On the Generalized Nullspace of *M*-Matrices and *Z*-Matrices*

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ABSTRACT

A proof is given for the preferred basis theorem for the generalized nullspace of a given M-matrix. The theorem is then generalized and extended to the case of a Z-matrix.

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1. INTRODUCTION

In this paper we give a proof for a known result, namely, the preferred basis theorem for the generalized nullspace of a given M-matrix. We then generalize and extend the theorem to the case of a Z-matrix.

Let A be a Z-matrix. A preferred set is an ordered set of semipositive vectors such that the image of each vector under $-A$ is a nonnegative linear combination of the subsequent vectors. Furthermore, the positivity of the

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entries of the vectors and the coefficients in the linear combinations depends entirely on the graph structure of A in a specified manner. The preferred basis theorem asserts that the generalized nullspace $E(A)$ of an M-matrix A has a basis which is a preferred set. As an immediate consequence of this theorem we obtain the index theorem for M-matrices, which asserts that the index of an M-matrix equals the maximal length of a singular chain in the reduced graph of A.

In the case of a Z-matrix A, we prove a result concerning the support of a nonnegative vector in the generalized nullspace. We deduce that the subspace of the generalized nullspace $E(A)$ which is spanned by the nonnegative vectors in $E(A)$ has a basis which is a preferred set. It follows that $E(A)$ has a basis consisting of nonnegative vectors if and only if a certain principal submatrix of A is an M-matrix. This condition is clearly satisfied when A is an M-matrix. Thus our results generalize the preferred basis theorem. We also obtain a graph theoretic lower bound for the index of a Z-matrix.

It should be noted that our formulation of these results in terms of M-matrices or Z-matrices rather than nonnegative matrices is in fact a technicality. Thus the preferred basis theorem for M-matrices may be stated as a theorem on the generalized eigenspace associated with the spectral radius of a nonnegative matrix. Similarly, our results concerning generalized nullspaces of Z-matrices may be considered as results on the generalized eigenspaces associated with the spectral radii of the diagonal blocks in the Frobenius normal form of a nonnegative matrix.

The discussion of such problems originated in Section 11 of [1]. Here Frobenius determines those eigenvalues of a nonnegative matrix which have an associated nonnegative eigenvector. He proves the following result: *Let A be a nonnegative matrix in (lower triangular) Frobenius normal form, and let ρ_i be the spectral radius of the diagonal block A_{ii}. Let λ be an eigenvalue of A. If, for some j, $\lambda = \rho_j$ and $\rho_k < \rho_j$ whenever $k > j$, then A has a semipositive eigenvector associated with λ.* He observed that the converse is true up to a permutation similarity that keeps the matrix in Frobenius normal form. Frobenius also proved that each component of a semipositive eigenvector of a nonnegative matrix, which is partitioned conformably with the Frobenius normal form, is either strictly positive or zero. His proofs were by the "tracedown method," which is essentially induction on the number of diagonal blocks in the Frobenius normal form. Some graph theoretical considerations or their equivalent are required to distinguish between those components that are positive and those that are zero. For the case of the spectral radius, this was done in [6]. In the general case a precise graph theoretic version of the result was recently proved by Victory [8]. However, by use of well-known results on permutations of partially ordered sets, it may be seen that Frobenius's theorem is equivalent to our Corollary 5.12 below, which is part of the result in [8].

In [6, Theorems 5 and 6], the preferred basis theorem is stated under the assumption that the singular graph of the matrix is linearly ordered, but actually the methods developed there are applicable to the general case. The first part of the theorem in full generality is proved by Rothblum [5], and the complete theorem is proved in [4]. However, it is one of the characteristics of proofs in this area that the basic ideas are simple but that the details are complicated and tend to obscure the underlying principles. Our main goal in re-proving the preferred basis theorem is to simplify the proof significantly in a manner which displays the underlying ideas.

Section 2 below contains most of our definitions and notation. In Section 3 we give some lemmas on general matrices. The preferred basis theorem is proved in Section 4, and Section 5 contains the new extensions to Z-matrices.

Further discussion of the history of the preferred basis theorem and of related results may be found in the survey paper [7].

This paper is the first in a sequence of related papers. This sequence will organize important known results and prove new results in the graph theoretic theory of (reducible) M-matrice sand Z-matrices. The next papers in the sequence are [3] and [2], which contain some applications of the results in the present paper.

2. NOTATION AND DEFINITIONS

This section contains most of the definitions and notation used in this paper. In the main we follow the definitions and notation used in [7].

