Determinantal Identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley*

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Dedicated to Alexander M. Ostrowski
on the occasion of his ninetieth birthday.

Submitted by George P. Barker

ABSTRACT

We give a common, concise derivation of some important determinantal identities attributed to the mathematicians in the title. We also give a formal treatment of determinantal identities of the minors of a matrix.

0. INTRODUCTION

In this report we show how some important determinantal identities associated with the gentlemen in the title admit a common derivation. Each of these identities can be obtained in several ways. The novelty of our approach is not with any individual proof but with the conciseness of the collective derivation of the identities. Our development is self-contained except for use of two basic results which may be found in any elementary textbook. These are first, that Gaussian elimination does not alter a determinant and, second, that a determinant has a Laplacian expansion. We give a formal treatment of determinantal identities of the minors of a matrix and then provide a careful exposition of two methods for obtaining from a given determinantal identity another determinantal identity. These are the not

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particularly well-known "law of extensible minors" and "law of complementarities." Many other results would readily follow, some of which are contained in the interesting and extensive surveys of Ouellette [23], Cottle [8], or Henderson and Searle [15].

In what follows we use names to describe the identities attributed to these various 18th and 19th century mathematicians by writers in the present century.

1. GAUSS, SCHUR, AND CAUCHY

Our first character is *Gauss*, for it is Gaussian elimination that constitutes our basic technique. Let $A = [a_{ij}]$ be an $n \times n$ matrix over a field $\mathbf{F}$ which is partitioned as

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where $E = A[1,\dots,k|1,\dots,k]$ is the leading principal $k \times k$ submatrix of A and $1 \leqslant k \leqslant n$. We assume that E is invertible. We apply Gaussian elimination of the following type to reduce G to a zero matrix:

$$(1) \qquad \begin{bmatrix} I_k & 0 \\ -GE^{-1} & I_{n-k} \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} E & F \\ 0 & H' \end{bmatrix}.$$

In this elimination, linear combinations of the first k rows of A are added to the last $n-k$ rows. From (1) we obtain that $H' = H - GE^{-1}F$, the *Schur complement* A/E of E in A:

$$(2) \qquad H' = A/E = H - GE^{-1}F.$$

It follows from (1) that the Schur complement A/E is invertible when A is, and that Schur's identity

$$(3) \qquad \det A = (\det E)(\det A/E)$$

holds. We note for later use that the Schur complement always results in the lower right corner whenever G is reduced to a zero matrix by Gaussian elimination on A which does not add linear combinations of the last $n-k$

rows to any of rows $1,\dots,n$. That is, if

$$\begin{bmatrix} X & 0 \\ Y & I_{n-k} \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix}=\begin{bmatrix} E' & F' \\ 0 & H' \end{bmatrix},$$

then $Y=-GE^{-1}$ and $H'=A/E$. Returning to (1) and the Schur complement, we index the entries of $H'=A/E$ using $k+1,\dots,n$:

$$H'=A/E=\begin{bmatrix} h'_{k+1,k+1} & \cdots & h'_{k+1,n} \\ \vdots & & \vdots \\ h'_{n,k+1} & \cdots & h'_{nn} \end{bmatrix}$$

Let $k+1\leqslant i,j\leqslant n$. Since the Gaussian elimination in (1) has only added linear combinations of the first k rows of A to the other rows, it follows that the determinant of a $(k+s)\times(k+s)$ submatrix of A which contains E as a (principal) $k\times k$ submatrix does not change. So, if $k+1\leqslant i_1<\cdots<i_s\leqslant n$ and $k+1\leqslant j_1<\cdots<j_s\leqslant n$, and we consider the $s\times s$ submatrix B of the Schur complement whose rows are indexed by $i_1,\dots,i_s$ and whose columns are indexed by $j_1,\dots,j_s$:

$$B=H'[i_1,\dots,i_s|j_1,\dots,j_s],$$

we obtain

$$\det A[1,\dots,k,i_1,\dots,i_s|1,\dots,k,j_1,\dots,j_s]=\det\begin{bmatrix} E & * \\ 0 & B \end{bmatrix}=(\det E)(\det B).$$

From this we obtain a result which we shall apply several times:

$$(4)\quad \det H'[i_1,\dots,i_s|j_1,\dots,j_s]=\frac{\det A[1,\dots,k,i_1,\dots,i_s|1,\dots,k,j_1,\dots,j_s]}{\det A[1,\dots,k|1,\dots,k]}.$$

In particular, the entries of the Schur complement A/E satisfy

$$(5)\qquad h'_{ij}=\frac{\det A[1,\dots,k,i|1,\dots,k,j]}{\det E}\qquad (k+1\leqslant i,j\leqslant n).$$

We conclude this section by mentioning a special case of the Schur complement formula (3). Let $k=n-1$, so that H is the 1×1 matrix $[a_{nn}]$, and F and G are column and row vectors, respectively. We then obtain from (2) and (3) that $\det A = (\det E)(a_{nn} - GE^{-1}F)$. Since $E^{-1} = (\det E)\operatorname{adj} E$, where $\operatorname{adj} E$ is the adjoint (adjugate) of E, we obtain a formula called by Aitken [1, p. 74] the *Cauchy* expansion of $\det A$:

$$\det A = a_{nn}\det E - G(\operatorname{adj} E)F. \tag{6}$$

Let $k+1 \leqslant i \leqslant n$. We may regard h'_{ij}, $j = k+1,\ldots,n$, as the error in column j when the ith row of A is replaced by a linear combination of the first k rows of A so that the first k entries are unaltered. Indeed, the formula (5) gives the error even for $1 \leqslant j \leqslant k$, since then it yields 0. Further, (5) is symmetric in i and j, and therefore h'_{ij} $(k+1 \leqslant i, j \leqslant n)$ is also the error obtained when the jth column of A is linearly interpolated by means of the first k columns.

