in W,\ j =< i => \ i = j\},$$

Definition 2.8. A vertex i of $R(A)$ is said to be *singular* [*nonsingular*] if A_{ii} is singular [nonsingular]. The set of all singular vertices of $R(A)$ is denoted by S.

Definition 2.9. Let W be a set of vertices in $R(A)$. A sequence $\alpha_1, \ldots, \alpha_k$ of singular vertices in W is said to be a *singular chain in* W *of length* k if $\alpha_1 -< \cdots -< \alpha_k$.

Definition 2.10. Let i be a vertex in $R(A)$. The *level* of i is defined to be the maximal length of a singular chain in below(i).

Notation 2.11. Let k be a nonnegative integer. We denote by Λ_k the set of all vertices in $R(A)$ of level k.

Definition 2.12. Let x be a vector. The *level* of x, level(x), is defined to be the maximal level of a vertex i, $i \in \text{supp}(x)$.

Definition 2.13. A real (not necessarily square) matrix P will be called *nonnegative* $(P \geqslant 0)$ if all its entries are nonnegative, *semipositive* $(P > 0)$ if $P \geqslant 0$ but $P \neq 0$, and (*strictly*) *positive* $(P \gg 0)$ if all its entries are positive.

NOTATION 2.14. Let P be a nonnegative square matrix. We denote by $\rho(P)$ the spectral radius of P (its Perron-Frobenius root).

DEFINITION 2.15. A *Z-matrix* is a square matrix of form $A = \lambda I - P$, where P is nonnegative. Such a Z-matrix A is an *M-matrix* if $\lambda \geqslant \rho(P)$. The least real eigenvalue of a Z-matrix A is denoted by $l(A)$ [observe that $l(A) = \lambda - \rho(P)$].

NOTATION 2.16. Let A be a Z-matrix. We denote

$$T = \{ i \varepsilon \langle p \rangle : l(A_{ii}) < 0 \}.$$

NOTATION 2.17. Let A be a square matrix. We denote

ind(A) = the index of 0 as an eigenvalue of A, viz., the size of the largest Jordan block associated with 0;
$E(A)$ = the generalized nullspace of A, viz. $N(A^n)$, where n is the order of A.

DEFINITION 2.18. Let A be a square matrix and let $x \in E(A)$. The *height* of x, height(x), is defined to be the minimal nonnegative integer k such that $A^k x = 0$.

DEFINITION 2.19. Let A be a square matrix in Frobenius normal form, and let $H = \{\alpha_1, \dots, \alpha_q\}$, $\alpha_1 < \cdots < \alpha_q$, be a set of vertices in $R(A)$. A set of semipositive vectors $x^1, \dots, x^q$ is said to be an *H-preferred set* (for A) if

$$\left. \begin{aligned} x_j^i &\gg 0 \qquad \text{if} \quad \alpha_i = < j, \\ x_j^i &= 0 \qquad \text{if} \quad \alpha_i \neq < j, \end{aligned} \right\} i = 1, \dots, q, \quad j = 1, \dots, p,$$

and

$$-Ax^i = \sum_{k=1}^{q} c_{ik} x^k, \qquad i = 1, \dots, q$$

where the c_{ik} satisfy

$$\left.\begin{array}{lll} c_{ik} > 0 & \textit{if} & \alpha_i - \alpha_i - < \alpha_k, \\ c_{ik} = 0 & \text{if} & \alpha_i \not< \alpha_k, \end{array}\right\} i, k = 1, \ldots, q.$$

Definition 2.20. Let A be a square matrix in Frobenius normal form, and let H be a set of vertices in $R(A)$. An H-preferred set that forms a basis for a vector space V is called an *H-preferred basis* for V.