Let A be a square matrix with entries in some field. As is well known (see [7] for further details), after performing an identical permutation on the rows and the columns of A we may assume that A is in Frobenius normal form, namely a block (lower) triangular form where the diagonal blocks are square irreducible matrices.

NOTATION 2.1. For a positive integer n we denote $\langle n \rangle = \{1, \ldots, n\}$.

CONVENTION 2.2. We shall always assume that A is an $n \times n$ matrix in Frobenius normal form (A_{ij}), where the number of diagonal blocks is p. Also, every vector b with n entries will be assumed to be partitioned into p vector components b_i conformably with A.

NOTATION 2.3. Let b be a vector with n entries. We denote

$$\operatorname{supp}(b) = \{ i \in \langle p \rangle : b_i \neq 0 \}.$$

DEFINITION 2.4. The *reduced graph of* A is defined to be the graph $R(A)$ with vertices $1, \ldots, p$ and wherer (i, j) is an arc if and only if $A_{ij} \neq 0$.

DEFINITION 2.5. Let i and j be vertices in $R(A)$. We say that j *accesses* i if $i = j$ or there is a path in $R(A)$ from j to i. In this case we write $i =< j$. We write $i -< j$ for $i =< j$ but $i \neq j$. We write $i \neq< j$ $[i \not-< j]$ if $i =< j$ $[i -< j]$ is false.

DEFINITION 2.6. Let W be a set of vertices of $R(A)$, and let i be a vertex of $R(A)$. We say that i *accesses* W ($W =< i$) if i accesses (at least) one element of W. We say that W *accesses* i ($i =< W$) if i is accessed by (at least) one element of W.

DEFINITION 2.7. A set W of vertices of $R(A)$ is said to be final [initial] if for every vertex j of $R(A)$, $j =< W$ implies $j \in W$ $[W =< j$ implies $j \in W]$. Observe that by Convention 2.2, $\{1\}$ is a final set and $\{p\}$ an initial set of $R(A)$.

NOTATION 2.8. Let W be a set of vertices of $R(A)$. We denote

$$\operatorname{below}(W) = \{\text{vertices } i \text{ of } R(A) : W =< i\},$$

$$\operatorname{above}(W) = \{\text{vertices } i \text{ of } R(A) : i =< W\},$$

$$\operatorname{bot}(W) = \{ i \in W : j \in W,\ i =< j \Rightarrow i = j\},$$

DEFINITION 2.9. Let i and j be vertices of $R(A)$. The set $\operatorname{below}(i) \cap \operatorname{above}(j)$ is called the *convex hull of* i *and* j and is denoted by $\operatorname{hull}(i, j)$. Observe that in general $\operatorname{hull}(i, j) \neq \operatorname{hull}(j, i)$. Also, $\operatorname{hull}(i, j) \neq \varnothing$ if and only if $i =< j$.

DEFINITION 2.10. A vertex i of $R(A)$ is said to be *singular* [*nonsingular*] if A_{ii} is singular [nonsingular]. The set of all singular vertices of $R(A)$ is denoted by S.

NOTATION (2.11). Let W be a set of vertices of $R(A)$. We denote

$A[W]$ = the principal submatrix of A whose rows and columns are indexed by the vertices of $G(A)$ that belong to the strong components in W,
$A(W) = A[\langle p\rangle \setminus W]$.

NOTATION 2.12. Let W be a set of vertices of $R(A)$ and let b be a vector. We denote

$b[W]$ = the vector obtained by omitting all b_i, such that $i \notin W$,
$b(W) = b[\langle p\rangle \setminus W]$.

DEFINITION 2.13. A real (not necessarily square) matrix P will be called *nonnegative* ($P \geqslant 0$) if all its entries are nonnegative, *semipositive* ($P > 0$) if $P \geqslant 0$ but $P \neq 0$, and (*strictly*) *positive* ($P \gg 0$) if all its entries are positive.

NOTATION 2.14. Let P be a nonnegative square matrix. We denote by $\rho(P)$ the spectral radius of P (its Perron-Frobenius root).

DEFINITION 2.15. A *Z-matrix* is a square matrix of form $A = \lambda I - P$, where P is nonnegative. A Z-matrix A is an *M-matrix* if $\lambda \geqslant \rho(P)$. The least real eigenvalue of a Z-matrix A is denoted by $l(A)$ [observe that $(A) = \lambda - \rho(P)$].