2. SYLVESTER

Another form of (4) is the identity of *Sylvester* on bordered determinants: Let $C = [c_{ij}]$ be the $(n-k)\times(n-k)$ matrix where

$$c_{ij} = \det A[1,\ldots,k,i|1,\ldots,k,j] \qquad (k+1 \leqslant i, j \leqslant n). \tag{7}$$

Then, by (5),

$$C = (\det E)(A/E)$$

and hence with use of (3), we get

$$\det C = (\det E)^{n-k}\det A/E = (\det E)^{n-k-1}\det A. \tag{8}$$

3. KRONECKER

A special case of Sylvester is the following identity of Kronecker on bordered determinants: Let $B = [b_{ij}]$ be the $(n-k)\times(n-k)$ matrix where

$$b_{ij} = \det \begin{bmatrix} a_{11} & \cdots & a_{1k} & a_{1,j} \\ \vdots & & & \vdots \\ a_{k1} & \cdots & a_{kk} & a_{k,j} \\ a_{i,1} & \cdots & a_{i,k} & 0 \end{bmatrix} \qquad (k+1 \leqslant i, j \leqslant n).$$

Then every $(k+1)\times(k+1)$ submatrix of B has a vanishing determinant:

$$\det B[i_1,\dots,i_{k+1}|j_1,\dots,j_{k+1}]=0 \tag{9}$$

for $k+1\leqslant i_1<\cdots<i_{k+1}\leqslant n$ and $k+1\leqslant j_1<\cdots<j_{k+1}\leqslant n$. To prove (9) one may assume $n=2k+1$. Taking the submatrix H of A to be 0, we have $C=B$ with $\det A$ vanishing identically. Hence (9) is a consequence of (8). Alternatively, it follows from (5) and (2) that

$$B=(\det E)(A/E-H)=(\det E)(-GE^{-1}F)$$

Hence $\operatorname{rank} B\leqslant \operatorname{rank} G\leqslant k$, and (9) follows.

4. JACOBI

We now assume A is invertible and continue the Gaussian elimination on A begun in (1):

$$\begin{bmatrix} E^{-1} & -E^{-1}F(H')^{-1} \\ 0 & (H')^{-1}\end{bmatrix}\begin{bmatrix} I_k & 0 \\ -GE^{-1} & I_{n-k}\end{bmatrix}\begin{bmatrix} E & F \\ G & H\end{bmatrix}$$

$$=\begin{bmatrix} E^{-1} & -E^{-1}F(H')^{-1} \\ 0 & (H')^{-1}\end{bmatrix}\begin{bmatrix} E & F \\ 0 & H'\end{bmatrix}$$

$$=\begin{bmatrix} I_k & 0 \\ 0 & I_{n-k}\end{bmatrix}.$$

Hence, multiplying the first two matrices on the left, we obtain

$$A^{-1}=\begin{bmatrix} E^{-1}+E^{-1}F(H')^{-1}GE^{-1} & -E^{-1}F(H')^{-1} \\ -(H')^{-1}GE^{-1} & (H')^{-1}\end{bmatrix}. \tag{10}$$

Since $H'=A/E$, it follows from (10) that the inverse of the Schur complement of E in A is the submatrix of A^{-1} which is "complementary to E." From (3) we now obtain the identity of *Jacobi*, which can be stated in either of the forms

$$\det A^{-1}[k+1,\dots,n|k+1,\dots,n]=\frac{\det A[1,\dots,k|1,\dots,k]}{\det A},\tag{11}$$

or

$$\det(\operatorname{adj} A)[k+1,\dots,n|k+1,\dots,n]=(\det A)^{n-k-1}\det A[1,\dots,k|1,\dots,k].$$

Let $1\leqslant i_1<\cdots<i_k\leqslant n$ and $1\leqslant j_1<\cdots<j_k\leqslant n$, and let $1\leqslant i'_{k+1}<\cdots<i'_n\leqslant n$ and $1\leqslant j'_{k+1}<\cdots<j'_n\leqslant n$, where $\{i_1,\dots,i_k,i'_{k+1},\dots,i'_n\}=\{j_1,\dots,j_k,j'_{k+1},\dots,j'_n\}=\{1,\dots,n\}$. To obtain from (11) Jacobi's identity in its usual forms,

$$\det A^{-1}[j'_{k+1},\dots,j'_n|i'_{k+1},\dots,i'_n]=\frac{(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}}{\det A}\det A[i_1,\dots,i_k|j_1,\dots,j_k]\tag{12}$$

or

$$\det(\operatorname{adj} A)[j'_{k+1},\dots,j'_n|i'_{k+1},\dots,i'_n]=(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}(\det A)^{n-k-1}\det A[i_1,\dots,i_k|j_1,\dots,j_k],\tag{13}$$

we need only replace A by PAQ where P and Q are permutation matrices such that

$$PAQ=\begin{bmatrix} A[i_1,\dots,i_k|j_1,\dots,j_k] & * \\ * & A[i'_{k+1},\dots,i'_n|j'_{k+1},\dots,j'_n]\end{bmatrix}.$$

For then,

$$\det(PAQ)=(-1)^{i_1+\cdots+i_k+j_1+\cdots+j_k}\det A,$$

$$(PAQ)[1,\dots,k|1,\dots,k]=A[i_1,\dots,i_k|j_1,\dots,j_k],$$

and

$$(PAQ)^{-1}[k+1,\dots,n|k+1,\dots,n]=(Q^{-1}A^{-1}P^{-1})[k+1,\dots,n|k+1,\dots,n]$$
$$=A^{-1}[j'_{k+1},\dots,j'_n|i'_{k+1},\dots,i'_n].$$

5. BINET, CAUCHY, AND LAPLACE

Now let F be a $k \times l$ matrix and G an $l \times k$ matrix. Let $n = k + l$, and define an $n \times n$ matrix A by

$$A = \begin{bmatrix} -I_k & F \\ G & 0 \end{bmatrix}. \tag{14}$$

The Schur complement of $-I_k$ in A is

$$A/(-I_k) = 0 - G(-I_k)^{-1}F = GF,$$

and hence by (3),

$$\det A = \det(-I_n)[\det A/(-I_k)] = (-1)^n \det GF,$$

or, equivalently,

$$\det GF = (-1)^n \det A. \tag{15}$$

Upon applying the expansion of *Laplace* (e.g. Aitken [1, p. 78]) to evaluate $\det A$ in (15), we obtain the determinant formula of *Binet* and *Cauchy*:

$$\det GF = \sum_{1 \leqslant i_1 < \cdots < i_l \leqslant k} \det G[1,\dots,l|i_1,\dots,i_l] \times \det F[i_1,\dots,i_l|1,\dots,l]. \tag{16}$$

