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Olga Taussky-Todd's Influence on Matrix Theory and Matrix Theorists

A Discursive Personal Tribute

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0. QUOTATIONS

"For what is the theory of determinants? It is an algebra upon algebra; a calculus which enables us to combine and foretell the results of algebraical operations, in the same way as algebra itself enables us to dispense with the performance of the special operations of arithmetic. All analysis‡[1] must ultimately clothe itself under this form.

"I have in previous papers[2] defined a 'Matrix' as a rectangular array of terms, out of which different systems of determinants may be engendered, as from the womb of a common parent; these cognate determinants being by no means isolated in their relations to one another, but subject to certain simple laws of mutual dependence and simultaneous deperition."

J. J. Sylvester[3] (1851)

"It will be seen that matrices (attending only to those of the same order) comport themselves as single quantities[3]; they may be added, multiplied or compounded together, &c.: the law of the addition of matrices is precisely similar to that for the addition of ordinary algebraical quantities; as regards their multiplication (or composition), there is the peculiarity that matrices are not in general convertible; it is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix, and thence to arrive

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‡ Superscripts refer to notes collected in section 5.

at the notion of a rational and integral function, or generally of any algebraical function, of a matrix."

A. Cayley (1858)

"A matrix . . . regarded apart from the determinant . . . becomes an empty schema of operation, . . . only for a moment looses the attribute of quantity to emerge again as quantity, . . . of a higher and unthought of kind, . . . in a glorified shape—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. . . . The conception of multiple quantity thus rises on the field of vision.[3]

Apotheosis of Algebraical Quantity
Sylvester (1884)

"The members of a hierarchy, like the Roman god Janus, all have two faces looking in opposite directions: the face turned towards the subordinate levels is that of a self-contained whole; the face turned towards the apex, that of a dependent part. One is the face of the master, the other the face of the servant. This 'Janus effect' is a fundamental characteristic of sub-wholes in all types of hierarchies."

A. Koestler (1967)

1. OLGA TAUSSKY'S DIRECT AND INDIRECT CONTRIBUTIONS

I would like to begin with some remarks concerning mathematics and the contribution of individual mathematicians.

Nineteenth century mathematics appears to me like a collection of isolated mountains; some higher than others. Twentieth century mathematics seems like a continuous mountain range, where it is hard to isolate individual peaks. Perhaps this is partly a matter of perspective, the foothills of present day mathematics may obscure the mountains of the previous generation. But, I think, there is another effect. Obviously there are far more active mathematicians today than a hundred years ago, and the following model may apply: what might in the nineteenth century have been the work of one man over a period of twenty years, is today the work of four in five years. One mathematician's good idea is picked up by others and immediately developed. Thus in assessing the work of a mathematician, one must ask the classical questions: "what theorems did he prove?", "what theories did he initiate?", but one must also ask: "what is his influence on others?". In this talk I wish to concentrate on Olga Taussky's *influence* on matrix theorists, and hence on matrix theory, for I believe that her influence in this area has been outstanding and unmatched. One might call this her *indirect contribution* to mathematics.

Everyone who has had students (and Taussky has had 11 at California Institute of Technology) has had some influence on others, but Taussky's influence has been far wider than this. Yet it is hard to document her indirect contributions, for the source of ideas is usually only hinted at in mathematical papers, and quite often no indication is given how or why an author began his investigations. Though there are a few ways of documenting Olga Taussky's influence [e.g. at least five of her research problems in the Bulletin of the AMS have led to publications, see also Cooper (1975) for her role in a paper by Stein and Rosenberg (1948)], by necessity, I must stress her influence on those whose work I have seen in actual progress; and this explains the word *personal* in the subtitle of this talk. I hope you will not think me particularly self-centered if the last part of this talk examines some ways Olga Taussky has influenced the research of my former students and myself.

But first I will briefly list her most important direct contributions to matrix theory. Note that her work in number theory including class field theory, group theory, topological algebra and differential equations is omitted. (I thank Jack Todd for helping me construct this list.)

Olga Taussky's direct contributions to matrix theory

I. *Algebraic* Commutativity and generalized commutativity, properties L & P, commutators, Shoda's theorem concerning commutators.

II. *Analytic* Positivity, Gershgorin circles, Lyapunov–Stein theory.

III. *Number Theoretic* Integral matrices, integral matrices connected with number theory, integral group rings, norms for algebraic number fields, quadratic forms.