NOTATION 2.16. Let A be a square matrix. We denote

$m(A)$ = the algebraic multiplicity of 0 as an eigenvalue of A,
$\operatorname{ind}(A)$ = the index of 0 as an eigenvalue of A, viz., the size of the largest Jordan block associated with 0,
$N(A)$ = the nullspace of A,
$E(A)$ = the generalized nullspace of A, viz. $N(A^n)$, where n is the order of A,
$F(A)$ = the subspace of $E(A)$ which is spanned by the nonnegative vectors in $E(A)$,
$K(A)$ = the cone of nonnegative vectors in $F(A)$.

3. LEMMAS ON GENERAL MATRICES

This section contains lemmas to be used in the next sections. They are given in a separate section because they refer to general square matrices.

LEMMA 3.1. *Let A be a square matrix in Frobenius normal form. Let W be a final set of vertices of $R(A)$, and let x be a vector such that*

$$x[W]=0.$$

Then

$$(Ax)[W]=0$$

and

$$(Ax)(W)=A(W)x(W).$$

Proof. Immediate. ∎

LEMMA 3.2. *Let A be a square matrix in Frobenius normal form. Let i be a vertex in $R(A)$, and let x be a vector such that*

$$x_j=0 \qquad \text{whenever} \quad i \not< j.$$

Then

$$(Ax)_j=0 \qquad \text{whenever} \quad i \not< j.$$

Proof. The set of all j satisfying $i \not< j$ is final. Our claim now follows from Lemma 3.1. ∎

LEMMA 3.3. *Let A be a square matrix in Frobenius normal form, and let A_{ii} be singular. Then there exists a vector x in $E(A)$ such that*

$$x_i \neq 0 \tag{3.4}$$

and

$$x_j=0 \qquad \text{whenever} \quad i \not< j. \tag{3.5}$$

Proof. Let W be the set of all vertices j of $R(A)$ such that $i \not< j$, and let $B=A(W)$. Let $v(W)\in E(B)$. Observe that if $v_i=0$ then $v(W)\setminus\{i\}\in E(B(i))$. Therefore, since $m(B(i))<m(B)$, it follows that $E(B)$ must contain a vector $x(W)$ such that $x_i\neq 0$. Since W is a final set of vertices of $R(A)$, it follows from Lemma 3.1 that the vector x obtained by putting $x_j=0$ for all $j\in W$ is in $E(A)$, and satisfies the required conditions. ∎

4. PREFERRED BASIS FOR THE GENERALIZED NULLSPACE OF AN M-MATRIX

DEFINITION 4.1. Let A be a square matrix in Frobenius normal form, and let $H = \{\alpha,\dots,\alpha_q\}$, $\alpha_1 < \cdots < \alpha_q$, be a set of vertices in $R(A)$. A set of semipositive vectors $x^1,\dots,x^q$ is said to be an *H-preferred set* (for A) if

$$x_j^i \gg 0 \quad \text{if} \quad \alpha_i =< j, \tag{4.2}$$

$$x_j^i = 0 \quad \text{if} \quad \alpha_i \neq< j, \tag{4.3}$$

$$i = 1,\dots,q, \quad j = 1,\dots,p,$$

and

$$-Ax^i = \sum_{k=1}^{q} c_{ik}x^k, \qquad i = 1,\dots,q \tag{4.4}$$

where the c_{ik} satisfy

$$\left.\begin{aligned} c_{ik} &> 0 \quad \text{if} \quad \alpha_i -< \alpha_k \\ c_{ik} &= 0 \quad \text{if} \quad \alpha_i \not-< \alpha_k \end{aligned}\right\} \qquad i,k = 1,\dots,q. \tag{4.5}$$

REMARK 4.6. Observe that an H-preferred set is a set of linearly independent vectors which spans an invariant subspace of $E(A)$.

DEFINITION 4.7. Let A be a square matrix in Frobenius normal form, and let H be a set of vertices in $R(A)$. An H-preferred set that forms a basis for a vector space V is called an *H-preferred basis* for V.

Note that for $H = S$, H-preferred bases may be found in [4] under the name S^+-bases (see also Section 7 of [7]).