Definition 2.21. Let A be a square matrix. A splitting $A = M - N$ is said to be a *Z-splitting* if M is a nonsingular matrix, M^{-1} is nonnegative, and the matrix $I - M^{-1}N$ is a Z-matrix. A splitting $A = M - N$ is said to be a *weakly regular splitting* if it is a Z-splitting and also $M^{-1}N$ is nonnegative.

3. ALTERNATING SEQUENCES

Definition 3.1. Let $A \in \mathbb{C}^{nn}$ and let $x \in \mathbb{C}^n$. The sequence $x, Ax, \ldots, A^k x$ is said to be *semipositive sequence for A of length k* if

$$A^r x > 0, \qquad r = 0, \ldots, k-1,$$

$$A^k x \geqslant 0.$$

The sequence $x, Ax, \ldots, A^k x$ is said to be an *alternating sequence for A of length k* if the sequence $x, Bx, \ldots, B^k x$ is a semipositive sequence for B, where $B = -A$.

Definition 3.2. Let $A \in \mathbb{C}^{nn}$ and let $x \in \mathbb{C}^n$. An infinite sequence $x, Ax, A^2x, \ldots$ is said to be an *infinite semipositive sequence for A* if

$$A^r x > 0, \qquad r = 0, 1, 2, \ldots.$$

The infinite sequence $x, Ax, A^2x, \ldots$ is said to be an *infinite alternating sequence for A* if it is an infinite semipositive sequence for $-A$.

LEMMA 3.3. *Let A be an M-matrix, and let x be a seminegative vector such that* $Ax = b \geqslant 0$. *Then* $\mathrm{level}(x) > \mathrm{level}(b)$.

Proof. Let $i \in \mathrm{top}(x)$. Observe that $A_{ii}x_i = b_i$. If i is a nonsingular vertex, then it follows that $b_i > 0$. Hence, since A_{ii} is an irreducible M-matrix, it follows that $x_i \gg 0$, contrary to assumption. Thus i is a singular vertex. Furthermore, since A_{ii} is an irreducible singular M-matrix and $b_i \geqslant 0$, it follows that $b_i = 0$. Hence, an examination of the accessibility relations (cf. Lemma (3.1) in [3]) shows that $\mathrm{top}(x) -\!\!< \mathrm{supp}(b)$. Since $\mathrm{top}(x)$ consists of singular vertices only, the result follows. ■

THEOREM 3.4. *Let A be an Z-matrix. Then*

(i) *if A is not an M-matrix, there exists an infinite alternating sequence for A;*

(ii) *if A is an M-matrix, the index of A is equal to the maximal length of an alternating sequence for A.*

Proof. (i): If A is not an M-matrix, then choose x to be a semipositive eigenvector associated with the least real (negative) eigenvalue of A. Observe that the sequence $x, Ax, \dots$ is an infinite alternating sequence for A.

(ii): Let A be an M-matrix, and let $\mathrm{ind}(A) = k$. By the preferred basis theorem (e.g., Theorem (4.14) in [2]) there exists an alternating sequence for A of length k (see also Theorem 3.1 in [4]). To show that k is the maximal length of such a sequence, let m be a positive integer, and assume that $x, Ax, \dots, A^m x$ is an alternating sequence for A. It follows from Lemma 3.3 that the $\mathrm{level}(x) \geqslant m$. Since by the index theorem for M-matrices (e.g., Theorem 3.1 in [4]; see also Corollary (4.37) in [2]) we have $\mathrm{ind}(A) \geqslant \mathrm{level}(x)$, it follows that $k \geqslant m$. ■

The following characterization of M-matrices is an immediate consequence of Theorem 3.4.

COROLLARY 3.5. *Let A be a Z-matrix. Then A is an M-matrix if and only if every alternating sequence for A is of finite length.*

4. DECOMPOSITIONS

THEOREM 4.1. *Let A be a Z-matrix. Then the following are equivalent:*

(i) *A is an M-matrix.*

(ii) $Ax \geqslant 0$ *implies that there exists a nonnegative vector u and a nonnegative vector v,* $v \in E(A)$, $Av \leqslant 0$, *such that* $x = u - v$.