In particular we have shown that for square matrices F and G we have

$$\det GF = \det G \det F.$$

6. THE QUOTIENT PROPERTY

We give two proofs of the quotient property of Schur complements. The first proof makes uses of the remark on uniqueness following (3). Suppose the $k \times k$ submatrix E of A is also partitioned so that

$$A = \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} K & L & F_1 \\ M & N & F_2 \\ G_1 & G_2 & H \end{bmatrix},$$

where K is $t \times t$ and N is $(k-t)\times(k-t)$. Assume that both E and K are invertible. Reducing M to a zero matrix by Gaussian elimination, we obtain

$$\begin{bmatrix} K & L & F_1 \\ 0 & E/K & F_2^* \\ G_1 & G_2 & H \end{bmatrix}$$

where E/K is invertible. Reducing G_1 to a zero matrix, we then obtain

$$\begin{bmatrix} K & L & F_1 \\ 0 & E/K & F_2^* \\ 0 & G_2^* & H^* \end{bmatrix} \tag{17}$$

where by our remark

$$A/K = \begin{bmatrix} E/K & F_2^* \\ G_2^* & H^* \end{bmatrix}. \tag{18}$$

Hence E/K is an invertible, principal submatrix of A/K. By reducing G_2^* to 0 in (17) and in (18) we arrive at

$$\begin{bmatrix} K & L & F_1 \\ 0 & E/K & F_2^* \\ 0 & 0 & H' \end{bmatrix},$$

where, using our remark again, $H' = A/E$. Hence we obtain the quotient property

$$A/E = (A/K)/(E/K). \tag{19}$$

The second proof of (19) uses the quotients (4) and (5) directly. Let $H'' = [h''_{ij}] = A/K$. Then $H''[t+1,\dots,k|t+1,\dots,k] = E/K$, and applying Schur's identity we obtain

$$\det E = (\det K)(\det E/K);$$

in particular E/K is invertible. Let

$$H''' = \left[h'''_{ij}\right] = H''/H'[t+1,\dots,k|t+1,\dots,k] = (A/K)/(E/K)$$

Applying (5) to the Schur complement of E/K in $A/K = H''$, we obtain

$$\text{(20)}\quad h_{ij}''' = \frac{\det H''[t+1,\ldots,k,i|t+1,\ldots,k,j]}{\det H''[t+1,\ldots,k|t+1,\ldots,k]} \qquad (k+1 \leqslant i, j \leqslant n).$$

Applying (4) to the Schur complement H'' of K in A and using (20), we get

$$h_{ij}''' = \frac{\det A[1,\ldots,k,i|1,\ldots,k,j]}{\det A[1,\ldots,t|1,\ldots,t]}\left(\frac{\det A[1,\ldots,k|1,\ldots,k]}{\det A[1,\ldots,t|1,\ldots,t]}\right)^{-1}$$

$$= \frac{\det A[1,\ldots,k,i|1,\ldots,k,j]}{\det A[1,\ldots,k|1,\ldots,k]} \qquad (k+1 \leqslant i, j \leqslant n).$$

It now follows from (5) that $h_{ij}''' = h_{ij}'$ $(k+1 \leqslant i, j \leqslant n)$, which is the quotient property (19).

7. MUIR AND CAYLEY

In Sylvester's identity (8) we may replace $\det C$ by its defining expansion in terms of its entries as given in (7) and obtain

$$\text{(21)}\quad \det A[1,\ldots,k|1,\ldots,k]^{n-k-1}\det A[1,\ldots,n|1,\ldots,n]$$
$$= \sum_{\sigma}(\operatorname{sign}\sigma)\prod_{i=k+1}^{n}\det A[1,\ldots,k,i|1,\ldots,k,\sigma(i)],$$

where the summation is over all permutations σ of $k+1,\ldots,n$. We observe that (21) may be obtained formally by adjoining the sequence $1,\ldots,k$ to each sequence in

$$\text{(22)}\quad \det A[\varnothing|\varnothing]^{n-k-1}\det A[k+1,\ldots,n|k+1,\ldots,n]$$
$$= \sum_{\sigma}(\operatorname{sign}\sigma)\prod_{i=k+1}^{n}\det A[i|\sigma(i)]$$

Of course, (22) is just the usual expansion of $\det A[k+1,\ldots,n|k+1,\ldots,n]$ made homogeneous by inserting a factor of $\det A[\varnothing|\varnothing]^{n-k-1}$. Here $\varnothing$ is the empty set and det $A[\varnothing|\varnothing]$ is defined to be 1. This method of obtaining an $n \times n$ determinantal identity from an $l \times l$ determinantal identity ($l = n - k$) is called *extension* and works in a general setting. *Muir* [20] called it the "law of extensible minors," and it leads to a somewhat bewildering variety of identities. We now formalize Muir's law and give a proof. We begin by attempting a definition of a determinantal identity for the minors of a matrix, a matter which appears to have been largely ignored.