IV. Also: Cramped matrices, Hilbert matrices, etc.

Everyone will have his own opinion which of these contributions might be emphasized. Let me quote one view:

> However, for me, the central core of her work is best typified by her continuing interest in the relationship between "classical" mathematics, especially algebra and matrix theory, and her insistence on the relevance of beauty in mathematics. In particular, I think her work on commutativity and generalizations, and on the form of certain mappings, are most representative. She poses a simple question—what does a particular form of commutativity imply? What are the characteristic polynomial, the characteristic roots and vectors, of a particular operator; be it $X \to AX-XA$, $X \to AX+X'A'$, or $X \to TXT'$, and then answers the question, in an elegant manner, ever mindful of the proper algebraic setting, and the classical origins of the topic.
>
> David Carlson (letter, September 1976)

Now I shall turn to the main subject of my talk: Olga Taussky's influence on matrix theory. Had I chosen to talk about her direct contributions I would have selected some of her papers for detailed examination different from those that are discussed in sections 2, 3 and 4. All the selected papers are in the area I called "analytic matrix theory" in the list above, a type of mathematics that Wisconsin algebraists occasionally call "hybrid analysis"—for they do not quite consider it within algebra, a point which I shall return to and which has some relevance to my main theme.

2. DIAGONAL DOMINANCE

2.1 In (1949) Olga Taussky published her five page note "A recurring theorem in determinants" in the *American Mathematical Monthly*. Her theorems were well-known, and the proofs were so easy as to be accessible to undergraduates, or so it seems in 1977. But the apparent simplicity conceals unsuspected depths. Let us begin with the fundamental result, her Theorem 1.

DIAGONAL DOMINANCE THEOREM *Let A be a complex $n \times n$ matrix and let A_i be the sum of the absolute values of the non-diagonal elements in the i-th row:*

$$A_i = \sum_{j \neq i} |a_{ij}|, \qquad i = 1, \ldots, n.$$

If A is diagonally dominant, viz.

$$|a_{ii}| > A_i, \qquad i = 1, \ldots, n,$$

then

$$\det A \neq 0.$$

Proof Assume that $\det A = 0$. Then the system of equations

$$\sum_{j=1}^{n} a_{ij} x_j = 0, \qquad i = 1, \ldots, n,$$

has a non-trivial solution $x_1, \ldots, x_n$. Let r be one of the indices for which $|x_i|$, $i = 1, \ldots, n$, is maximal. The rth equation implies

$$|a_{rr}||x_r| \leqslant \sum_{j \neq r} |a_{rj}||x_j| \leqslant A_r |x_r|,$$

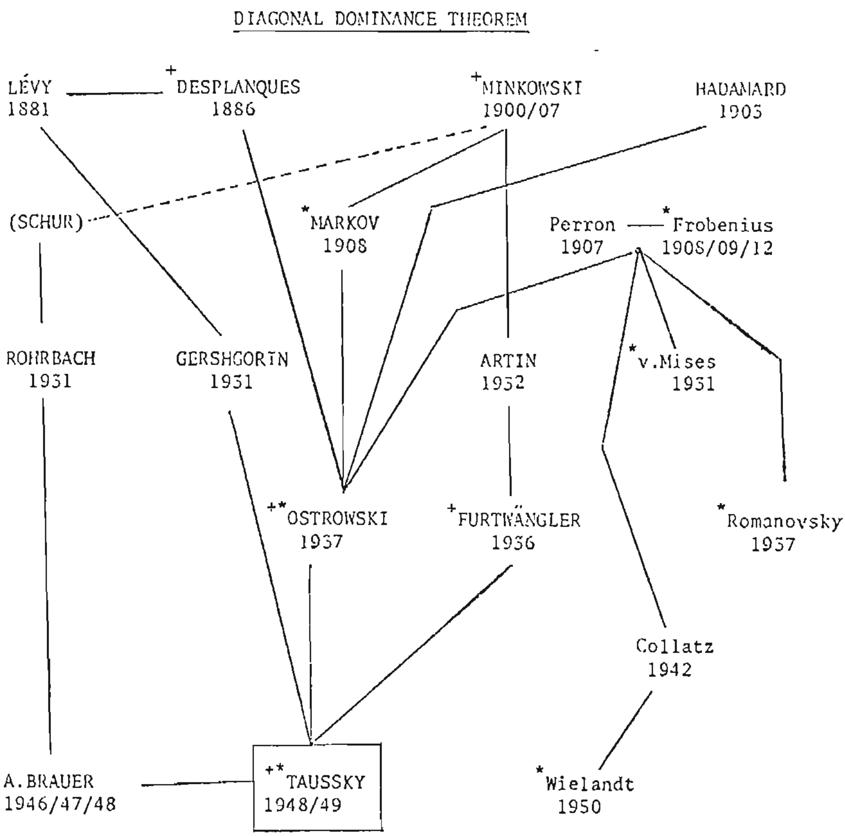
which contradicts the hypothesis.