(iii) $Ax \geqslant 0$ *implies that there exists a nonnegative vector* u *and a vector* v, $v \in E(A)$, *such that* $x = u - v$.

Proof. (i) $\Rightarrow$ (ii): Let A be an M-matrix, and let i be a vertex in $R(A)$. If $i \notin \text{below}(S)$, then it follows from Proposition (4.2) in [3] that $x_i \geqslant 0$. Furthermore, by the preferred basis theorem, $E(A)$ contains a semipositive vector w such that $w_i \gg 0$ for all $i \in \text{below}(S)$ and $Aw \leqslant 0$. Therefore, for a sufficiently large positive c, the vector $x + cw$ is semipositive. Hence, the decomposition $x = u - v$, where $u = x + cw$ and $v = cw$, satisfies the required conditions.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): Suppose that (iii) holds, and assume that A is not an M-matrix. By the preferred basis theorem there exists a seminegative eigenvector x for A associated with $l(A)$ such that

$$\text{supp}(x) \cap T \neq \varnothing. \tag{4.2}$$

Observe that $Ax > 0$. By (iii), x may be written as $x = u - v$, where $u \geqslant 0$ and $v \in E(A)$. Since $x < 0$ we have $v > 0$. Furthermore, by (4.2) we have

$$\text{supp}(v) \cap T \neq \varnothing. \tag{4.3}$$

However, since v is a nonnegative vector in $E(A)$ it follows from Theorem (5.3) in [2] that $\text{supp}(v) \cap T = \varnothing$, in contradiction to (4.3). Therefore, our assumption that A is not an M-matrix is false. ■

LEMMA 4.4. *Let* A *be an* M*-matrix and let* $x \in E(A)$. *Then* $\text{top}(x) \subseteq S$.

Proof. The claim clearly holds for all elements in the S-preferred basis for $E(A)$ (e.g., Theorem (4.14) in [2]), and hence it holds for all $x \in E(A)$. ■

LEMMA 4.5. *Let* A *be a singular* M*-matrix. Let* x *be a vector of level* k *in* $E(A)$ *such that* $x_i > 0$ *for all* $i \in \Lambda_k \cap \text{supp}(x) \cap S$. *Then* $\text{height}(x) = k$.

Proof. We prove our assertion by induction on k. The case $k = 0$ is obvious, since then $x = 0$ by Lemma 4.4. Assume the claim holds for $k < m$ where $m > 0$, and let $k = m$. By the preferred basis theorem, x is a linear combination of the S-preferred basis elements, where the coefficients corresponding to the k-level vectors are nonnegative and not all zero. Also, the coefficients corresponding to vectors of level greater than k (if any) are zero. Thus, by the preferred basis theorem, $y = -Ax$ is a $(k-1)$-level vector in

$E(A)$, where $y_i > 0$ for all $i \in \Lambda_{k-1} \cap \text{supp}(y) \cap S$. By the inductive assumption we have height$(y) = k - 1$, and hence height$(x) = k$. ■

Corollary 4.6. *Let A be an M-matrix. Then there exists a vector x such that $Ax \geq 0$, and for every decomposition $x = u - v$ where $u \geq 0$ and $v \in E(A)$ we have* height$(v) = $ ind(A).

Proof. Let $k = $ ind(A). Choose $-x$ to be one of the k-level vectors in an S-preferred basis for $E(A)$. Then $Ax \geq 0$. Let $x = u - v$, where $u \geq 0$ and $v \in E(A)$. Observe that $v > 0$ and level$(v) = k$. By Lemma 4.5 we have height$(v) = k$. ■

Remark 4.7. Using similar arguments we can prove the following statement: *Let x be a k-level vector such that $x_i > 0$ for all $i \in \Lambda_k \cap$ supp$(x) \cap S$. Then for every decomposition $x = u - v$ where $u \geq 0$ and $v \in E(A)$ we have* height$(v) \geq k$.