We now suppose that $X = [x_{ij}]$ is an $l \times l$ matrix whose entries are pairwise commuting indeterminates over the field F. For the sake of future convenience, we assume that the rows and columns of X are indexed by $k+1,\dots,n = k + l$ for some integer $k \geqslant 0$. By a *term* of an $l \times l$ determinantal identity we shall mean a (possibly empty) product $T(X)$ of the form

$$\det X\left[\alpha^{(1)}|\beta^{(1)}\right]\cdots\det X\left[\alpha^{(p)}|\beta^{(p)}\right], \tag{23}$$

where $\alpha^{(i)}$ and $\beta^{(i)}$ are naturally ordered subsequences of the same length of the sequence $k+1,\dots,n$ for each $i = 1,\dots,p$. Thus a term is a product of *minors* of X, possibly of different orders. A formula of the form

$$\sum_{q=1}^{t} c_q T_q(X) = 0, \tag{24}$$

where $T_q(X)$ is a term and $c_q \in \mathrm{F}$ for $q = 1,\dots,t$, is an $l \times l$ *determinantal identity* provided (24) becomes an identity in x_{ij} $(k+1 \leqslant i, j \leqslant n)$ when each factor $\det X[\alpha^{(i)}|\beta^{(i)}]$ of each term $T_q(X)$ is replaced by its expansion in terms of the entries of X. We observe that if (24) is an $l \times l$ determinantal identity, then for any $l \times l$ matrix $B = [b_{ij}]$ with entries from an extension field F′ of F (in particular with the entries from F), we have

$$\sum_{q=1}^{t} c_q T_q(B) = 0. \tag{25}$$

This is so because the mapping $x_{ij} \to b_{ij}$ $(k+1 \leqslant i, j \leqslant n)$ induces a homomorphism of the *ring* $\mathrm{F}[x_{k+1,k+1}, x_{k+1,k+2},\dots,x_{nn}]$ into the field F′.

Now suppose that (24) is an $l \times l$ determinantal identity. By incorporating a suitable power of det $X[\varnothing|\varnothing]$ in each term, we may assume that all terms have the same number p of factors with at least one term having no factor det $X[\varnothing|\varnothing]$. Let $A = [a_{ij}]$ be an $n \times n$ matrix whose entries are pairwise commuting indeterminates over F. Applying the identity (24) to the matrix

$B = H'$, where $H' = A/E$ is the Schur complement of $E = A[1,\ldots,k|1,\ldots,k]$ (evidently E is invertible), we obtain (25). If we now replace each minor of H' by its value given by (4) and multiply by $(\det E)^p$, we obtain from (25) an $n \times n$ determinantal identity

$$\sum_{q=1}^{t} c_q \tilde{T}_q(A) = 0. \tag{26}$$

In (26) each minor in $\tilde{T}_q(A)$ is the extension of the corresponding minor in $T_q(X)$ by $\gamma = (1,\ldots,k)$. Thus (26) is the *extension* of (24) obtained by replacing each term $T_q(X)$ of the form (22) by

$$\tilde{T}_q(A) = \det A\left[\gamma \cup \alpha^{(1)}|\gamma \cup \beta^{(1)}\right] \cdots \det A\left[\gamma \cup \alpha^{(p)}|\gamma \cup \beta^{(p)}\right].$$

Now suppose (24) is an $n \times n$ determinantal identity. We may apply (24) to the matrix $B = \operatorname{adj} X$ to obtain (25). Let $\det X[\alpha'|\beta']$ be a minor in a term $T_q(X)$ given by (22) where $\alpha' = (j'_{k+1},\ldots,j'_n)$ and $\beta' = (i'_{k+1},\ldots,i'_n)$. By Jacobi (13) we may replace the minor $\det B[\alpha'|\beta']$ in $T_q(B)$ by

$$(-1)^{i_1 + \cdots + i_k + j_1 + \cdots + j_k}(\det X)^{n-k-1}\det X[\beta|\alpha],$$

where $X[\beta|\alpha]$ is the $k \times k$ submatrix of X which is "complementary to $X[\alpha'|\beta']$ in X^t." If we do this for each minor in every term (23), we obtain another $n \times n$ determinantal identity

$$\sum_{q=1}^{t} c'_q T'_q(X) = 0. \tag{27}$$

The identity (27) is called the *complementary identity* of (24). The fact that (27) is a determinantal identity when (24) is, is called by Muir [20] the "law of complementarities," and he attributes this law to *Cayley*. For example, the complementary identity of the defining expansion

$$\det X[1,\ldots,n|1,\ldots,n] = \sum_{\sigma} (\operatorname{sign}\sigma) \prod_{i=1}^{n} \det X[i|\sigma(i)]$$

is

$$(\det X)^{n-1}\det X[\varnothing|\varnothing] = \sum_{\sigma} (\operatorname{sign}\sigma) \prod_{i=1}^{n} (-1)^{i+\sigma(i)}\det X(\sigma(i)|i),$$

where $X(\sigma(i)|i)$ is the matrix obtained from X by eliminating row $\sigma(i)$ and column i. Since $\det X[\varnothing|\varnothing] = 1$, this is equivalent to

$$(28) \qquad (\det X)^{n-1} = \det(\operatorname{adj} X),$$

a theorem of Cauchy (see Aitken [1, p. 104]), which in turn is a special case of Jacobi (13). It is a consequence of the development above that Cayley's "law of complementarities" has a relation to Jacobi's theorem and the adjoint which is similar to the relation of Muir's "law of extensible minors" to Sylvester's identity and the Schur complement.

8. MUIR AND CAYLEY AS MODERN ALGEBRAISTS

We give in this section a more precise and formal treatment of determinantal identities and the replacement processes involved in Muir and Cayley laws. We continue to use the notation in Section 7.

Let integers $k \geqslant 0$ and $l \geqslant 1$ be given. For $p = 0, 1, \dots, l$ we denote by $S^p_{k,l}$, or for brevity S^p_l, the set of all sequences of integers $\alpha = (i_1, \dots, i_p)$ where $k+1 \leqslant i_1 < \cdots < i_p \leqslant k+l = n$. In particular we note that S^0_l consists only of the empty sequence $\varnothing$. If $\alpha \in S^p_{k,l}$, then we say α has *length* p and write $|\alpha| = p$.