The diagonal dominance theorem was first published by L. Lévy (1881), under the assumption that $a_{ii} < 0$, $a_{ij} \geqslant 0$, $i \neq j$, $i,j = 1, \ldots, n$. In a subsequent paper, Desplanques[6] (1886) gave a proof of the general case, which in essence was the same as Olga Taussky's, paraphrased above. In Nekrasov (1892) the theorem is all but stated, that is it results from a combination of two theorems quoted in the paper. One of these is to be found in a letter from Mehmke (1892) to Nekrasov: if A is diagonally dominant, then an associated Seidel iteration converges. The diagonal dominance theorem was rediscovered independently, if the absence of references is an indication by Hadamard (1903) and a related theorem was found by Minkowski (1900, 1907). He showed that with the assumptions[7] $a_{ii} > A_i$, $a_{ij} \leqslant 0$, $i \neq j$, $i,j = 1, \ldots, n$, in Theorem I, one may conclude that $\det A > 0$. (We shall call matrices satisfying these assumptions *Minkowski matrices*).

Some subsequent papers in this area derive from Minkowski (1900). His proof was not particularly illuminating and this occasioned Artin's (1932) remark, the first sentence of which seems truly amazing today:

> A frequent objection to the use of this theorem on determinants (Minkowski's diagonal dominance theorem) is that its proof is too complicated. But this is by no means the case.

It is most interesting to observe that Hopf[8] (1929), who may have discovered the theorem independently, used the theorem to give an elementary proof that an affine map of the n-simplex into itself has a fixed point. Bankwitz (1930), who refers to Perron–Frobenius, proved the theorem, and applied it to knot theory, cf. also Reidemeister (1932, p. 32).



Capital letters: Diagonal dominance or circles theorem
Lower case letters: Perron–Frobenius theorem
*: Irreducible matrices
†: Minkowski matrices and det $A > 0$.

FIGURE 1

The theorem was also rediscovered independently on several other occasions. One of these[9] deserves mention. A Belgian civil engineer, C. Massonnet (1945), rediscovered Minkowski's theorem and proved as a corollary that det A is positive if $a_{11}a_{22}\ldots a_{nn}$ is positive and

$$\sum_{\substack{j\neq i\\ j=1}}^{n} \frac{a_{ij}}{\sqrt{|a_{ii}a_{jj}|}} \leqslant 1, \qquad i = 1, \ldots, n.$$

(He does not distinguish between "positive" and "non-negative".) We should not be too critical that Massonnet failed to search the mathematical literature for references, for the paper bears the address and date Oflag II (Prenzlau), 1942.

It should be noted that from the time of Perron–Frobenius and Markov (1908) until Ostrowski's papers (1937a, 1937b) there were no new results in this area, (the Gerschgorin circles theorem, first published in this gap was apparently known much earlier, cf. (2.2) and (2.3) below). We display the relations of the various papers in a diagram. Lines denote references to or use of that paper—but here I may have missed some connections. Also there is no attempt to list all papers referenced by Taussky (1949).

2.2 We now turn to a second feature of Olga Taussky's paper: it goes straight to an important point, which most previous authors had ignored and none had made as simple. We call a complex $n\times n$ matrix *irreducible* if it cannot be transformed into the form

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11}, A_{22} are square, by the same permutation of rows and columns.

The following result is called *Olga Taussky's theorem* in Parodi (1959):

THEOREM II (diagonal dominance theorem for irreducible matrices) *Let A be a complex $n\times n$ matrix such that*

$$|a_{ii}| \geqslant A_i, \qquad i = 1, \ldots, n$$

where the equality holds in at most $n-1$ cases. If A is irreducible, then det $A \neq 0$.