In view of Corollary 4.6, Theorem 4.1 can be stated in a slightly stronger version:

Theorem 4.8. *Let A be a Z-matrix and let k be a nonnegative integer. Then the following are equivalent:*

(i) *A is an M-matrix with* ind$(A) = k$.

(ii) $Ax \geq 0$ *implies that there exists a nonnegative vector u and a nonnegative vector v, $v \in E(A)$, $Av \leq 0$,* height$(v) = k$, *such that $x = u - v$.*

(iii) $Ax \geq 0$ *implies that there exists a nonnegative vector u and a vector v, $v \in E(A)$,* height$(v) = k$, *such that $x = u - v$.*

Proof. (i) ⇒ (ii): The proof is identical to the proof of the corresponding implication in Theorem 4.1. Note that the vector w chosen there is of height k.

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (i): By Theorem 4.1, A is an M-matrix. Also, clearly, ind$(A) \geq k$. By Corollary 4.6 it follows from (iii) that ind$(A) = k$. ■

In the case $k \geq 1$, the implication (iii) ⇒ (i) in Theorem 4.8 may be found as (E_{12}) ⇒ (i) in Theorem 2 of [5]; see also [1, p. 154].

We have the following extension of Theorem 4.1 to Z-matrices. We let $F(A)$ be the subspace spanned by the nonnegative vectors in $E(A)$.

THEOREM 4.9. *Let A be a Z-matrix, and suppose that $Ax \geqslant 0$. Then the following are equivalent*:

(i) $\operatorname{supp}(x) \cap \operatorname{above}(T) = \varnothing$.

(ii) *There exists a nonnegative vector u and a nonnegative vector v, $v \in E(A)$, $Av \leqslant 0$, such that $x = u - v$.*

(iii) *There exists a nonnegative vector u and a vector v, $v \in F(A)$, such that $x = u - v$.*

Proof. (i) $\Rightarrow$ (ii): Let $W = \operatorname{below}(\operatorname{supp}(x))$. If (i) holds, then $A[W]$ is an M-matrix. Hence (ii) follows by an application of Theorem 4.1 to $A[W]$.

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): Suppose (i) is false and that (iii) holds. Since (i) is false, $\operatorname{below}(\operatorname{supp}(x)) \cap T \neq \varnothing$. Let

$$V' = \operatorname{top}(\operatorname{below}(\operatorname{supp}(x)) \cap T),$$

and let $V = \operatorname{above}(V')$. By Corollary (5.8) in [3] we have $v[V] = 0$, and it follows from (iii) that

$$x[V] \geqslant 0. \tag{4.10}$$

Since, by its definition, V does not access any vertex outside V, we have $(Ax)[V] = A[V]x[V]$. Hence, $A[V]x[V] \geqslant 0$. Since every initial vertex of $A[V]$ belongs to T, it follows from Theorem (5.1) in [3] that

$$A[V]x[V] = 0. \tag{4.11}$$

By Corollary (5.9) in [2] it now follows from (4.10) and (4.11) that $x[V] = 0$. But this is absurd, since by the definition of V we have $\operatorname{supp}(x) \cap V \neq \varnothing$. ■

We note that we have found another proof of the implication (iii) $\Rightarrow$ (i) of Theorem 4.9 which uses Proposition 3.6 of [3] in place of Corollary 5.9 of [2].

REMARK 4.12. In condition (iii) of Theorem 4.9 it is impossible to replace $F(A)$ by $E(A)$. This may be seen by the following example. Let

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix},$$

and let $x = [1, -1]^T$. Then x is a nullvector of A and obviously $x = 0 - v$, where $v = -x \in E(A)$. But (i) of Theorem 4.9 is false.