Now let $\mathbf{F}$ be a field, and let Π_l be a set of pairwise commuting, algebraically independent indeterminates $\pi[\alpha|\beta]$ indexed by the set of all ordered pairs $[\alpha|\beta]$ with $[\alpha|\beta] \in \cup^l_{p=0}(S^p_l \times S^p_l)$. Let $\mathbf{F}[\Pi_l]$ be the polynomial domain over $\mathbf{F}$ generated by the indeterminates in Π_l. We refer to the elements of $\mathbf{F}[\Pi_l]$ as *formulas*. Let f be a nonzero formula. Then

$$(29) \qquad f = \sum_q c_q \psi_q,$$

where each c_q is a nonzero element of $\mathbf{F}$ and each ψ_q is a term of the form

$$(30) \qquad \psi_q = \pi[\alpha_1|\beta_1] \cdots \pi[\alpha_t|\beta_t]$$

for some $t \geqslant 0$. (If $t = 0$, then ψ_q is an empty product and equals 1.) We assume that in the representation (29) no two of the terms (30) are equal. The formula f is called *t-homogeneous* if each of its terms ψ_q has the same number

t of factors. We define the *weight of the term* ψ_q in (30) to be

$$w(\psi_q) = \sum_{i=1}^{t} |\alpha_i| = \sum_{i=1}^{t} |\beta_i|.$$

More generally, if $k+1 \leqslant j \leqslant n$, we define the *row weight of the index* j in ψ_q, denoted by $w(j;\psi_q)$, to be the number of occurrences of j in the sequences $\alpha_1,\dots,\alpha_t$. The *column weight of the index* j in ψ_q is the number $w(\psi_q; j)$ of occurrences of j in the sequences $\beta_1,\dots,\beta_t$. We note that

$$w(\psi_q) = \sum_{j=k+1}^{n} w(j;\psi_q) = \sum_{j=k+1}^{n} w(\psi_q; j).$$

Now let $X = [x_{ij}]$ be an $l \times l$ matrix whose entries are l^2 pairwise commuting, algebraically independent indeterminates over $\mathbf{F}$. As before, we assume that the rows and columns of X are indexed by $k+1,\dots,n = k+l$ for some $k \geqslant 0$. The mapping

$$\pi[\alpha|\beta] \to \det X[\alpha|\beta] \qquad (\pi[\alpha|\beta] \in \Pi_l)$$

(where $\det X[\phi|\phi] = 1$) induces a homomorphism $\mathfrak{D}$ from $F[\Pi_l]$ to $\mathbf{F}[x_{k+1,k+1}, x_{k+1,k+2},\dots,x_{n,n}]$ which we call the *determinantal homomorphism*. If f is a formula, then we write

$$\mathfrak{D}(f) = f(X).$$

Thus if f is given by (29) and (30), then

$$f(X) = \sum_q c_q \psi_q(X),$$

where

$$\psi_q(X) = \det X[\alpha_1|\beta_1] \cdots \det X[\alpha_t|\beta_t].$$

The formulas f in the kernel $\mathfrak{I}_l$ of $\mathfrak{D}$, that is, for which $f(X) = 0$, are the $l \times l$ *determinantal identities.*

By combining terms with the same number of factors, every formula $f \in \mathbf{F}[\Pi_l]$ can be written as a sum of homogeneous polynomials of different degrees. If t is the largest degree of a term of f, then by introducing factors of $\pi[\varnothing|\varnothing]$ in the terms of f, we can convert f into a t-homogeneous formula $\hat{f}$.

(31) f is a determinantal identity if and only $\hat{f}$ is.

The formula f is said to have *constant weight* if each of its terms has the same weight, denoted $w(f)$. By combining terms of the same weight, each formula f can be uniquely written as a sum of formulas, $f = f_1 + \cdots + f_r$, where each f_i has constant weight and $w(f_i) \neq w(f_j)$ for $i \neq j$.

(32) If f is a determinantal identity, then each of $f_1, \ldots, f_r$ is also a determinantal identity.

To verify (32), we choose an indeterminate y, independent of $x_{k+1,k+1}, x_{k+1,k+2}, \ldots, x_{n,n}$, and let $Y = yX$. It then follows that

$$0 = f(Y) = y^{w(f_1)} f_1(X) + \cdots + y^{w(f_r)} f_r(X).$$

Since y is independent of $x_{k+1,k+1}, x_{k+1,k+2}, \ldots, x_{n,n}$ and since $w(f_i) \neq w(f_j)$ for $i \neq j$, it follows that $f_i(X) = 0$, that is, f_i is a determinantal identity for $i = 1, \ldots, r$. We note that since the sum and product of determinantal identities are determinantal identities, there do exist identities having terms of different weight.

Constant weight identities can be further classified by consideration of the row weight and column weight of indices. Let f be a formula and let $k+1 \leqslant j \leqslant n$. By combining terms of f for which j has the same row weight, f can be uniquely written as a sum of formulas, $f = g_1 + \cdots + g_s$, where for each i, each term of g_i has the same row weight $a_{j,i}$ for the index j, and $a_{j,i} \neq a_{j,k}$ for $1 \leqslant i < k \leqslant s$.

(33) If f is a determinantal identity, then each of $g_1, \ldots, g_s$ is a determinantal identity.

For proof, we choose an indeterminate z, independent of $x_{k+1,k+1}, x_{k+1,k+2}, \ldots, x_{n,n}$, and let Z be the matrix obtained from X by multiplying the entries of row j by z. It then follows that

$$0 = f(Z) = z^{a_{j,1}} g_1(X) + \cdots + z^{a_{j,s}} g_s(X)$$

and $g_i(X) = 0$ for $i = 1, \ldots, s$.

By considering each row index and each column index we arrive at the following. Let f be a nonzero formula. Then f can be written as a sum of formulas, $f = h_1 + \cdots + h_p$, such that the row weight of j for each term of h_u is b_j^u and the column weight of j for each term is c_j^u $(1 \leqslant u \leqslant p, k+1 \leqslant j \leqslant n)$, where

$$(b_{k+1}^u,\ldots,b_n^u,c_{k+1}^u,\ldots,c_n^u) \neq (b_{k+1}^v,\ldots,b_n^v,c_{k+1}^v,\ldots,c_n^v)$$

for $1 \leqslant u < v \leqslant p$.