The first appearance of the theorem seems to be in Markov (1908); however the hypothesis of irreducibility must be inferred there from the proof and other circumstances [see Schneider (1977)]. The result is stated and proved in Bankwitz (1930, p. 156) and in Hilde Geiringer (1949, p. 379) where it is applied to the solution of linear equations by iteration methods. It may also be found in Ostrowski (1937a, p. 89), where it is derived from a more general result (see p. 88). Since it was not Ostrowski's purpose to go straight to the theorem, it would require an astute reader to spot Taussky's simple proof of her Theorem II. The point is not unimportant, for the diagonal dominance theorem is connected with the Perron–Frobenius theorem, and to make this connection one must consider irreducible matrices: if A is an irreducible Minkowski matrix then *all* the equalities hold in Theorem II if and only if A is singular.

Markov (1908) used precisely this argument to prove the simplicity of the eigenvalue 1 of an irreducible stochastic matrix, and a much subtler form of the argument is the basis of Wielandt's (1950) proof of Perron–Frobenius. Taussky does not mention Perron–Frobenius, but the essential tools for understanding Wielandt's paper are there, see the beginning of (8.29) in Taussky (1962).

2.3 A simple consequence of the diagonal dominance theorem is the now famous:

CIRCLES THEOREM *Let A be an $n \times n$ complex matrix. Then the spectrum of A is contained in the union of the n circles in the complex plane:*

$$|z - a_{ii}| \leqslant A_i, \qquad i = 1, \ldots, n.$$

The weaker inequality $|\lambda| \leqslant \max \{ \sum_{j=1}^{n} |a_{ij}| : \; i = 1, \ldots, n\}$ was presumably known to Frobenius [one combines a remark in Frobenius (1908) with another in (1909)]. It seems that the circles theorem was known to Schur [cf. Brauer (1973, p.v)]; the theorem is certainly stated and proved in Rohrbach (1931) in the course of a proof of Minkowski's theorem attributed to Schur: For a Minkowski matrix the circles lie in the right half plane, hence the real eigenvalues are positive, the complex eigenvalues occur in conjugate pairs. Nowadays the theorem is usually called the Gershgorin theorem in view of Gershgorin (1931), where indeed the result is published for the first time as a separate theorem.

Gershgorin's paper is imprecise. It begins with a statement of Taussky's Theorem II, but without the hypothesis that A is irreducible. (I have often wondered how this could have happened.) This omission now appears glaring, but neither of the excellent reviews in the Zentralblatt (Wegner) or Jahrbuch (Wielandt) noted it (or a resulting mistake). One can only suppose that the omission was far from obvious in 1931. Taussky is obviously aware of the mistake, she does not comment on it, but quietly states the result correctly.

Though (for stochastic matrices) the result can be found in Fréchet (1933, 1938) [he refers to Tambs–Lyche (1929)] and the result was undoubtedly known to several mathematicians in the 1930s and 1940s, I know of no further developments until Brauer (1946, 1947, 1948) and Taussky (1948, 1949). In particular, I have found no published reference to Gershgorin (1931) before Taussky (1949) with the exception of Wittmeyer (1936). Perhaps this point illustrates the very great change that has occurred in the status of Gershgorin's theorem since 1949.

2.4 There are few occasions where a mathematician records in print the events that led him to study a certain problem. Thus most accounts of the development of a topic omit a certain level of personal history, as anyone who is active in research surely knows. I was most curious about the events that resulted in Olga Taussky's (1949) note,[10] for I knew only that somehow her interest in this subject was connected with her war-time work for the British Ministry of Aircraft Production. I have asked her to describe how she got interested in the (Gerschgorin) circles theorem, which she calls Γ.