LEMMA 4.8. *Let A be an M-matrix, and assume that* 1 *is a singular vertex of $R(A)$. Let u be a vector satisfying*

$$u_1 \gg 0, \tag{4.9}$$

and

$$u_j = 0 \qquad \textit{if} \quad 1 \neq< j. \tag{4.10}$$

Let G be the set of all vertices j in $R(A)$ such that $\mathrm{hull}(1, j) \cap S = \{1\}$. *If*

$$(Au)_j = 0 \qquad \text{for all} \quad j \text{ in } G, \tag{4.11}$$

then

$$u_j \gg 0 \qquad \text{for all} \quad j \text{ in } G. \tag{4.12}$$

Proof. We prove the lemma by induction on the number p of diagonal blocks in the Frobenius normal form of A. If $p = 1$, there is nothing to prove. Assume that the lemma holds for $p < m$ where $m > 1$, and let $p = m$. Since $A(m)$ and $u(m)$ satisfy the conditions of the lemma, then by the inductive assumption we need to prove (4.12) only for $j = m$. So assume that $m \in G$, and denote by Q the set of all k such that $k < m$ and $A_{mk} \neq 0$. Since by (4.11), $(Au)_m = 0$, we have

$$A_m u_m = - \sum_{k \in Q} A_{mk} u_k. \tag{4.13}$$

Let $k \in Q$. If $1 \neq< k$, then by (4.10), we have $u_k = 0$. If $1 =< k$, then necessarily $k \in G$, and by the inductive assumption we have $u_k \gg 0$. Furthermore, since $1 =< m$, there exists $k \in Q$ such that $1 =< k$. Therefore, it follows from (4.13) that $A_{mm} u_m > 0$. Since A_{mm} is a nonsingular irreducible M-matrix, it follows that $u_m \gg 0$. ■

We now prove the preferred basis theorem for M-matrices (see Theorem 7.1 in [7] for another statement of this result). In the course of the proof we shall use certain subsets of $R(A)$. An example illustrating these sets will be given after the proof.

THEOREM 4.14. *Let A be an M-matrix. Then there exists an S-preferred basis for $E(A)$.*

Proof. We prove the theorem by induction on p. For $p = 1$ the claim follows from the Perron-Frobenius theorem on spectra of nonnegative matrices. Assume the claim holds for $p < m$ where $m > 1$, and let $p = m$. Let $S = \{\alpha_1, \dots, \alpha_q\}$, where $\alpha_1 < \cdots < \alpha_q$. By the inductive assumption we can find an $S \setminus \{1\}$-preferred basis for $E(A(1))$. We now adjoin zero components at the top of the basis elements and so obtain linearly independent vectors in $E(A)$. If A_{11} is nonsingular, then this is the required basis. In the rest of this proof we assume that A_{11} is singular, and we denote the $q-1$ vectors just obtained by $x^2,\ldots,x^q$. Since the multiplicity of 0 as an eigenvalue of A is q, it follows that we need to add one more vector u in order to obtain a basis for $E(A)$. By Lemma 3.3 we choose a vector u in $E(A)$ satisfying

$$u_1 \neq 0 \tag{4.15}$$

and

$$u_j = 0 \qquad \text{whenever} \quad 1 \not< j. \tag{4.16}$$

It now follows from (4.16) by Lemma 3.2 that

$$(Au)_j = 0 \qquad \text{whenever} \quad 1 \not< j. \tag{4.17}$$

By (4.15) the vector u is linearly independent of $x^2,\ldots,x^q$. Observe that $u_1 \in (A_{11}) = N(A_{11})$. By the Perron-Frobenius theorem for irreducible nonnegative matrices, we may thus assume that

$$u_1 \gg 0. \tag{4.18}$$

Since

$$(Au)_1 = 0, \tag{4.19}$$

we have

$$(Au)(1) \in E(A(1)). \tag{4.20}$$

Since $Au \in E(A)$, we have

$$-Au = d_1 u + \sum_{k=2}^{q} d_k x^k. \tag{4.21}$$

By (4.18) and (4.19), and since the first components of $x^2,\ldots,x^q$ are zero, we have

$$d_1 = 0. \tag{4.22}$$

We now prove that

$$d_k = 0 \qquad \text{whenever} \quad k \in \langle q \rangle, \quad 1 \not< \alpha_k. \tag{4.23}$$

Suppose that (4.23) is false. Let $j \in \langle q \rangle$ be the smallest integer such that

$$1 \neq< \alpha_j \tag{4.24}$$

and $d_j \neq 0$. By the structure of $x^2, \ldots, x^q$ it follows from (4.21) that

$$(Au)_{\alpha_j} = -d_j x^j_{\alpha_j} \neq 0. \tag{4.25}$$

However, by (4.17) and (4.24) we have

$$(Au)_{\alpha_j} = 0,$$

contradicting (4.25). Therefore, (4.23) holds.