(34) If f is a determinantal identity, then each of $h_1,\ldots,h_p$ is also a determinantal identity.

Less formally, (34) says that a determinantal identity can be written as a sum of identities $h_1,\ldots,h_p$ such that each h_i has the property that for each index j, the row weight of j is the same in each term of h_i and the column weight of j is the same in each term of h_i. We call such identities *constant row and column weighted identities.*

Many familiar determinantal identities are constant row and column weighted identities. For example, such an identity is the identity corresponding to the Laplace expansion on row i of a matrix:

$$f = \pi[1,\ldots,n|1,\ldots,n] - \sum_{j=1}^{n} (-1)^{i+j}\pi[i|j]\pi(i|j)$$

where we write $\pi(i|j)$ for $\pi[1,\ldots,i-1,i+1,\ldots,n|1,\ldots,j-1,j+1,\ldots,n]$. However, $f+f^2$ is not a constant row and column weighted identity. The identity corresponding to the defining expansion of the determinant,

$$g = \pi[1,\ldots,n|1,\ldots,n] - \sum_{\sigma} (\operatorname{sign}\sigma)\pi[1|\sigma(1)]\cdots\pi[n|\sigma(n)],$$

where the summation extends over all permutations σ of $\langle 1,\ldots,n\rangle$, is also a constant row and column weighted identity. Since g is an identity,

$$\det X = \sum (\operatorname{sign}\sigma)x_{1\sigma(1)}\cdots x_{n\sigma(n)}.$$

One of the purposes of our symbolism is to have a clear distinction between

$$\det X, \qquad \text{alias} \quad \pi[1,\ldots,n|1,\ldots,n],$$

and

$$\sum_{\sigma} (\operatorname{sign}\sigma)x_{1\sigma(1)}\cdots x_{n\sigma(n)}, \qquad \text{alias} \quad \sum_{\sigma} (\operatorname{sign}\sigma)\pi[1|\sigma(1)]\cdots\pi[n|\sigma(n)]$$

The left hand sides are equal as elements of $\mathbf{F}[x_{11}, x_{12}, \ldots, x_{nn}]$; the right hand sides are different elements of $\mathbf{F}[\Pi_n]$.

We now mention two basic properties of the ideal $\mathcal{I}_l$ of determinantal identities.

(35) $\mathcal{I}_l$ is a prime ideal.

To verify (35) we suppose f is an identity and $f = gh$ where g and h are formulas. Then $0 = f(X) = g(X)h(X)$. Since $\mathbf{F}[x_{k+1,k+1}, x_{k+1,k+2}, \ldots, x_{n,n}]$ is an integral domain, $g(X) = 0$ or $h(X) = 0$. Hence g or h is an identity, and it follows that $\mathcal{I}_l$ is a prime ideal.

For $[\alpha|\beta] \in \cup_{p \neq 1}(S_l^p \times S_l^p)$, let

$$g_{\alpha,\beta} = \pi[\alpha|\beta] - \sum_{\sigma} (\operatorname{sign} \sigma)\pi[i_1|\sigma(i_1)] \cdots \pi[i_p|\sigma(i_p)] \tag{36}$$

where $\alpha = (i_1, \ldots, i_p)$, $\beta = (j_1, \ldots, j_p)$, and the summation extends over all bijections $\sigma: \{i_1, \ldots, i_p\} \to \{j_1, \ldots, j_p\}$. By convention, when $\alpha = \beta = \varnothing$, the summation in (36) is 1 and $g_{\varnothing,\varnothing} = \pi[\varnothing|\varnothing] - 1$.

(37) $G = \{g_{\alpha,\beta} : |\alpha| = |\beta| \neq 1\}$ is an irredundant set of generators of the ideal $\mathcal{I}_l$ of determinantal identities.

We first show that the formulas in G generate $\mathcal{I}_l$. It follows from the defining expansion of the determinant that each $g_{\alpha,\beta} \in G$ is an identity. Let f be any nonzero identity given by (29) and (30). Replacing in (29) each $\pi[\alpha_i|\beta_i]$ for which $|\alpha_i| = |\beta_i| \neq 1$ by

$$g_{\alpha_i,\beta_i} + \sum_{\sigma} (\operatorname{sign} \sigma)\pi[i_1|\sigma(i_1)] \cdots \pi[i_p|\sigma(i_p)],$$

we obtain,

$$f = \sum_{[\alpha|\beta]} h_{\alpha,\beta} g_{\alpha,\beta} + h,$$

where $h_{\alpha,\beta} \in \mathbf{F}[\Pi_l]$ and $h \in \mathbf{F}[\pi[i|j] : k+1 \leq i, j \leq n]$. Since f and $g_{\alpha,\beta}$ are identities, it follows that h is an identity. Hence $h(X) = 0$. Since $h(X) \in F[x_{ij} : k+1 \leq i, j \leq n]$, it follows that h is identically zero and

$$f = \sum_{[\alpha|\beta]} h_{\alpha,\beta} g_{\alpha,\beta}.$$

We conclude that G generates $\mathcal{I}_l$.

We now establish the irredundancy of G as generators of $\mathcal{I}_l$. Suppose to the contrary that for some $[\alpha_0|\beta_0]$,

$$g_{\alpha_0,\beta_0} = \sum_{[\alpha|\beta] \neq [\alpha_0|\beta_0]} p_{\alpha,\beta} g_{\alpha,\beta}$$

where $p_{\alpha,\beta} \in \mathbf{F}[\Pi_l]$. First suppose $[\alpha_0|\beta_0] = [\varnothing|\varnothing]$. Then

$$\pi[\varnothing|\varnothing] = 1 + \sum_{[\alpha|\beta] \neq [\varnothing|\varnothing]} p_{\alpha,\beta} g_{\alpha,\beta}.$$

Substituting for each $p_{\alpha,\beta}$ the expression given by (36), we obtain

$$\pi[\varnothing|\varnothing] = 1 + h,$$

where h is a formula in $F[\Pi_l]$ each term of which has as a factor at least one of the indeterminates in Π_l different from $\pi[\varnothing|\varnothing]$. This contradicts the algebraic independence of the indeterminates in Π_l. Now suppose $[\alpha_0|\beta_0] \neq [\varnothing, \varnothing]$. Then using (36) again, we obtain

$$\pi[\alpha_0|\beta_0] = h + p_{\varnothing,\varnothing}(\pi[\varnothing|\varnothing] - 1)$$

where each term of h has as a factor at least one of the indeterminates different from $\pi[\alpha_0|\beta_0]$ and $\pi[\varnothing|\varnothing]$. It follows that the coefficient of $\pi[\alpha_0|\beta_0]$ on the right hand side of the above equation is not 1, contradicting again the algebraic independence of Π_l.