Γ: First of all, I seemed unusually interested in Vienna in the "recurring theorem" as soon as Furtwängler got to it in his course on algebraic number theory, when proving Dirichlet's unit theorem. He proved it by induction and it is really quite an interesting proof, but I did not like it then. I think the next proof was by Artin (1932). I observed people refinding this theorem. The one evening Aronszajn visited our apartment in London during the war and application of functional analysis to numerical analysis was the main issue. I suddenly pricked my ears when he mentioned there was a good approximation to eigenvalues via Γ-theorem, and I questioned him for I thought that such a result for flutter matrices would interest my boss, R. A. Frazer. He gave me the year 1931 and the Zentralblatt review. This is a very well written review. I saw the actual paper much later in the British Museum. I immediately tinkered with the theorem, applying it to a very nasty looking 6×6 matrix of complex elements given with many decimals, revealing off hand nothing about its stability (which for practical purposes means, no eigenvalues with large negative real parts, for a plane is not required to fly at ∞ speed). I am enclosing zerox copies of the circles. Number I gives the 6 circles, one containing points with large negative real part; Number II comes after a diagonal similarity which already excludes a large part of the negative real axis; Number III then shows what happens if we expand the shrunk circle again and shrink the other 5. This seemed great fun. I then realized an optimum for this process must be possible, this was carried out much later by Henrici, Jack [11] (1965), Varga (1965), etc. At the time I wrote a report for the Aeronautics Research Council (1947) which contains most of what is in the monthly article, apart from the equality case which I carried out under prodding from G. B. Price. It was he, who in 1947 pushed me into writing the article. [12]

2.5 To appreciate the wider significance of Taussky's (1949) note, we briefly and necessarily inadequately, discuss the history of matrix theory. Recent historians [Kline (1972), Hawkins (1975, 1977)] have stressed that even before Sylvester's (1850) definition of matrix there were results which today are considered matrix theoretic, though obviously they were then expressed in different terms. There was a well developed theory of determinants, and there was some spectral theory, for example, the remarkably early Cauchy (1829) interlacing theorem for the eigenvalues of submatrices of a symmetric matrix. Both of these belong to an aspect of the subject I shall call *inward matrix theory*. The choice of name is nicely illustrated by the second part of the quotation from Sylvester (1851) in section 0: here matrices look downward and inward to their children the minors. While I am not ready to give a characterization of inward matrix theory, one may describe some attributes: one is usually concerned with properties of a single matrix (e.g. the location of its spectrum), consideration of its elements is important (so one tends to forget that the matrix is also a linear operator on some space), while multiplication of matrices plays little role and addition hardly any at all. For the sake of contrast, we shall also speak of *outward matrix theory*, though as we shall presently explain, the outward aspects of the subject have largely ceased to be part of matrix theory, as presently understood.

In the 1850s there was an outward turn of matrix theory, and one may speculate that this was due in part to the profound psychological effect of the definition of "matrix" as a separate and independent entity—I am tempted to say matrix theory turned outward at least partly because matrices had been named.[13] In distinction to its previous role, a matrix was no longer merely a "schema" for writing a determinant or linear substitution but a "quantity", "if it be allowed that that term is properly applied to whatever is subject of functional operation" [in the words of Sylvester (1884)]. The future is foreshadowed in our quotation from Cayley[14] (1858), see section 0, and the magnitude of the change as it appeared to his contemporaries is stressed in Sylvester's (1884) "Apotheosis of Algebraic Quantity".

> At first undifferentiated from their content, matrices came to be regarded as susceptible of being multiplied together, the word multiplication, strictly applicable at that stage of evolution to the content alone, getting transferred by a fortunate confusion of language to the schema . . ., but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema, and stood revealed as bona-fide multiple quantity . . .. This revolution was effected by a forcible injection into the subject of the concept of addition . . . a notion, as it seems tome, quite foreign to the idea of substitution, the *nidus* in which that of multiple quantity was laid, hatched and reared.[15]

A revolution indeed! In outward matrix theory, matrices ("single quantities") look outward and upward to those great societies of groups and algebras of which they are members. A good example is the theory of (linear associative) algebras in the late nineteenth century culminating in Wedderburn's (1908) structure theorem on simple and semi-simple finite dimensional algebras.

In this century, new levels of abstraction have transformed mathematics. In algebra, the lectures of E. Artin and Emmy Noether in Goettingen in the 1920s led to v. d. Waerden's (1930) famous book *Moderne Algebra*—and the abstract axiomatic approach to algebra continued to be known by the name of modern algebra until quite recently [see Birkhoff (1976a,b) for an interesting account]. Matrices play an important role here, but they are the servants,[16] not the masters. Thus outward matrix theory was absorbed by modern algebra and disappeared as an area of mathematics. For example, the proper setting of Wedderburn's structure theorem is now no longer finite dimensional algebras but Artinian rings. It is even possible to view the theorem as a special case of the Jacobson density theorem for primitive rings[17] [e.g. Herstein (1964, Ch. 2)].

