

The Influence of the Marked Reduced Graph of a Nonnegative Matrix on the Jordan Form and on Related Properties: A Survey

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ABSTRACT

For a nonnegative matrix P , we discuss the relation of its marked reduced graph to that part of the Jordan form that is associated with the Perron-Frobenius root, to the nonnegativity of the eigenvectors and generalized eigenvectors, to the nonnegativity of solutions of linear equations, and to the asymptotic growth of powers of the matrix. Results are often stated in terms of M -matrices, and standard results on irreducible matrices are assumed. We give examples to illustrate the theorems surveyed.

1. INTRODUCTION

The graph theoretic properties of a nonnegative matrix and its algebraic (spectral) and analytic (growth) properties are intimately connected. We sketch the history of some results, though we do not necessarily state these in the manner of their original occurrence. We follow certain themes and do not give complete summaries of the papers discussed. References are appended in an order close to chronological. We take for granted standard properties of graphs and of complex matrices, irreducible nonnegative matrices, and M -matrices that can be found in many textbooks.

Let P be an (elementwise) nonnegative (square) matrix, and let $\rho = \rho(P)$ be the Perron-Frobenius root (spectral radius) of P . Let λ be a real number. We are principally concerned with the relation of each of the following topics to the (marked reduced) graph of P .

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Area I. The positivity properties of the eigenvectors and generalized eigenvectors of P belonging to an eigenvalue λ , especially for $\lambda = \rho$.

Area II. If b is a nonnegative vector, the nonnegative solutions x of $(\lambda I - P)x = b$.

Area III. The Jordan blocks associated with ρ in the Jordan form of P .

Area IV. The growth (asymptotic behavior) of the elements of P^m as m grows.

As is common, many results are stated in terms of the associated M -matrix $A = \rho I - P$.

The original and fundamental contribution to this subject is that of Frobenius in [F12, Section 11]. While other parts of his paper are now among the standard results on nonnegative matrices found in many texts, this is not true of the results of Section 11 on the existence of nonnegative eigenvectors. There seem to have been no references to these results for 40 years after publication, and even today they and their subsequent development are largely unfamiliar except to specialists. Perhaps this may have occurred because Frobenius did not state the results formally as theorems, and did not formulate them with his usual clarity. This would require the use of graphs or some equivalent concept as, for example, in our Section 3 below. Or perhaps the lack of familiarity is due to the apparently technical nature of the results and their proofs. In spite of its technical appearance, the method developed by Frobenius, which we call the trace down method, and which has remained a basic tool for proofs in the first three areas of this subject listed above, rests on a trivial lemma: see [S56, Lemma 2]. Once the trace down method is grasped, many results become intuitively clear and indeed beautiful. We discuss this no further, and we give no proofs in this survey.

We now describe the contents of our paper in more detail. We require many definitions, and therefore we have put those definitions necessary throughout the paper in Section 2 and postponed the others to Sections 5 and 9. Sections 3, 4, and 9 do not use the definitions of Section 5.

Section 3 contains results on nonnegative eigenvectors of a nonnegative matrix P . In Section 4 we discuss nonnegative solutions of $(\lambda I - P)x = b$, where λ is a real number and b is a nonnegative vector. In Section 6 we turn to the fascinating subject of the relation of the singular graph of P to the Jordan form associated with the Perron-Frobenius root of P , a topic to which we return in Section 8. This topic is closely related to the existence of nonnegative eigenvectors, nonnegative generalized eigenvectors, and nonnegative Jordan chains, which forms the material of Section 7. In Section 9 we turn to the growth of the elements of powers of a nonnegative matrix. There is a list of papers discussed at the end of each section.

We have attempted to present results in historical order, but have deviated from this when necessary to unify our exposition and to state results

in their current form. Also we have reversed the historical order of much of Sections 4 and 6 so as to place the latter close to subsequent developments described in Sections 7 and 8.

2. DEFINITIONS AND NOTATION

We begin with some concepts from combinatorial matrix theory that lead to the Frobenius normal form and to the reduced graph. Let $\langle n \rangle = \{1, \dots, n\}$. Let A be an $n \times n$ matrix with entries in some field. As usual, we define the (directed) *graph of A* to be the graph $G(A)$ with vertices $1, \dots, n$ where (i, j) is an arc if and only if $a_{ij} \neq 0$. The graph is *strongly connected* if either it has only one vertex or there is a path in $G(A)$ from i to j , for all $i, j \in \langle n \rangle$, and in this case the matrix A is called *irreducible*. A *strong component* of $G(A)$ is a maximal strongly connected subgraph of $G(A)$. We index the strong components of $G(A)$ by $1, \dots, p$. We now define a partial order on the set of strong components of $G(A)$ —identified with $\langle p \rangle$ —which we call the *reduced graph $R(A)$* of A . Let $i, j \in \langle p \rangle$. We let $i = < j$ ($j = > i$) if and only if $i = j$ or there is a path in $G(A)$ from a vertex of the j th strong component of $G(A)$ to a vertex of the i th strong component. We also say that j *has access to i* (or i is accessed from j) in $R(A)$ if $i = < j$. We write $i - < j$ for $i = < j$ but $i \neq j$. (Think: i is less than j if $i - < j$.) A *chain* of length t of $R(A)$ is a sequence of t vertices (i_1, \dots, i_t) such that

$$(2.1) \quad i_1 > \dots > i_t.$$

We may renumber the strong components so that $i = < j$ implies that $i \leq j$ but not (necessarily) conversely. Without loss of generality, we assume that this has been done. Our renumbering corresponds to a permutation similarity applied to the matrix A that puts A into the familiar (lower triangular) *Frobenius normal form*

$$(2.2) \quad A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix},$$

where the diagonal blocks A_{11}, \dots, A_{pp} are irreducible. Our use of the partial order $R(A)$ allows us to state results in a form that is essentially invariant under permutation similarity of A , while at the same time making use of the Frobenius normal form to partition matrices and vectors. It should be noted

that in general certain permutation similarities may be applied to a matrix in Frobenius normal form that keep the matrix in this form. Some additional insights may be obtained into the results below by considering permutation similarities; see [S56] or [RiS78] for details. Most of the above may be found in [Ro75], [RiS78] and in many other papers and books.

(2.3) CONVENTION. We shall always assume that a matrix is given (partitioned) in Frobenius normal form (2.2).