A less formal way of stating (36) is that the defining expansions of the minors of order different from 1 of X generate irredundantly all the identities satisfied by the minors of X.

We now turn to a precise formulation of Muir and Cayley laws. Let $\mathcal{I}_l^*$ be the set of all homogeneous determinantal identities [cf. (31)]. We define the *Muir monomorphism* $\mathfrak{M}_k$ from $\mathbf{F}[\Pi_l]$ to $\mathbf{F}[\Pi_n]$ as the mapping induced by

$$\pi[\alpha|\beta] \to \pi[\gamma \cup \alpha|\gamma \cup \beta],$$

where $\gamma = (1, \ldots, k)$. Then arguments in the previous section show that

$$\mathfrak{M}_k(\mathcal{I}_l^*) \subseteq \mathcal{I}_n^*,$$

which is a reformulation of Muir.

The *Cayley isomorphism* $\mathcal{C}_l$ of $\mathbf{F}[\Pi_l]$ *into* itself is defined to be the mapping induced by

$$\pi[\alpha|\beta] \to (-1)^{i_1+\cdots+i_p+j_1+\cdots+j_p}\pi[1,\ldots,n|1,\ldots,n]^{p-1}\pi[\beta'|\alpha']$$

where $\alpha=(i_1,\ldots,i_p)$, $\beta=(j_1,\ldots,j_p)$, and α' and β' are complementary to α and β, respectively. We then obtain Cayley in the form

$$\mathcal{C}_l(\mathcal{I}_l)\subseteq \mathcal{I}_l.$$

Using (34), we can give a simpler version of Cayley. Suppose h is a constant row and column weighted t-homogeneous identity. Let the row weight of i in h be c_i, and the column weight be d_i. Then each term of $\mathcal{C}_l(h)$ contains as a factor

$$(-1)^{\mu}\pi[1,\ldots,n|1,\ldots,n]^{\mu-t}$$

where $\mu=\Sigma_i ic_i+\Sigma_i id_i$. Hence we may write

$$\mathcal{C}_l(h)=(-1)^{\mu}\pi[1,\ldots,n|1,\ldots,n]^{\mu-t}\mathcal{C}_l'(h).$$

Now let f be a t-homogeneous identity. Then from (34) it follows that $f=h_1+\cdots+h_p$, where each h_i is a constant row and column weighted, t-homogeneous identity. Hence if we define $\mathcal{C}_l'(f)$ by

$$\mathcal{C}_l'(f)=\mathcal{C}_l'(h_1)+\cdots+\mathcal{C}_l'(h_p),$$

then we see that $\mathcal{C}_l'(f)$ is also a t-homogeneous identity obtained from f by replacing each $\pi[\alpha|\beta]$ by $\pi[\beta'|\alpha']$ where α' and β' are complementary to α and β, respectively. It follows that

$$\pi[\alpha|\beta]\to\pi[\beta'|\alpha']$$

induces an isomorphism of $\mathbf{F}[\Pi_l]$ onto itself, denoted $\mathcal{C}_l'$, where

$$\mathcal{C}_l'(\mathcal{I}_l^*)=\mathcal{I}_l^*.$$

We call $\mathcal{C}_l'$ the *modified Cayley isomorphism.*

We now denote by ε_k the natural embedding of $F[\Pi_l]$ into $F[\Pi_n]$ induced by $\pi[\alpha|\beta]\to\pi[\alpha|\beta]$ whenever α and β are subsequences of $\{k+$

$1,\ldots,n\}$ of the same length. It now follows from the definition of the Muir and the modified Cayley mappings that

$$\mathfrak{M}_k = \mathcal{C}'_n \varepsilon_k \mathcal{C}'_l. \tag{38}$$

Thus Muir can be regarded as a consequence of (modified) Cayley.

Finally, we mention that there are identities, which one generally would regard as determinantal identities, that are not included in our definition as given in this section. For example, there is no formula (and hence no identity) corresponding to the fact that the determinant changes sign when two rows are interchanged. Such a formula could be obtained by not insisting that the indices in the indeterminantes $\pi[\alpha|\beta]$ be in strictly increasing order. Other identities—in particular, those involving more than one matrix—could be included by expanding our definition of determinantal identities.

9. HISTORY

We begin this section with references to the theorems discussed as they appear in Muir's history [21]. Muir heads each paper (or group of papers) reviewed by the author, title, and the date, which may differ by a year or two from the date of publication (cf. Muir's own article [20]). We follow this procedure, deleting the title but adding the page reference in [21]. Early theorems on determinants are numbered by Muir; where available, we append the number.

Laplace (1772), I, 24–33, Theorem 14, our Reference [18].
Binet (1812), I, 80–92, Theorems 17 and 18, our [3].
Cauchy (1812), I, 92–131, our [5].
Here Theorem 37 is the Cauchy expansion (6), Theorems 18 and 42 are Binet-Cauchy (15), and Theorem 17 is the special case of the product theorem for square matrices. Theorem 21 is the special case (28) of Jacobi.

Jacobi (1841), I, 258–272, our [16].
Sylvester (1851), II, 58–61, our [27]. (see also *Chiò* (1853), I, 79–81, our [7]).
Kronecker (1870), III, 191, our [17].
Cayley, see *Tanner* (1878), III, 277, our [6].
Muir (1881), IV, 7–8, our [20].