Abstract algebra represents one of the major advances in mathematics in the first half of this century. Yet it implied a restriction of the term "algebra" and a completion of the separation[18] of "algebra" and "analysis". Inward matrix theory, which had flourished through the time of Perron–Frobenius, drew much of its strength from classical analysis. [Observe that Perron–Frobenius first appeared in Perron's (1907a) paper on continued fractions, see also Lyapunov's (1892) theorem discussed in section 3.] Thus inward matrix theory could not be axiomatized in the modern manner, and since algebra was now identified with modern algebra, the former became homeless,[19] old-fashioned and unimportant, and thus it withered. In the period 1918–45 there were few significant papers, and, unlike in the nineteenth century, the exceptions were written by men who were primarily analysts: Ostrowski (1937a), and Gantmacher and Krein (1937).

Those of us who worked on matrices in the 1950s and early 1960s had a feeling of being in a no-man's land in the center of a triangle whose vertices were algebra, analysis (e.g. generalizations of Perron–Frobenius in functional analysis) and applied mathematics (e.g., applications of diagonal dominance to economics). Somehow, we were being ignored by other mathematicians.

In 1976, matrix theory may still be in the center of this triangle, but the center is now a vantage point from which one may stimulate research in the areas of algebra, functional analysis and applications, and is again a discipline in its own right.[20] Undoubtedly the need for numerical linear algebra adapted to computers has inspired much study of the theory of matrices [cf. Birkhoff (1976b)]. The subject is now "classical" rather than "old-fashioned"—two words without a difference in meaning but reflecting different attitudes. A fresh Ph.D. in matrix theory is more in demand than a group-theorist, and I shall let the reader decide whether this is cause or effect. But there is no doubt about one phenomenon: the period since 1950, particularly since 1960, has seen a spectacular increase in publication of papers concerning properties of a single matrix.[21] This resurgence of inward matrix theory can be traced directly to papers we have mentioned[22]: A. Ostrowski (1937a), A. Brauer (1946, 1947, 1948), O. Taussky (1948, 1949) and H. Wielandt (1950). We may note an interesting phenomenon: none of these papers used a major result unknown to Frobenius (1912). Thus the branch point of inward matrix theory from abstract algebra, is not 1950 but 1907 or earlier. For, surely Perron–Frobenius belongs to inward matrix theory, but Frobenius' (1908) proof depended critically on the intermediate value property of a continuous function which is by no means algebra in the modern sense.

A renaissance must start with a renewed and novel appreciation of the classics. This was provided by Olga Taussky's (1949) note and that is the significance in the development of matrix theory of that short contribution in a journal largely devoted to exposition at the college level.

2.6 We make no attempt to trace the history of the "recurring theorem" beyond 1949, for it is virtually impossible to make a complete bibliography of related results that have appeared since then. However, as a footnote to the influence of Olga Taussky's (1949) note, we point to a paper whose title is reminiscent of Taussky's note and which may be as influential to the next generation of mathematicians as Taussky's note was to the last. We refer to Varga's (1976) "On recurring theorems on diagonal dominance".

3. INERTIA THEORY

3.1 We shall let Olga Taussky tell what led her to work on Lyapunov's theorem, which she calls Λ.

You asked me concerning the origin of my interest in Lyapunov's theorem. This is a strange story. Well, this theorem really closed in on me from various sides, actually before I saw it in Gantmacher (1953), Bellman (1960), W. Hahn (1955, 1959). It had an unusual attraction for me. By the way, from Arrow and McManus (1959) papers I took over only the concept of D-stability. However, there was a paper by D. C. Lewis (1951) in *American J. Math.* and which comes somehow near to it. I do not know whether I learnt about D. C. Lewis's paper via Bass' report or vice versa, or through a M.R. review. I immediately realized that I could reprove Lewis' theorem and in my joint paper with him we found a much simpler approach. I suppose you heard Antosiewicz' talk at Argonne in 1967 at the Γ–Λ conference and maybe you saw my remarks about his work in my Besançon (1968a) lecture. He pointed out that the needs of the analysts are much more complicated. He wanted me to do more on the algebraic side. In, I think, 1960 I gave a course on matrix theory mainly to geophysics students (the algebraists did not come to a matrix course) and it suddenly came to me to lecture on Λ. By then Gantmacher (1953) and Bellman's (1960) books existed and I saw some of Hahn's work and Arrow's papers (which I think Jack[11] pointed out to me) and while preparing my course I wrote my little note (1961a) for the Bellman Journal (*J. Math. Anal. Appl.*) which pleased me a lot. I showed it to Ostrowski who made no comments, but as I found out later, was quite interested. Givens noticed my abstract in the Notices about it and you wrote to me for a preprint. Then Householder invited me to lecture on it at Gatlinburg[23] and I tried to complete something new. I completed this about a day before our departure. We had to give a party in our house and Antosiewicz was present at it and I remembered that I made him check what later became my SIAM note (1961b) in our kitchen during the party. The remainder is known to you. However, there is still one other sign of my interest in Λ a year prior to my matrix course. Zassenhaus visited here during that year and Erdélyi needed a proof of a Λ-related equation for general matrices, in stability, found in a book by Lefschetz. Zassenhaus proved what was needed in an algebraic way and I think, if you want it, I ought to be able to dig it out. But, what is amusing, is the fact that I felt really excited about it.