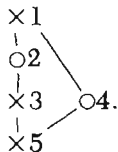
Let b be a column vector with n entries. We partition b conformably with A into vector components $b^T = (b_1^T, \dots, b_p^T)$. Then we define

$$\text{supp}(b) = \{i \in \langle p \rangle : b_i \neq 0\}.$$

(2.4) EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -a & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -b & -c & -1 & -1 & 1 \end{bmatrix},$$

where a, b, c , are nonnegative (possibly 0). Then the partial order $R(A)$ is the reflexive transitive closure of the following diagram. We draw “skeletons” throughout, and arcs are directed upwards. The meaning of \times and \circ will become clear presently.



The above example can be generalized without affecting $R(A)$. Thus the entries can be replaced by matrices of appropriate size, where the blocks on the diagonal are irreducible, and the off-diagonal blocks are zero or nonzero according as the corresponding entry is zero or nonzero.

Next, we explain our terminology and notation for nonnegative matrices.

A matrix $P \in \mathbb{R}^{n \times n}$ (the set of $n \times n$ matrices with real entries) will be called *nonnegative* ($P \geq 0$) if all its entries are nonnegative, *semipositive* if $P \geq 0$ but $P \neq 0$, and *strictly positive* ($P \gg 0$) if all its entries are positive.

Similar terminology and notation will be used for vectors in \mathbb{R}^n . If P is a nonnegative (square) matrix, we denote by $\rho = \rho(P)$ the spectral radius of P (its Perron-Frobenius root). A Z -matrix is a matrix of form $A = \lambda I - P$, where P is nonnegative, and a Z -matrix A is an M -matrix if $\lambda \geq \rho(P)$. It is often convenient to state results on a nonnegative matrix P in terms of the associated singular M -matrix $A = \rho(P)I - P$. The matrix of Example (2.4) is an M -matrix.

We now mark the reduced graph. Let P be a nonnegative matrix. We put $\rho_i = \rho_i(P) = \rho(P_{ii})$, $i = 1, \dots, p$. We often identify the vertex i with the block P_{ii} . Thus we call the vertex i of $R(A)$ a λ -vertex if $\rho_i = \lambda$. The graph $R(A)$ with each vertex marked by the corresponding Frobenius root is called the *marked reduced graph* of P , and is also denoted by $R(P)$. If A is the associated M -matrix $\rho I - P$, then a ρ -vertex of $R(P)$ is called a *singular vertex* of $R(A)$. But since $R(P) = R(A)$, we may also call a ρ -vertex of $R(P)$ a singular vertex. Ambiguity arises (only) for nonnegative diagonal matrices, but that causes no difficulty. In Example (2.4) each singular vertex of $R(A)$ is indicated by an \circ , and every other (or nonsingular) vertex by an \times , a convention we follow throughout.

We call a vertex i of $R(A)$ *final (initial)* if there is no j in $R(A)$ such that $j \rightarrow i$ ($j \leftarrow i$). In view of Convention (2.3), 1 is always a final vertex, and p an initial vertex of $R(A)$. The blocks A_{ii} corresponding to final i in $R(A)$ are also called *row isolated*: see [S53].

Let P be a nonnegative matrix. A vertex i of $R(P)$, or of the reduced graph of an associated Z -matrix $A = \lambda I - P$, is called a *distinguished vertex* if $\rho_i > \rho_j$ whenever $i \rightarrow j$ in $R(P)$. Thus a singular vertex i of an M -matrix is distinguished if and only if j is a nonsingular vertex whenever $i \rightarrow j$.

Further definitions will be found in Sections 5 and 9.

3. SEMIPOSITIVE EIGENVECTORS

Many results in this section are closely related to [F12, Section 11], though none were stated there in the form below, as we explained in the introduction. Our first theorem below generalizes the familiar result from [F12] that an irreducible nonnegative matrix has a strictly positive eigenvector associated with its Perron-Frobenius root. Note that the marked reduced graph of a nonnegative matrix has at least one distinguished singular vertex, and therefore Theorem (3.1) guarantees the existence of a semipositive eigenvector for the Perron-Frobenius root.

(3.1) THEOREM. *Let A be an M -matrix, and let $\delta_1 < \dots < \delta_r$ be the distinguished singular vertices of $R(A)$.*

(i) *There exist (up to scalar multiples) unique vectors x^1, \dots, x^r such that $Ax^i = 0$ and*

$$(3.2) \quad x_j^i \begin{cases} \gg 0 & \text{if } \delta_i = \langle j, \\ = 0 & \text{otherwise} \end{cases}$$

for $j = 1, \dots, p$ and $i = 1, \dots, r$.

(ii) *Every nonnegative vector x satisfying $Ax = 0$ is a linear combination with nonnegative coefficients of x^1, \dots, x^r .*

The first part of this theorem is contained in [S52] and [S56, Theorem 2], and part (ii) was added in [Ca63, Theorem 2].

We shall illustrate our results by means of very simple examples (plus a few others). If A is a singular M -matrix with two diagonal blocks in its Frobenius normal form, then five (nonisomorphic) possibilities arise for $R(A)$ if we distinguish only between singular and nonsingular vertices. We shall give examples of 2×2 singular M -matrices with graphs corresponding to four of these graphs in Examples (3.3), (3.4), (3.6), and (4.11). (The 2×2 matrix 0 gives the fifth possibility.) These matrices illustrate Theorem (3.1) and other results of Sections 3 and 4, as do the 3×3 matrices of Examples (7.7) and (7.8) and the 4×4 matrix of Example (6.6).

(3.3) **EXAMPLE.** Let A be the M -matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

with $R(A)$ given by

$$\circ 1 \quad \times 2.$$

Then 1 is the unique (distinguished) singular vertex of $R(A)$, and

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is the unique eigenvector belonging to the eigenvalue 0. Note that $1 = \langle 1$, but 1 is not accessed by 2.

(3.4) **EXAMPLE.** Let

$$A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

with $R(A)$



Then 1 and 2 are singular vertices, but only 2 is a distinguished singular vertex. Note that

$$x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the unique (semipositive) eigenvector belonging to 0.

In Examples (3.3) and (3.4) it so happens that the nullspace of A has a basis of semipositive vectors, but this is false for M -matrices in general: see Examples (7.7) and (7.10).

Our next result is an application of Theorem (3.1) concerning the existence of a strictly positive eigenvector of a nonnegative matrix P . In view of Theorem (3.7) below and the remark immediately following it, this can happen only for the eigenvalue $\rho(P)$. An equivalent result is proved in [G59, Vol. II, p. 77], which is formulated in terms of the existence of a certain permutation similarity applied to the Frobenius normal form of P ; for the present formulation see [Ro79]. Again, we state the theorem in terms of an M -matrix.