Denote by R the sets below(1)$\cap S \setminus \{1\}$. In view of (4.21), (4.22), and (4.23), we have to prove that u may be chosen such that

$$d_k > 0 \qquad \text{whenever} \quad k \in R. \tag{4.26}$$

We shall first prove (4.26) for singular vertices k such that hull$(1, k) \cap S = \{1, k\}$. Then we shall show that u can be modified so that (4.26) holds, and remodified (and renamed x^1) so that (4.2) and (4.3) hold.

We first show that u satisfies the conditions of Lemma 4.8. Note that (4.9) and (4.10) are (4.18) and (4.16) respectively. Now let

$$G = \{\text{vertices } j \text{ in } R(A) : \text{hull}(1, j) \cap S = \{1\}\}$$

and let $j \in G$. Assume that $(Au)_j \neq 0$. By (4.19), $j > 1$. By (4.20) and by the structure of the $S \setminus \{1\}$-preferred set $x^2, \ldots, x^q$ there must exist $h \in S$, $h > 1$, such that $h =< j$ and $(Au)_h \neq 0$. By (4.17) we have $1 -< h$ and hence $h \in$ hull$(1, j) \cap S$, in contradiction to $j \in G$. Therefore, our assumption that $(Au)_j \neq 0$ is false and (4.11) is satisfied. Thus the vector u indeed satisfies the conditions of Lemma 4.8.

We now define the sets

$$R_1 = \{j \in R : \text{hull}(1, j) \cap S = \{1, j\}\},$$

$$R_2 = R \setminus R_1.$$

Let $t \in R_1$ and let $t = \alpha_r$. It follows from (4.21) that

$$A_{tt}u_t = y_t - z_t, \tag{4.27}$$

where

$$y_t = -\sum_{j=1}^{t-1} A_{tj}u_j, \tag{4.28}$$

and by (4.21) and (4.22)

$$z_t = \sum_{k=2}^{r} d_k x_t^k. \tag{4.29}$$

Let k be in $\{2,\dots,r-1\}$. If $\alpha_k \not\prec t$, then by the inductive assumption we have $x_t^k = 0$. If $\alpha_k -\!\!< t$, then since $t \in R_1$, we have $1 \neq\!< \alpha_k$ and by (4.23) $d_k = 0$. Thus, it follows from (4.29) that

$$z_t = d_r x_t^r. \tag{4.30}$$

Let $Q = \{ j \in \langle t-1 \rangle : A_{tj} \neq 0\}$. Since $t \in R_1$ it follows that for each $j \in Q$ we have either $1 \neq\!< j$, in which case $u_j = 0$ by (4.16), or $1 =\!< j$, in which case $j \in G$ and by Lemma 4.8 $u_j \gg 0$. Further, since $1 -\!\!< t$ there exists $j \in Q$ such that $1 =\!< j$. Hence, it follows from (4.28) that

$$y_t > 0. \tag{4.31}$$

Assume that $d_r \leqslant 0$. Since $x_t^r \gg 0$, it follows from (4.30) and (4.31) that the right-hand side of (4.27) is semipositive, which is impossible, since A_{tt} is a singular irreducible M-matrix. Hence our assumption that $d_r \leqslant 0$ is false and we have

$$d_r > 0.$$

Now let $t \in R_2$, and let $t = \alpha_r$. Then there exists an integer $s(t) \in \{2,\dots,r-1\}$ such that $\alpha_{s(t)} \in R_1$ and $\alpha_{s(t)} -\!\!< t$. We choose a positive constant b_t which is greater than $|d_r|/c_{s(t),r}$, where $c_{s(t),r}$ is the positive coefficient defined in (4.4) for the $S \setminus \{1\}$-preferred basis $x^2,\dots,x^q$. Observe that the vector

$$v = u + \sum_{t \in R_2} b_t x^{s(t)}$$

satisfies

$$-Av = \sum_{k=2}^{q} d'_k x^k, \tag{4.32}$$

where

$$\begin{aligned} d'_k &= 0 \quad \text{if} \quad 1 \neq\!< \alpha_k, \\ d'_k &> 0 \quad \text{if} \quad 1 =\!< \alpha_k, \end{aligned} \tag{4.33}$$