We have relied on the attributions in Muir's remarkably complete history [21] and have made only a modest attempt at independent verification of historical accuracy. As Muir [21, I, p. 33] points out in remarks on Laplace's contribution to the expansion that bears his name, some attributions may be a matter of judgement. Other histories that we have consulted are Spottiswoode (1856) [25; 21, II, p. 81], Studnička (1876) [26; 21, III, p. 58], and Günther (1877) [13; 21, III, p. 66].

The above list of attributions omits Schur, who published the determinantal identity (3) in [24]. This paper [24] is devoted to power series, and perhaps for this reason, there is no reference to [24] in Muir [21], or in some standard books on matrices first published before 1940, e.g. Macduffee (1933) [19], Wedderburn (1934) [29], both of which contain many references. Schur's identity (3) may be found in Banachiewicz [2], who apparently rediscovered it. The result is mentioned in Gantmacher (1953) [12, pp. 45–46]. The introduction by Haynsworth [14] in 1968 of the term "Schur complement" has influenced subsequent exposition.

We end this section with brief remarks on the history of determinants as it relates to the theorems we have discussed and the proofs we have given. The appearance of determinants in the 17th and 18th centuries was closely related to the elimination of unknowns in systems of linear equations and the solution of such equations: e.g. Leibnitz (1693), Cramer (1750), Vandermonde (1771), Laplace (1772) among others; see [21, I, Chapter 1]. However, in the first systematic expositions of the theory of determinants the sections on systems of linear equations are of the nature of applications: see Cauchy (1812) [5; 21, I, pp. 92–131] and Jacobi (1841) [16; 21, I, pp. 253–272]. Here the determinant's character as an alternating function of its rows is heavily exploited. Cauchy and Jacobi use the multilinearity of a determinant as a function of its rows as a natural consequence of the defining expansion $\det A = \Sigma_\sigma(\operatorname{sign}\sigma)a_{1\sigma(1)}\cdots a_{n\sigma(n)}$. However, the explicit formulation of the multilinear property appears to be due to Scherk (1825) [21, I, pp. 150–159, Theorems 46 and 47], while the theorem that addition of a multiple of one row to another does not alter a determinant apparently was not formulated until the third of Jacobi's memoirs of 1841; see [21, I, p. 272, Theorem 59]. In each case Muir expresses surprise at the late date of these theorems [21, I, pp. 155, 272]. The second of these theorems, viz. Theorem 59, is of course the basis for the use of Gaussian elimination in proving theorems on determinants. Aitken [1, p. 46] uses Gaussian elimination, calling it *condensation*, for this purpose and states that the method is ascribed to Chiò in 1853, although he adds it was "virtually used by Gauss more than forty years earlier in evaluating symmetric determinants."

Unfortunately Aitken does not give a precise reference to the result of Gauss he had in mind, but apparently he is referring to Chiò's 1853 paper [7; 21, II, p. 79–81]. In that paper Chiò proves that $\det B = a_{11}^{n-2}\det A$ where $A=[a_{ij}]$ is an $n\times n$ matrix and $B=[b_{ij}]$ is the $(n-1)\times(n-1)$ matrix defined by $b_{ij} = \det A[1, i|1, j]$ $(i, j = 2, \ldots, n)$. Chiò uses this identity to express the solution of a system of $2n$ nonlinear equations as the roots of a polynomial of degree n. However, we do not know who first proposed the technique of Gaussian elimination as a systematic technique for proving theorems on determinants.

10. CODA

We now compare our proofs with some of those found in the literature. Our basic technique of Gaussian elimination leads to Schur and Sylvester. Here we seem to be following an outline of a proof of Sylvester in de Boor and Pinkus [4, p. 83] where (4) and (5) may be found. We then use Schur to obtain Jacobi, Jacobi to obtain Cayley, and Sylvester to obtain Kronecker and Muir. We combine Schur and Laplace to obtain Binet-Cauchy. Two proofs of Sylvester are given by Frobenius [10, 11]. He also points out that Kronecker is a special case of Sylvester. Our first proof of Kronecker is similar to that of Frobenius; our second proof is similar in spirit to Kronecker's, since he too obtains a rank inequality from a factorization of the matrix B.

Aitken [1] does not state Schur except for the special case of Cauchy [1, p. 74]. Our proof of Binet-Cauchy is, nevertheless, essentially the same as Aitken's [1, p. 85], since Aitken in effect computes the Schur complement of $-I_k$ in (14). Note however that Aitken evaluates $\det A$ in (15), using what he calls the extended Cauchy expansion. But for A given by (14) this reduces to the Laplace expansion. Thereafter Aitken applies Binet-Cauchy to obtain Jacobi [1, pp. 98–99] and uses Jacobi to obtain Cayley. A double application of Cayley then yields Muir and hence Sylvester [1, pp. 103–105]. The explanations of Muir and Aitken in deriving Muir from Cayley seem to be incomplete, since one apparently has to use (34) in the derivation.

We conclude with some other brief references to the literature. Gantmacher proves Schur [12, pp. 45–46], Sylvester [12, pp. 31–33], and Jacobi [12, pp. 21–22]; Jacobi is also proved in Ouellette [23, pp. 205–206]. The entries of the Schur complement are obtained in Crabtree and Haynsworth [9] by appealing to the special case of Schur which we have identified as the Cauchy expansion. The formula (10) for A^{-1} may be found in Banachiewicz [2], to whom it is attributed by Ouellette [23, p. 201]. The quotient property is proved in Crabtree and Haynsworth [9], Ostrowski [22], and Ouellette [23, p. 210]. Our second proof is similar to that given in [9]. Our first proof uses Gaussian elimination and is similar in spirit but not in detail to that in [22] which is couched in terms of successive transformation of variables. Many additional properties and applications of the Schur complement can be found in Cottle [8], Ouellette [20], and Henderson and Searle [15]. These contain extensive bibliographies. Cauchy's identity (28) has been generalized by Taussky [28], who proves that adj A can be factored into a product of $n-1$ matrices each of which has det A as its determinant.

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