(3.5) COROLLARY. *Let A be an M -matrix. Then the following are equivalent:*

- (a) *There exists a vector x such that $x \gg 0$ and $Ax = 0$.*
- (b) *The set of singular vertices of $R(A)$ is equal to the set of final vertices of $R(A)$.*

Observe that a stochastic matrix P (a nonnegative matrix with all row sums equal to 1) has eigenvector $(1, 1, \dots, 1)^T$ for the eigenvalue 1. Hence by Corollary (3.5), P satisfies (3.5)(b), as is well known.

(3.6) EXAMPLE. Let

$$A = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix},$$

with $R(A)$



Evidently, if $x = (1, 1)^T$, then $x \gg 0$ and $Ax = 0$. Also 1 is the only singular vertex and the only final vertex of $R(A)$, and so (3.5)(b) holds.

We end this section with a generalization of Theorem (3.1) that is found in [V85, Lemma 1]. However, most of Section 11 of [F12] is devoted to what may, with hindsight, be regarded as its proof. Thus Theorem (3.7) may be called both the oldest and currently (in 1986) the newest result in this area, depending on taste.

(3.7) THEOREM. *Let P be a nonnegative matrix (in Frobenius normal form). Let λ be a real number.*

(i) *The following are equivalent:*

(a) *There exists an eigenvector vector x such that*

$$(3.8) \quad Px = \lambda x, \quad x \text{ semipositive.}$$

(b) *There is a distinguished vertex i of $R(P)$ such that*

$$(3.9) \quad \rho_i = \lambda.$$

(ii) *If i is a distinguished vertex satisfying (3.9), then there is a (up to scalar multiples) unique vector x that satisfies (3.8) and*

$$(3.10) \quad x_j \begin{cases} \gg 0 & \text{if } i \leq j, \\ = 0 & \text{otherwise.} \end{cases}$$

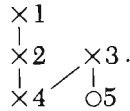
Thus in (3.10), $\text{supp}(x)$ is the set of vertices with access to i .

A third part may be added to this theorem corresponding to Theorem (3.1)(ii): Every nonnegative eigenvector of P is a linear combination with nonnegative coefficients of the eigenvectors determined by the distinguished vertices in Theorem (3.7)(ii).

(3.11) EXAMPLE. Let

$$P = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 1 & 1 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 & 6 \end{bmatrix}.$$

Then $R(P)$ is (the transitive closure of)



Then 1 is a distinguished 5-vertex but 3 is not. Thus the unique semipositive eigenvector for the eigenvalue 5 is $x^1 = (5, 1, 0, 6, 0)^T$. Note that $(0, 0, 1, 1, -1)^T$ is also an eigenvector for this eigenvalue.

Papers discussed: [F12], [S52], [S56], [G59], [Ca63], [Ro79], [V85].

4. NONNEGATIVE SOLUTIONS OF LINEAR EQUATIONS

The following result is stated in [S53] under the assumption that A is an M -matrix. It generalizes the well-known result that for a nonsingular or irreducible singular M -matrix A there exists a vector $x \gg 0$ such that $Ax \geq 0$.

(4.1) THEOREM. *Let A be a Z -matrix. Then the following are equivalent:*

- (a) *There exists an $x \gg 0$ such that $Ax \geq 0$.*
- (b) *A is an M -matrix and every singular vertex of $R(A)$ is final.*

For an M -matrix, Condition (4.1)(b) is weaker than condition (3.5)(b), as is shown by the next example.

(4.2) EXAMPLE. Let A be the M -matrix of Example (3.3). Then there is no $x \gg 0$ such that $Ax = 0$, but for $x = (1, 1)^T$ we have $Ax \geq 0$. Observe that 1 and 2 are final vertices of $R(A)$, but that 1 is the unique singular vertex. Hence (4.1)(b) holds, but (3.5)(b) does not.

Our next theorem is a basic result in this area. It is due to Carlson [Ca63, Theorems 1 and 2]; see also [FrS80, Theorem 7.1].

(4.3) THEOREM. *Let A be an M -matrix and let b be a nonnegative vector.*

- (i) *The following are equivalent:*
 - (a) *There is a vector x such that*

$$(4.4) \quad Ax = b, \quad x \geq 0.$$

- (b) *No vertex in $\text{supp}(b)$ is accessed by a singular vertex of $R(A)$.*

(ii) If (4.3)(b) holds, then there is a unique $x = x^0$ that satisfies (4.4) and

$$(4.5) \quad x_j^0 \begin{cases} \geq 0 & \text{when } i = < j \text{ for some } i \in \text{supp}(b), \\ = 0 & \text{otherwise.} \end{cases}$$

(iii) The vector x satisfies (4.4) if and only if x is the sum of x^0 satisfying (4.5) and a linear combination with nonnegative coefficients of the vectors x^1, \dots, x^r of Theorem (3.1).

Note that Theorem (4.3) allows the solution $x = 0$ when $b = 0$, in contrast to Theorem (3.1)(i). The vector x^0 obtained in (4.3)(ii) is the nonnegative vector of minimal support satisfying $Ax = b$. One obtains a vector x of maximal support satisfying (4.4) if all the nonnegative coefficients in the third part of the theorem are chosen positive.

(4.6) EXAMPLE. Let A be the M -matrix of Example (2.4). If $b \geq 0$, then $Ax = b$ has a nonnegative solution x if and only if $\text{supp}(b) \subseteq \{3, 5\}$. For example, if $b = (0, 0, 1, 0, 1)^T$ then the vector x^0 that satisfies (4.4) and (4.5) is $x^0 = (0, 0, 1, 0, 2)^T$. The general nonnegative solution is $x = x^0 + d_1 x^1 + d_2 x^2$, where $x^1 = (0, 1, 1, 0, 1 + c)$, $x^2 = (0, 0, 0, 1, 1)$, and $d_i \geq 0$ for $i = 1, 2$.

Note that Theorem (4.3) characterizes only the nonnegative solutions of $Ax = b$, where $b \geq 0$. There may be other solutions: see our next example.

(4.7) EXAMPLE [FrS80]. Let A be the M -matrix of Example (3.4), and let

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then there exists no $x \geq 0$ such that $Ax = b$. Note that the vertex 2 is singular and belongs to $\text{supp}(b)$, and hence, as required by Theorem (4.3), condition (4.3)(b) does not hold. But $Az = b$, where $z = (-1, 0)^T$.

Our next two results may not have been stated previously. They are dual to Corollary (3.5) and Theorem (4.1), respectively. Corollary (4.8) is a simple application of Theorem (4.3), but some results of Section 7 are used in the proof of Theorem (4.9).

(4.8) COROLLARY. Let A be an M -matrix and let z be a vector. Then the following are equivalent:

- (a) $z \geq 0$ and $Az \geq 0$ imply that $Az = 0$.
- (b) Every initial vertex of $R(A)$ is singular.