Now let $j \in \text{below}(1)$. We have either $j \in G$, in which case $v_j = u_j \gg 0$ by Lemma 4.8, or there exists $k(j) \in \{2,\dots,q\}$ such that $1 -\!< \alpha_{k(j)} -\!< j$. In the latter case we choose a positive b'_j such that $b'_j x_j^{k(j)} - v_j \gg 0$. Define the vector

$$x^1 = v + \sum_{j \in \text{below}(1) \setminus G} b'_j x^{k(j)}. \tag{4.34}$$

By the choice of the b'_j's it follows from (4.34) that (4.2) is satisfied for $i = 1$. Also, since all the $\alpha_{k(j)}$ chosen above access 1, it follows from (4.32) and (4.33) that

$$-Ax^1 = \sum_{k=1}^{q} c_{1k} x^k,$$

where

$$\begin{aligned} c_{1k} &= 0 \quad \text{if} \quad 1 \neq\, < \alpha_k, \\ c_{1k} &> 0 \quad \text{otherwise}, \end{aligned}$$

and hence (4.4) and (4.5) are satisfied for $i = 1$. Let $j \notin \text{below}(1)$. Then j does not access any of the $\alpha_{s(t)}$ and $\alpha_{k(j)}$ discussed above, since they access 1. Thus the jth components of the corresponding basis elements are zero. In view of (4.16), it follows that (4.3) holds also for $i = 1$. Therefore, $x^1, \dots, x^q$ is an S-preferred basis for $E(A)$. ■

The following example illustrates subsets of $R(A)$ used in the proof of the preferred basis theorem.

EXAMPLE 4.35. Let A be an M-matrix, and let $R(A)$ be the graph in Figure 1, where 0 denotes a singular vertex in $R(A)$ and + denotes a

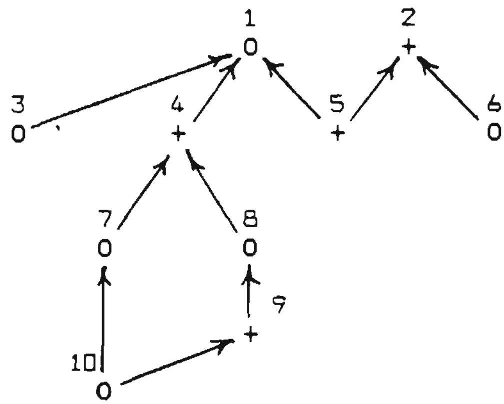


FIG. 1.

nonsingular vertex in $R(A)$. We have

$$S = \{1,3,6,7,8,10\},$$

$$\text{below}(1) = \{1,3,4,5,7,8,9,10\},$$

$$\text{hull}(1,9) = \{1,4,8,9\},$$

$$G = \{1,4,5\},$$

$$R = \{3,7,8,10\},$$

$$R_1 = \{3,7,8\},$$

$$R_2 = \{10\}.$$

DEFINITION 4.36. Let A be a square matrix, and let $x \in E(A)$. We define the ***height*** of x to be the minimal nonnegative integer k such that $A^k x = 0$.

DEFINITION 4.37. Let A be a square matrix in Frobenius normal form, and let H be a set of vertices in $R(A)$. A sequence $\alpha_1, \dots, \alpha_k$ of vertices in H is said to be an *H-chain in $R(A)$ of length k* if $\alpha_1 -\!< \cdots -\!< \alpha_k$. An S-chain in $R(A)$ is called a *singular chain in $R(A)$*.

Let A be an M-matrix. Observe that it follows from the preferred basis theorem that the maximal height of an element in an S-preferred basis of A

equals the maximal length of a singular chain in $R(A)$. Hence we obtain the index theorem for M-matrices due to Rothblum ([5]; see also [7, Corollary 7.5]), as a corollary to the preferred basis theorem.

COROLLARY 4.38. *Let A be an M-matrix. Then the index of A equals the maximal length of a singular chain in* $R(A)$.

5. NONNEGATIVE GENERALIZED EIGENVECTORS OF A Z-MATRIX

NOTATION 5.1. Let A be a Z-matrix. Denote

$$T = \{i : l(A_{ii}) < 0\},$$

$$U = S \setminus \text{above}(T),$$

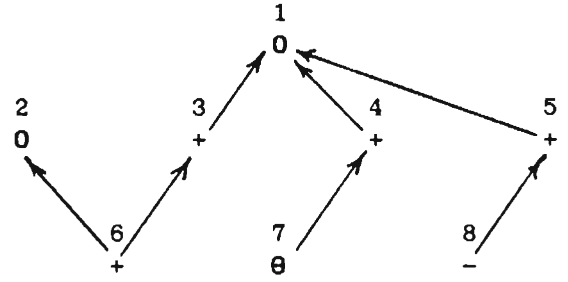
$$T_0 = \{i : l(A_{ii}) = 0\}.$$

Observe that $T_0 \subseteq S$, and that this inclusion may be proper.