5. SPLITTINGS

In this section we consider Z-splittings, introduced in Definition 2.21. By definition, every weakly regular splitting is a Z-splitting. It is easy to extend Lemma 4.2 of [8] to show that if A is an M-matrix for which there exists a positive vector x such that Ax is nonnegative and if $A = M - N$ is a Z-splitting, then the matrix $B = I - M^{-1}N$ is an M-matrix. It now follows that if A is either a nonsingular M-matrix or an irreducible singular M-matrix and if $A = M - N$ is a Z-splitting, then the matrix $B = I - M^{-1}N$ is an M-matrix. However, in general if A is an M-matrix and if $A = M - N$ is a Z-splitting (or even a weakly regular splitting), then the matrix $B = I - M^{-1}N$ need not be an M-matrix, as demonstrated by the weakly regular splitting

$$\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$$

discussed in [4].

Proposition 5.1. *Let A be an M-matrix. Then there exists a weakly regular splitting $A = M - N$ for which $B = I - M^{-1}N$ is an M-matrix and* $\operatorname{ind}(B) = \operatorname{ind}(A)$.

Proof. Since A is an M-matrix, it can be written as $A = sI - P$, where P is a nonnegative matrix and $\rho(P) \leqslant s$. Evidently, the splitting where $M = sI$ and $N = P$ has the required properties. ■

Conversely we have for Z-splittings.

Proposition 5.2. *Let A be a Z-matrix, and let $A = M - N$ be a Z-splitting. If the matrix $B = I - M^{-1}N$ is an M-matrix with* $\operatorname{ind}(B) \leqslant 1$, *then A is an M-matrix and* $\operatorname{ind}(A) = \operatorname{ind}(B)$.

Proof. Let x be such that $Ax \geqslant 0$. Then $Bx = M^{-1}Ax \geqslant 0$. Since $\operatorname{ind}(B) \leqslant 1$, it follows from Theorem 4.1 that $x = u - v$, where $u \geqslant 0$ an $Bv = 0$. Observe that $Av = MBv = 0$, and hence by Theorem 4.8 the matrix A is an M-matrix and $\operatorname{ind}(A) \leqslant 1$. Clearly $\operatorname{ind}(A) = \operatorname{ind}(B)$, since both indices are less than or equal to 1 and both matrices are either singular or nonsingular. ■

As a corollary of Propositions 5.1 and 5.2 we now obtain the following theorem.

THEOREM 5.3. *Let A be a Z-matrix, and let k be either* 0 *or* 1. *Then the following are equivalent*:

(i) *A is an M-matrix and* $\mathrm{ind}(A) = k$.

(ii) *There exists a weakly regular splitting* $A = M - N$ *for which* $B = I - M^{-1}N$ *is an M-matrix and* $\mathrm{ind}(B) = k$.

(iii) *There exists a Z-splitting* $A = M - N$ *for which* $B = I - M^{-1}N$ *is an M-matrix and* $\mathrm{ind}(B) = k$.

Observe that the implication (ii) ⇒ (i) in Theorem (5.3) improves the implication $(C_9) \Rightarrow$ (i) in Theorem 2 of [5]; see also [1, p. 154].

REMARK 5.4. Philip Kavanagh [private communication] informs us that there are several examples of weakly regular splittings $A = M - N$ where $B = I - M^{-1}N$ is an M-matrix and $\mathrm{ind}(B) > \mathrm{ind}(A)$ or $\mathrm{ind}(B) < \mathrm{ind}(A)$. Thus although by Proposition 5.1 the implication (i) ⇒ (ii) in Theorem 5.3 holds for all k, the reverse implication holds in general only for $k \leqslant 1$. These examples also show that we cannot replace "there exists" by "for every" in statements (ii) and (iii) of Theorem 5.3.

REMARK 5.5. Michael Neumann has shown us an alternative proof of Theorem 5.3 which is related to the proof of one direction of Theorem 1 in [4].

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