(4.9) THEOREM. *Let A be an M -matrix and let z be a vector. Then the following are equivalent:*

- (a) $Az \geq 0$ implies that $Az = 0$.
- (b) *The set of initial vertices of $R(A)$ is equal to the set of singular vertices of $R(A)$.*

(4.10) EXAMPLE. Let A be the matrix of Example (3.4). Then A satisfies (4.8)(b) but not (4.9)(b). As observed in Example (4.7), there exists a vector z such that Az is semipositive, but $x \geq 0$ and $Ax \geq 0$ imply that x is a multiple of $(0, 1)^T$. Hence $Ax = 0$.

(4.11) EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

with $R(A)$

$$\begin{array}{c} \times 1 \\ | \\ \circ 1 \end{array}.$$

Then (4.9)(b) holds. If $Ax \geq 0$, then x is a multiple of $(0, 1)^T$ and hence $Ax = 0$.

We remark that an irreducible singular M -matrix has $R(A)$

$$\circ 1$$

and hence trivially satisfies all of the following conditions: (3.5)(b), (4.1)(b), (4.8)(b), and (4.9)(b).

We again end this section with a recent result due to Victory [V85, Theorem 1]. Theorem (4.12) generalizes Carlson's Theorem (4.3).

(4.12) THEOREM. *Let P be a nonnegative matrix, and let b be a nonnegative vector. Let λ be a real number.*

(i) *The following are equivalent:*

- (a) *There is a vector x such that*

$$(4.13) \quad (\lambda I - P)x = b, \quad x \geq 0.$$

- (b) *No vertex in $\text{supp}(b)$ is accessed by a vertex j in $R(A)$ for which $\rho_j \geq \lambda$.*

(ii) *If (4.12)(b) holds, then there is a unique $x = x^0$ that satisfies (4.13) and (4.5).*

A third part may be added to Theorem (4.12) along the lines of Theorem (4.3)(iii).

(4.14) **EXAMPLE.** Let P be the matrix of Example (3.11), and let $b \geq 0$. Let $\lambda = 5$. Then there is a vector x satisfying (4.13) if and only if the vertices 1,3,5 do not have access to any vertex of $\text{supp}(b)$, viz. if and only if $\text{supp}(b) \subseteq \{2,4\}$. Suppose $b = (0,5,0,1,0)^T$. Then $x^0 = (0,1,0,2,0)^T$ satisfies $Ax^0 = b$ and (4.5). The general nonnegative solution of $Ax = b$ is $x = x^0 + dx^1$, where x^1 is given in Example (3.11) and $d \geq 0$.

Further interesting examples may be found in [V85].

Papers discussed: [S53], [Ca63], [FrS80], [V85].

5. FURTHER DEFINITIONS AND NOTATION

We present our terminology for spectral properties of a matrix. Since we shall confine ourselves to an M -matrix A and consider only the eigenvalue 0, we shall usually omit mention of this eigenvalue.

Let A be an M -matrix. Suppose the Jordan blocks of A (associated with the eigenvalue 0) have sizes $\sigma_1, \dots, \sigma_s$, where $\sigma_1 \geq \dots \geq \sigma_s > 0$. Then the *Segré characteristic* of A is the sequence $(\sigma_1, \dots, \sigma_s)$. The *Jordan diagram* of A (for 0), denoted by $J(A)$, is the diagram formed by s columns of stars such that the j th column (from the left) has σ_j stars. The sequence $(\omega_1, \dots, \omega_u)$ of row lengths of $J(A)$ (read upwards) is called the *Weyr characteristic* of A . Note that $\omega_1 \geq \dots \geq \omega_u > 0$. As is well known, $\omega_1 + \dots + \omega_k$ is the dimension of the nullspace of A^k , $k = 1, \dots, u$. The diagram $J(A)$ is nonempty if and only if 0 is an eigenvalue of A .

The *index* of (0 in) A , denoted by $\text{ind}(A)$, is the first term σ_1 of the Segré characteristic, viz. the size of the largest Jordan block (and therefore the *height* of the Jordan diagram). Thus $\text{ind}(A) > 0$ if and only if 0 is eigenvalue of A ; otherwise $\text{ind}(A) = 0$.

(5.1) **EXAMPLE.** Suppose there are 4 Jordan blocks belonging to the eigenvalue 0 of A of sizes 3,2,2,1. Then $J(A)$ is



The Segré characteristic of A is $(3,2,2,1)$, and the Weyr characteristic is $(4,3,1)$. Further, $\text{ind}(A) = 3$.

We denote the nullspace of A by $\text{Ker } A$. The *generalized eigenspace* $E(A)$ of A is the nullspace $\text{Ker}(A^u)$, where $u = \text{ind}(A)$. A *Jordan chain* for A of length k is a sequence of k nonzero vectors $x, (-A)x, \dots, (-A)^{k-1}x$, such that $(-A)^kx = 0$. A *Jordan basis* for $E(A)$ is basis for $E(A)$ consisting of unions of Jordan chains for 0. A Jordan basis (Jordan chain) is called *semipositive* if each vector in the basis (chain) is semipositive.

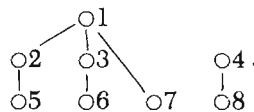
Next, we introduce more graph theoretic concepts. We denote by $\alpha_1, \dots, \alpha_q$, where $\alpha_1 < \dots < \alpha_q$, the singular vertices of $R(A)$ defined in Section 2. The *singular graph* $S(A)$ has vertex set $\{\alpha_1, \dots, \alpha_q\}$ with partial order induced by $R(A)$, viz., $\alpha_i = < \alpha_j$ in $S(A)$ if and only if the same relation holds in $R(A)$.

We call $S(A)$ a *rooted forest* if for each vertex j of $S(A)$ the set of vertices i such that $i = < j$ is linearly ordered. (In our diagrams forests appear to grow downwards with roots at the top.)

We now arrange $S(A)$ according to *level*, viz., on the lowest row Λ_1 we put the maximal elements of $S(A)$ (in the $=<$ order). Note that the elements of Λ_1 are the distinguished singular vertices of $R(A)$. In the next row up the maximal elements of $S(A) \setminus \Lambda_1$, and so on inductively. We then erase the arrows and call the resulting diagram the *level diagram* $S_*(A)$ —i.e., there is no graph structure or partial order in $S_*(A)$. We shall assume that the levels of $S_*(A)$ are indexed by $1, \dots, h$, and we call h the *height* of S_* . [Thus levels are counted upwards up in our diagrams. Also, the height of $S_*(A)$ is the maximal length of a chain of singular vertices in $R(A)$.] We shall denote the cardinality of Λ_k by ν_k for $k = 1, \dots, h$, and we call the sequence (ν_1, \dots, ν_h) of row lengths of $S_*(A)$ the *level characteristic* of A .