To illustrate the sets discussed in Notation 5.1, consider the following example.

EXAMPLE 5.2. Let A be a Z-matrix and let $R(A)$ be given by the following graph, where

+ denotes a nonsingular M-matrix vertex,
0 denotes a singular M-matrix vertex,
$\ominus$ denotes a singular component A_{ii} with $l(A_{ii}) < 0$, and
$-$ denotes a nonsingular component A_{ii} with $l(A_{ii}) < 0$:



Here we have

$$T = \{7,8\},$$

$$\text{above}(T) = \{1,4,5,7,8\},$$

$$S = \{1,2,7\},$$

$$U = \{2\}$$

$$T_0 = \{1,2\}.$$

THEOREM 5.3. *Let A be a Z-matrix. Then for every nonnegative vector x in $E(A)$ we have*

$$\text{supp}(x) \subseteq \text{below}(U). \tag{5.4}$$

Proof. We prove our assertion by complete induction on the number p of diagonal blocks in the Frobenius normal form of A. Assume that our claim holds for $p < m$ where $m > 0$, and let $p = m$. Let x be a nonnegative vector in $E(A)$. If $T = \varnothing$, then the result follows from the preferred basis theorem (Theorem 4.14). Let $T \neq \varnothing$, and let j be the smallest integer such that $j \in T$. Let $\mu = l(A_{jj})$ and let $J = \text{above}(j)$. It follows from Theorem 4.14 applied to the matrix $A^T[J] - \mu I$ that $A^T[J]$ has a (strictly) positive eigenvector u associated with μ. By adjoining zero components to u we obtain a semipositive eigenvector v for A^T associated with μ which satisfies

$$\begin{aligned} v_h &\gg 0, \qquad h \in j, \\ v_h &= 0 \qquad \text{otherwise}. \end{aligned} \tag{5.5}$$

Let $k = \text{ind}(A)$. We have

$$\mu^k v^T x = v^T A^k x = 0. \tag{5.6}$$

Since $\mu < 0$, it follows from (5.6) that $v^T x = 0$, and hence, since $x \geqslant 0$, it follows from (5.5) that

$$\text{supp}(x) \cap J = \varnothing. \tag{5.7}$$

Observe that J is a final set of vertices of $R(A)$. Therefore, it follows from (5.7) and Lemma 3.1 that $x(J)$ is a nonnegative vector in $E(A(J))$. Let

$S' = S \setminus J$, $T' = T \setminus J$, and $U' = S' \setminus \text{above}(T')$. By the inductive assumption we have

$$\text{supp}(x(J)) \subseteq \text{below}(U'). \tag{5.8}$$

Observe that $\text{above}(T') \setminus J = \text{above}(T) \setminus J$, and hence it follows that $U' = U$. Therefore it follows from (5.7) and (5.8) that (5.4) holds. ∎

We now obtain several corollaries to Theorems 4.14 and 5.3.

COROLLARY 5.9. *Let A be a Z-matrix. Then $F(A)$ has a U-preferred basis.*

Proof. Observe that by adjoining zero components to a generalized nullvector of $A[\text{below}(U)]$ we obtain a generalized nullvector of A. Conversely, by Theorem 5.3, every nonnegative vector in $E(A)$ has its support contained in $\text{below}(U)$. Note that $A[\text{below}(U)]$ is an M-matrix and therefore by Theorem 4.14, $E(A[\text{below}(u)])$ has a U-preferred basis. Our result thus follows. ∎

In view of Corollary 5.9 we can now obtain a more complete version of Theorem 5.3.

THEOREM 5.10. *Let A be a Z-matrix. Then for every nonnegative vector x in $E(A)$ we have* $\text{supp}(x) \subseteq \text{below}(U)$. *Furthermore, there exists a nonnegative vector x in $E(A)$ such that* $\text{supp}(x) = \text{below}(U)$ *and* $x[\text{below}(U)] \gg 0$.

It follows from Corollary 5.9 that the maximal height of a vector in $F(A)$ equals the maximal length of a U-chain in $R(A)$. Thus, Corollary 5.9 yields the following lower bound for the index of a Z-matrix.