We define a *successor operator* Δ on $S(A)$. Suppose that $2 \leq k \leq h$ and that Q is a subset of Λ_k . Then $\Delta(Q)$ consists of all vertices j that belong to Λ_{k-1} such that $i = < j$ in $S(A)$ for some $i \in Q$.

(5.2) EXAMPLE. Suppose $S(A)$ is



Note that $S(A)$ is a rooted forest. Further $S_*(A)$ is the diagram of Example (5.1). We have $\Lambda_1 = \{5, 6, 7, 8\}$, $\Lambda_2 = \{2, 3, 4\}$, $\Lambda_3 = \{1\}$. Thus $\Delta(1) = \{2, 3\}$,

but $7 \notin \Delta(1)$, and $\Delta(2,3) = \{5,6\}$. Note that the height of $S_*(A)$ is $h = 3$. The level characteristic of A is $(4,3,1)$.

6. THE RELATION BETWEEN THE JORDAN AND LEVEL DIAGRAMS—PART I

In this section we state some results from [S52] and [S56]. Their formulation in terms of singular graphs, level diagrams, and Jordan diagrams is natural. They were not so stated in [S56], where graphs do not appear explicitly, though constructions easily seen to be equivalent do occur.

(6.1) THEOREM [S56, Theorem 3]. *Let A be a M -matrix. Then the following are equivalent:*

(a) *The Jordan blocks (for the eigenvalue 0) are all of size 1, i.e., the Segré characteristic is $(1, \dots, 1)$.*

(b) *No singular vertex in $R(A)$ has access to any other singular vertex, i.e., the singular graph $S(A)$ is trivially ordered.*

Thus the level diagram $S_*(A)$ is of form

$$* \quad * \quad \dots \quad *$$

if and only if the Jordan diagram $J(A)$ is of the same form.

If all singular vertices of $R(A)$ are final (or initial), then condition (6.1)(b) holds. Hence Theorem (6.1) generalizes the well-known result that for a stochastic matrix all Jordan blocks for the eigenvalue 1 are of size 1; see the remarks following Corollary (3.5). Further, if $S(A)$ is trivially ordered, then all singular vertices of $R(A)$ are distinguished and by Theorem (3.1), $\text{Ker}(A)$ has as a basis the semipositive vectors x^k , $k = 1, \dots, q$, which satisfy (3.2). The eigenprojection corresponding to 0 (the projection on the range of A that annihilates the nullspace) is semipositive [S56, Theorem 4].

(6.2) EXAMPLE. Let A be the matrix of Example (2.4). Then $S(A)$ is

$$\circ 2 \quad \circ 4,$$

and hence $S_*(A)$ and $J(A)$ are

$$* \quad * .$$

(6.3) THEOREM [S56, Theorem 5]. *Let A be an M -matrix. Then the following are equivalent:*

(a) *There is at most one Jordan block for the eigenvalue 0, i.e., the Weyr characteristic is $(1, \dots, 1)$.*

(b) *The singular graph $S(A)$ is linearly ordered.*

In other words, the level diagram $S_*(A)$ is of form



if and only if the Jordan diagram is of the same form.

When $S(A)$ is linearly ordered we can choose a semipositive Jordan basis (a Jordan chain) for $E(A)$; see [S52, p. 182] and [S56, Theorem 6].

(6.4) EXAMPLE. Let A be the M -matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ -b & -u & 1 & 0 & 0 \\ -c & -v & -w & 0 & 0 \\ -d & -e & -f & -g & 1 \end{bmatrix},$$

where all entries below the main diagonal are nonpositive and where either $uw > 0$ or $v > 0$. Then $S(A)$ is



Hence $S_*(A)$ and $J(A)$ are



Thus there is just one Jordan block (for 0), and it must be 2×2 .

Conditions (6.1)(b) and (6.3)(b) imply that

(6.5)
$$S_*(A) = J(A).$$

Now suppose $S(A)$ has q elements. In [S56] it is shown that if $q \leq 3$, then $S_*(A)$ [and hence $S(A)$] determines $J(A)$. In fact, for such q , (6.5) holds

unless $S_*(A)$ is

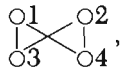
$$\begin{matrix} * & * \\ * & \end{matrix},$$

in which case $J(A) = S_*(A^T)$. See Examples (7.7) and (7.8) below for the cases when $q = 3$ that are not covered by Theorem (6.1) and (6.3). Observe that the size of $R(A)$ is not involved. We now give an example from [S56] (with a misprint corrected) to show that for $q = 4$ there exist two M -matrices with $S(A) = S(B)$, yet $J(A) \neq J(B)$.

(6.6) EXAMPLE. Let

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -a & -1 & 0 & 0 \end{bmatrix},$$

where $a > 0$. Then $S(A)$ is



and hence $S_*(A)$ is

$$\begin{matrix} * & * \\ * & * \end{matrix}.$$

If $a \neq 1$, then there are two Jordan blocks of size 2, and hence $J(A) = S_*(A)$. But for $a = 1$, the Segré characteristic is $(2, 1, 1)$ and hence we have an exceptional case of

$$\begin{matrix} * \\ * & * & * \end{matrix}$$

for $J(A)$.

Thus in general $S(A)$ does not determine $J(A)$. Theorems (6.1) and (6.3) and Example (6.6) raise a central question:

QUESTION (6.7). What is the relation of $S(A)$ to $J(A)$?

More specifically:

QUESTION (6.8). When does the equality (6.5) hold?

For the equality (6.5) one clearly requires that the row lengths of $S_*(A)$ be nonincreasing in an upward direction, since this always holds for $J(A)$. Questions (6.7) and (6.8) are still partly open, though considerable progress has been made in the last 30 years, as will be shown in Sections 7 and 8, which are devoted to results related to these questions and certain closely associated nonnegativity properties. There we shall state generalizations of Theorems (6.1) and (6.3) and of the other results of this section.

Papers discussed: [S52]/[S56]

7. SEMIPOSITIVE BASES

(7.1) THEOREM. *Let A be an M -matrix.*

Let $\alpha_1 < \dots < \alpha_q$ denote the singular vertices of $R(A)$.

(i) *The generalized eigenspace $E(A)$ (for the eigenvalue 0) has a semi-positive basis of vectors x^1, \dots, x^q such that*

$$(7.2) \quad x_j^i \begin{cases} \gg 0 & \text{if } \alpha_i = \langle j, \\ = 0 & \text{otherwise} \end{cases}$$

for $j = 1, \dots, p, i = 1, \dots, q$, and every set of vectors in $E(A)$ that satisfies (7.2) is a basis for $E(A)$.

(ii) *There exists a semipositive basis for $E(A)$ that satisfies both (7.2) and*

$$(7.3) \quad c_{ik} \begin{cases} > 0 & \text{if } \alpha_i = \langle \alpha_k, \\ = 0 & \text{otherwise,} \end{cases}$$

$i, k = 1, \dots, q$, where the c_{ik} are determined by

$$(7.4) \quad -Ax^i = \sum_{k \in \langle q \rangle} c_{ik} x^k, \quad i = 1, \dots, q.$$

The nonnegative basis theorem, Theorem (7.1)(i), is due to Rothblum [Ro75, Theorem 3.1, Part 1]. The additional precision of (7.1)(ii) was added in [Ris78, Theorem 6.2]. We then obtain as corollaries the next two results, which are also contained in [Ro75, Theorem 3.1, Parts 2 and 3].

(7.5) COROLLARY. *The index $\text{ind}(A)$ (of the eigenvalue 0 of A) is equal to the height h of $S_*(A)$, viz. to the length of the longest chain of singular vertices in $R(A)$.*

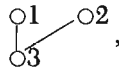
Corollary (7.5) is usually called Rothblum's index theorem. It implies Theorems (6.1) and (6.3).

(7.6) COROLLARY. *There is a semipositive Jordan chain for A of h vectors.*

(7.7) EXAMPLE. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

Then $S(A)$ is



and $S_*(A)$ is



while $J(A)$ is



The three unit vectors form a basis of semipositive vectors for $E(A) = \mathbb{R}^3$; but these do not satisfy (7.2) or (7.3). However, the vectors

$$x^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad x^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

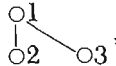
form a basis for $E(A)$ that satisfies (7.2) and (7.3). Note that $-Ax^1 = -Ax^2 = x^3$. The vectors (x^1, x^3) form a Jordan chain of length $h = 2$. Evidently $\text{ind}(A) = 2$.

Observe further that $\dim \text{Ker}(A) = 2$, while the number of elements of L_1 is equal to 1. Also note that a basis for $\text{Ker}(A)$ is given by x^3 and $(1, -1, 0)^T$. Thus there cannot be a semipositive basis for $\text{Ker}(A)$ or, *a fortiori*, a semipositive Jordan basis.

(7.8) EXAMPLE. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Then $S(A)$ is



while $S_*(A)$ and $J(A)$ are



The three unit vectors form a basis of semipositive vectors for $E(A) = \mathbb{R}^3$; but again these do not satisfy (7.2). However, the vectors

$$x^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

form a basis for $E(A)$ that satisfies (7.2) and (7.3), since $(-A)x^1 = x^2 + x^3$. Indeed, in the present case there is even a semipositive Jordan basis [that cannot satisfy (7.3)], viz. the two chains $(x^1, x^2 + x^3)$ and (x^3) . Note also that $\text{Ker}(A)$ has a semipositive basis, viz. (x^2, x^3) .

The contrast between Examples (7.7) and (7.8) is explained by the following result, which guarantees the existence of a semipositive Jordan basis in the case of Example (7.8) and shows why it cannot exist in the case of Example (7.7).

(7.9) THEOREM [RiS78, Theorem 6.5]. *Let A be an M -matrix. Then the following are equivalent:*

- (a) *There exists a semipositive Jordan basis for the generalized eigenspace $E(A)$.*
- (b) *For $k = 1, \dots, h$, $\text{Ker}(A^k)$ has a semipositive basis.*
- (c) $S_*(A) = J(A)$.

(7.10) EXAMPLE. Let A be the matrix of Example (6.6).

(i) For all $a > 0$, a semipositive basis for $E(A)$ that satisfies (7.2) and (7.3) is given by

$$x^1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad x^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad x^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(ii) For all $a > 0$ we have $\text{ind}(A) = 2$, as shown by the Jordan diagrams in Example (6.6). But the height h of $S_*(A)$ is also 2. This verifies Corollary (7.5) for this example. Further, $(x^1, x^3 + ax^4)$ is a semipositive Jordan chain of length 2 whose existence is required by Corollary (7.6), and $(x^2, x^3 + x^4)$ is another such chain.

(iii) Let $a \neq 1$. Then a semipositive Jordan basis for $E(A)$ may be obtained by adjoining the two Jordan chains mentioned in (ii). Evidently (x^3, x^4) is a semipositive basis for $\text{Ker}(A)$ and, as noted in Example (6.6), $S_*(A) = J(A)$. Thus the equivalent conditions (7.9)(a), (7.9)(b) and (7.9)(c) all hold in this case.

(iv) Now let $a = 1$. Then a basis for $\text{Ker}(A)$ is given by (x^3, x^4, z) , where $z^T = (-1, 1, 0, 0)$. Thus, in this case, there cannot be a semipositive basis for $\text{Ker}(A)$. It follows that neither (7.9)(a) nor (7.9)(b) can hold. As noted in Example (6.6), condition (7.9)(c) does not hold.

Papers discussed: [Ro75], [RiS78].

8. THE RELATION BETWEEN THE JORDAN AND LEVEL DIAGRAMS—PART II

We continue the discussion of Questions (6.7) and (6.8) concerning the relation of $S_*(A)$ and $J(A)$ for an M -matrix A . The following majorization result on the row lengths of $S_*(A)$ and $J(A)$ is found in [RiS78, Corollary (4.5)]. It can be generalized to other classes of matrices.

(8.1) THEOREM. Let A be an M -matrix, and suppose that (ν_1, \dots, ν_h) and $(\omega_1, \dots, \omega_h)$ are respectively the level and Weyr characteristics of A . Then

- (i) $\nu_1 + \dots + \nu_k \leq \omega_1 + \dots + \omega_k, \quad k = 1, \dots, h - 1,$
- (ii) $\nu_1 + \dots + \nu_h = \omega_1 + \dots + \omega_h.$

In the statement of Theorem (8.1) we have implicitly used the index theorem, Corollary (7.5).

(8.2) **EXAMPLE.** If A is the matrix of Example (7.7), then the level and Weyr characteristics are respectively (1,2) and (2,1). For the matrix of Example (6.6) the level characteristic is (2,2) and the Weyr characteristic is (2,2) or (3,1), depending on the parameter a . Thus Theorem (8.1) holds for these matrices.