COROLLARY 5.11. *Let A be a Z-matrix. Then the index of A is greater than or equal to the maximal length of a U-chain in $R(A)$.*

An upper bound for the index of a Z-matrix is proved in [8, Proposition 2].

COROLLARY 5.12. *Let A be a Z-matrix. Then the dimension of $F(A)$ is equal to the cardinality of U.*

Proof. Immediate by Corollary 5.9. ∎

The following corollary slightly generalizes Proposition 1 in [8].

COROLLARY 5.13. *Let A be a Z-matrix. Then A has a semipositive nullvector if and only if $U \neq \emptyset$. Furthermore, there is a* bot(U)*-preferred basis for $F(A) \cap N(A)$.*

Proof. It immediately follows from Corollary 5.9 that $K(A)$ consists of the nonnegative linear combinations of vectors in a U-preferred basis for $F(A)$. But such a vector is a nullvector if and only if it is a nonnegative linear combination of the vectors x^i in this basis for which $i \in \text{bot}(U)$. This proves the first part of the corollary. The second part follows by observing that every vector in $F(A) \cap N(A)$ is the difference of two vectors in $K(A) \cap N(A)$. ■

REMARK 5.14. Observe that there is a unique (up to positive scalar multiples) bot(U)-preferred basis for $F(A) \cap N(A)$, and the elements of this basis are the extremals of the cone of nonnegative nullvectors of A.

COROLLARY 5.15. *Let A be an irreducible Z-matrix. Then $E(A)$ contains a semipositive vector if and only if $l(A) = 0$.*

Proof. Immediate by Corollary 5.13 and Theorem 4.14. ■

COROLLARY 5.16. *Let A be a Z-matrix. Then the following are equivalent:*

(i) *$E(A)$ has a nonnegative basis;*
(ii) *$E(A)$ has an S-preferred basis;*
(iii) $A[\text{below}(S)]$ *is an M-matrix;*
(iv) $S = U$.

Proof. Immediate by Corollary 5.12. ■

Since condition (iii) and (iv) in Corollary 5.16 hold trivially when A is an M-matrix, it follows that Corollary 5.16 generalizes the preferred basis theorem.

REMARK 5.17. Since for every Z-matrix we have $U = T_0 \backslash \text{above}(T)$, and since for every M-matrix we have $T_0 = S$, it follows that all our results except for Corollary 5.16 remain valid if S is replaced by T_0 everywhere, including the definition of U. However, we cannot replace S by T_0 in condition (iv) of Corollary (5.16), as is shown by the matrix

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Observe that $E(A)$ is spanned by $[1, -1]^T$, and hence $E(A)$ does not have a nonnegative basis. Indeed, $\{1\} = S \neq U = \varnothing$, but $U = T_0$.

The contents of our paper raise some natural questions concerning the cone $K(A)$ of all nonnegative vectors in $E(A)$, where A is a given Z-matrix. For example, what are the extremals of the cone? Another question suggested by Theorem (4.14) is whether this cone is invariant under multiplication by $-A$. It can be shown that the answer is positive when $\text{ind}(A) \leqslant 2$. Nevertheless, in general the answer is negative, as demonstrated by the following example.

EXAMPLE 5.18. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -1 & 0 & 3 & -3 \\ 0 & -6 & -3 & 3 \end{bmatrix}.$$

The semipositive vector $x = [3,0,2,0]^T$ belongs to $E(A)$, since $A^3x = 0$. However, $-Ax = [0,9,-3,6]$. Thus the cone $K(A)$ is not invariant under $-A$.

Indeed, the invariance of the cone $K(A)$ of all nonnegative vectors in $E(A)$ under multiplication by $-A$ is not determined solely by the graph of A, as is shown by our final example. Note that the matrix B of Example 5.19 has the same graph and the singular block structure as the matrix A of Example 5.18.

EXAMPLE 5.19. Let

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 \\ -1 & 0 & 3 & -3 \\ 0 & -1 & 3 & 3 \end{bmatrix}.$$

Then $x' = [3,0,1,0]^T$, $x^2 = [0,1,0,1]^T$, $x^3 = [0,0,1,1]^T$ is a basis for $E(B)$. Let $z \in K(B)$. By examining the supports of the vectors x^1, x^2, x^3 we easily prove that $z = d_1x^1 + d_2x^2 + d_3x^3$, where d_1 and d_2 are nonnegative. Since $Bx^1 = -9x^2$, $Bx^2 = -3x^3$, and $Bx^3 = 0$, it follows that $-Bz \in K(B)$. Hence $K(B)$ is invariant under $-B$.

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