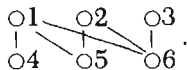
Given an M -matrix A , necessary and sufficient conditions are known for the equality (6.5) and may be found in [RiS78, Theorem 4.7]. There, a strictly lower triangular M -matrix C such that $S(C) = S(A)$ and $J(C) = J(A)$ is associated with A , and the conditions are stated in terms of ranks of certain submatrices of C . (In fact, C may be chosen as the transpose of the $q \times q$ matrix of coefficients C_{ik} in (7.3).) As shown by Example (6.6), such conditions cannot be purely graph theoretic. However, we have the following theorem, which employs the concept of successor operation Δ defined in Section 5.

(8.3) **THEOREM** [RiS78, THEOREM 5.6]. *Let T be a given (directed, finite) graph. The following are equivalent:*

(a) *For all M -matrices A (of all sizes) such that $S(A) = T$ we have $S_*(A) = J(A)$.*

(b) *If $k = 2, \dots, h$ (the height of $S_*(A)$), there do not exist nonempty disjoint subsets P, Q of the level Λ_k of T such that $\Delta(P) = \Delta(Q)$.*

(8.4) **EXAMPLE.** Let T be the graph



It is easily checked that condition (8.3)(b) holds for T . Hence all M -matrices A such that $S(A) = T$ have $J(A)$ equal to

$$\begin{matrix} * & * & * \\ * & * & * \end{matrix}$$

An example of such a matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

Note that nonsingular blocks may be added to A and block sizes may be arbitrary without changing $J(A)$ as long as $S(A)$ is preserved.

(8.5) **EXAMPLE.** In Example (6.6) the graph $S(A)$ does not satisfy condition (8.3)(b), since $\Delta(1) = \Delta(2) = \{3, 4\}$. As noted in that example, there is an M -matrix A with the given singular graph such that $S_*(A) \neq J(A)$.

Since a rooted forest satisfies condition (8.3)(b), we have the following corollary.

(8.6) **COROLLARY** [RiS78, Corollary 5.8]. *Let A be an M -matrix, and suppose that $S(A)$ is a rooted forest. Then $S_*(A) = J(A)$.*

We observe that Corollary (8.6) generalizes a previously known result that is due to Cooper [Co73], who proved that $\dim \text{Ker}(A)$ equals the number of distinguished singular vertices when $S(A)$ is a rooted forest, viz. $\nu_1 = \omega_1$. Cooper's result in turn implies one direction of Theorems (6.1) and (6.3).

Note that the graph of Example (5.2) is a rooted forest. Thus if A is an M -matrix whose singular graph $S(A)$ is given in Example (5.2), then $S_*(A) = J(A)$. Note that the graph T of Example (8.4) is not a rooted forest, even though (6.5) holds for all M -matrices A with $S(A) = T$.

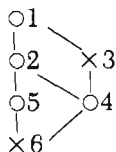
(8.7) **EXAMPLE.** We give the following matrix with six diagonal blocks in symbolic form. Let A be an M -matrix of form

$$\begin{bmatrix} \circ & 0 & 0 & 0 & 0 & 0 \\ < & \circ & 0 & 0 & 0 & 0 \\ < & 0 & \times & 0 & 0 & 0 \\ \leq & < & < & \circ & 0 & 0 \\ \leq & < & 0 & 0 & \circ & 0 \\ \leq & \leq & \leq & < & < & \times \end{bmatrix},$$

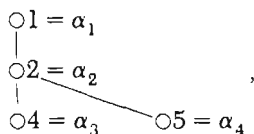
where

- \times = an irreducible nonsingular diagonal block,
- \circ = an irreducible singular diagonal block,
- 0 = a zero off-diagonal block,
- \leq = a nonpositive off-diagonal block,
- $<$ = a seminegative off-diagonal block.

Then $R(A)$ is (the transitive closure of)



$S(A)$ is



and so $S_*(A)$ is



By the nonnegative basis theorem, Theorem (7.1.i), we can choose a basis for the generalized eigenspace $E(A)$ such that

$$x^1 = \begin{bmatrix} + \\ + \\ + \\ + \\ + \\ + \end{bmatrix}, \quad x^2 = \begin{bmatrix} 0 \\ + \\ 0 \\ + \\ + \\ + \end{bmatrix}, \quad x^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ + \\ 0 \\ + \end{bmatrix}, \quad x^4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ + \\ + \end{bmatrix},$$

where

$+$ = a strictly positive vector component.

In addition, by Theorem (7.1)(ii) we can choose the above vectors so that for the matrix

$$X = [x^1, x^2, x^3, x^4]$$

we have

$$-AX = XC,$$

where C is the 4×4 matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_{12} & 0 & 0 & 0 \\ c_{13} & c_{23} & 0 & 0 \\ c_{14} & c_{24} & 0 & 0 \end{bmatrix}.$$

where $c_{12}, c_{13}, c_{14}, c_{23}, c_{24}$ are positive. For the sake of simplicity we assume that $c_{12} = c_{13} = c_{14} = c_{23} = c_{24} = 1$. Note that $S(C) = S(A)$ except for the labeling of the vertices (which is no coincidence).

Since $S(A)$ is a rooted forest, it follows by Corollary (8.6) that $J(A) = S_{\star}(A)$. By Theorem (7.9) it also follows that there exists a Jordan basis for $E(A)$ of semipositive vectors. In fact, such a basis is

$$(x^1, x^2 + x^3 + x^4, x^3 + x^4), (x^3).$$

We conclude this section by observing that the above results relate $S_{\star}(A)$ and $J(A)$ only in special cases. We may reformulate Question (6.8) more precisely as:

(8.8) QUESTION. Given $S(A)$, what are the possible $J(B)$ for M -matrices B such that $S(B) = S(A)$?

(8.9) QUESTION. Given $J(A)$, what are the possible $S(B)$ for M -matrices B such that $J(B) = J(A)$?

Another version of these questions is obtained by replacing $S(A)$ and $S(B)$ by $S_{\star}(A)$ and $S_{\star}(B)$, respectively.

Papers discussed: [Co73], [RiS78].

9. THE GROWTH OF POWERS

This section does not use the definitions of Section 5. We require the following definition, which is essentially found in [A86].

Let

$$(9.1) \quad B^{(1)}, B^{(2)}, \dots$$

be a sequence of nonnegative matrices. Let s be a nonnegative real number and d a nonnegative integer. If $s > 0$, we say that the sequence (9.1) has *pseudoexponential growth rate* (s, d) if there exists a strictly positive matrix L such that

$$(9.2) \quad \lim_{m \rightarrow \infty} \frac{B^{(m)}}{s^m m^d} = L,$$

and we write

$$(9.3) \quad B^{(m)} \sim s^m m^d$$

in this case. We write $B^{(m)} \sim 0^m m^d$ if $B^{(m)} = 0$ for all sufficiently large m . (The d is irrelevant here.) Note that when (9.3) holds, all elements of $B^{(m)}$ have the same pseudoexponential growth rate (s, d) .

Let P be an irreducible semipositive matrix. We define the *cycle index* $c(P)$ to be the g.c.d. (greatest common divisor) of the lengths of simple cycles in the graph $G(P)$ of P . As is very well known, there are exactly $c(P)$ eigenvalues of P whose absolute value equals $\rho(P)$. An irreducible matrix P is called *primitive* if $c(P) = 1$ or if P is the 1×1 matrix 0. Observe that the cycle index of an irreducible nonnegative matrix cannot be determined from the marked reduced graph that has just one vertex.

Now let P be an arbitrary nonnegative matrix (in Frobenius normal form). Let i, j be vertices of $R(P)$. We define the indices $s(i, j)$ and $d(i, j)$: If i has no access to j (and so in particular if $i < j$), we put $s(i, j) = 0$, $d(i, j) = 0$. If $i \geq j$, we let $s(i, j)$ be the maximum of ρ_k over all vertices k that lie on a chain from i to j in $R(P)$ (see the definition of chain in Section 2). Let $d(i, j) + 1$ be the maximum number of $s(i, j)$ -vertices k [viz. vertices such that $\rho_k = s(i, j)$] that lie on a chain from i to j in $R(P)$. Clearly $s(i, i) = \rho_i$ and $d(i, i) = 0$ for a vertex i of $R(P)$.

We first state a theorem under primitivity assumptions.

(9.4) THEOREM. *Let P be a nonnegative matrix in Frobenius normal form and suppose that all diagonal blocks are primitive. Then*

$$(9.5) \quad (P^m)_{ij} \sim s(i, j)^m m^{d(i, j)}.$$

Intuitively, if one thinks of the vertices k of $R(P)$ as being mountain peaks of height ρ_k , then (9.5) implies that the pseudoexponential growth rate of $(P^m)_{ij}$ is determined by a "hardest" route from i to j : a chain from i to j

in $R(P)$ that scales the peak of greatest height $s(i, j)$ and the most peaks [viz. $d(i, j) + 1$] of that height.

Theorem (9.4) generalizes the following well-known example.

(9.6) EXAMPLE. Let

$$P = \begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix},$$

where $0 \leq a$.

If $a = 0$, then $s(2, 1) = 1$, $d(2, 1) = 0$ and

$$(P^m)_{21} = 1 \sim 1.$$

If $0 < a < 1$, then $s(2, 1) = 1$, $d(2, 1) = 0$ and

$$(P^m)_{21} = \frac{1 - a^m}{1 - a} \sim 1.$$

If $a = 1$, then $s(2, 1) = 1$, $d(2, 1) = 1$ and

$$(P^m)_{21} = m - 1 \sim m.$$

If $1 < a$, then $s(2, 1) = a$, $d(2, 1) = 0$ and

$$(P^m)_{21} = \frac{a^m - 1}{a - 1} \sim a^m,$$

where we have used the obvious conventions $1^m = 1$, $m^0 = 1$.

In order to state a more general theorem we need to define a third index $g(i, j)$ for vertices i, j of $R(P)$. If $s(i, j) = 0$, we put $g(i, j) = 1$. Suppose that $s(i, j) > 0$. Let $\Pi(i, j)$ be the set of all chains from i to n in $R(P)$ such that each chain contains $d(i, j) + 1$ vertices that are $s(i, j)$ -vertices. Then for each chain γ in $\Pi(i, j)$ let $g(\gamma)$ be the g.c.d. of all the cycle indices $c(P_{kk})$ where k ranges over the vertices of γ . Then let $g(i, j)$ be the l.c.m. (least common multiple) of all $g(\gamma)$ for chains $\gamma \in \Pi(i, j)$.

We now define the *smoothing matrix* $M(i, j)$ by

$$(9.7) \quad M(i, j) = I + P + \dots + P^{g(i, j)-1}.$$

(9.8) THEOREM. *Let P be a nonnegative matrix in Frobenius normal form. Then*

$$(9.9) \quad (M(i, j)P^m)_{ij} \sim s(i, j)^m m^{d(i, j)}.$$

If $g(i, j) = 1$ then $M(i, j) = I$. Further, if all diagonal blocks of P are primitive, then $g(i, j) = 1$ for all vertices i, j of $R(P)$, and so theorem (9.8) reduces to Theorem (9.5).

Theorem (9.8) is essentially to be found in [FrS80, Theorem 5.10], where however the result is stated under the normalization $\rho(P) = 1$ and where it is proved that $(P^m)_{ij}$ tends to 0 at an exponential rate for $0 < s(i, j) < 1$. The proof in [FrS80] applies in the general case. In [Ro81] (based on a 1977 report) Rothblum proves a result of the same type as Theorem (9.8), where he uses polynomial expansions and Cesàro means of the powers P^m instead of the smoothing matrix $M(i, j)$. Under the assumption that all diagonal blocks of P are primitive, Artzroumi proves a result in [A86] that is equivalent to Theorem (9.5).

(9.10) EXAMPLE [FrS80]. Let

$$P = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$= \begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix},$$

where P_{11} is 2×2 . We shall display the coefficients $s(i, j)$, $d(i, j)$, and $g(i, j)$, $i = 1, 2$, in 2×2 matrices respectively denoted by S , D , and G . Thus

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

For $i, j = 1, 2$, let E_{ij} be the matrix of the same size as P_{ij} all of whose

entries are 1. Then, with the notation above, we have $M(1,0) = I + P$, and it is easy to see that, for $m = 0, 1, \dots$,

$$(M(1,1)P^m)_{11} = E_{11}.$$

Similarly $M(2,2) = I + P + P^2$, and

$$(M(2,2)P^m)_{22} = E_{22}.$$

Finally, $M(2,1) = I$, and if $0 \leq k < 6$ then it may be shown that

$$(P^{6m+k})_{21} = mE_{21} + N(k),$$

where $N(k)$ is a matrix whose entries are 0 or 1. Hence

$$(P^m)_{21} \sim m.$$

Our final example above illustrates the somewhat surprising result that it is the g.c.d. of cycle indices over a single chain that enters into the definition of the smoothing matrix, rather than the l.c.m. Of course, one could replace the g.c.d. by the l.c.m. in the definition of $g(\gamma)$, but the result would then be weaker and, for that matter, easier to prove. In Example (9.10) one would then obtain $g(2,1) = 6$ instead of $g(2,1) = 1$. It is an open question whether it is best possible to define $g(i, j)$ as the l.c.m. over the set of chains $\Pi(i, j)$, i.e., whether this is general yields the smallest $s(i, j)$ for which Theorem (9.8) holds.

Papers discussed: [FrS80], [Ro81], [A86].

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