Somewhat later, in February 1977, she wrote:

when I write to you I sometimes forget to mention the *most* important facts. One of these concerns Λ. The reason why Λ fascinates me so much is two-fold:

(1) My interest in criteria (computational if possible) for the stability of a complex matrix. This was stimulated in my war work.

(2) The *much* bigger reason is that the Δ expression is a generalized Jordan product.[24] There I have some great unfulfilled dreams (which I was able to transmit to Givens who studied Jordan and Lie algebras). Soon after I wrote my first Δ-paper I wrote to Jacobson asking whether generalized Jordan products had been used and he said "yes", in representation of Lie algebras. He gave me a reference to his own work. Two years ago in his retiring address at San Francisco he had expressions like the Lyapunov, or Stein transform. Now this appeared in *Advances of Mathematics* (1976), but is much more abstract, I have not studied it. [A more accessible thing for me is a paper by U. Hirzebruch (1974) published in LAA.]

Several years ago I became *more* than excited when I heard about a Minnesota report by Koecher linking Jordan products with positivity. I feel strongly that this, when transferred to generalized Jordan products, would give a wonderful insight into Δ. He gave an invited address at Nice (1971), and both Givens and I think there may be related ideas in it.

3.2 "Lyapunov's theorem"[25] here refers to a result in the great memoir on the stability of solutions of differential equations—Lyapunov (1892, Ch. 2, section 20, Theorems 1, 2, 3). A special case can be formulated in terms of matrices [e.g. Gantmacher (1953, Chelsea Ed. vol. 2, p. 189, Theorem 3′ and Note, p. 190)] and then the theorem deals, as in the case of the Gershgorin circles theorem, with the location of the spectrum of a matrix. But in this case the question concerns the location of the spectrum in the right (or left) half plane. Taussky's interest led to the papers Lewis–Taussky (1960) and Taussky (1961a), which were quickly followed by papers by herself (1961b, 1964a) and Taussky–Wielandt (1962) and by others: Givens' (1961) report, Ostrowski–Schneider (1962), and Carlson–Schneider (1963). After this, the papers become too numerous to list, though we must mention Taussky's general review of the state of the theory in her (1968a) Besançon lecture, see also her surveys (which always add something new) (1967, 1968b). For the years 1972–7, I have counted 17 papers in this area in just one journal (perhaps not chosen entirely at random, for the journal is *Linear Algebra and its Applications*). The subject takes its importance not least from its applications to other fields, e.g. control theory, see Barnett (1971). The generalizations of Lyapunov's theorem are often called "inertia theory", and the name derives from the fact that a theorem in Taussky (1961b) and the slightly more general, Ostrowski and Schneider (1962, Theorem 1) contain not only Lyapunov's theorem but also the even more classical Sylvester (1852) inertia theorem,[26] which can be found in any textbook on matrices. For this reason we called Ostrowski and Schneider (1962, Theorem 1) the *Main Inertia Theorem.*

It should be recorded that there were others who proved results closely related to the main inertia theorem, even before the work I have just described. First, a similar result was known to M. G. Krein in the U.S.S.R. in the 1950s and though he circulated some notes, his work was not published till Daleckii and Krein (1970, Ch. I, §7). Krein's work influenced the theory of operators on Hilbert space, but as far as I can tell, did not lead to results analogous to the many finite matrix theorems found in the West. Second Wielandt's (1951) report contained an extension of Sylvester's inertia theorem, and though Lyapunov's theorem was not mentioned by Wielandt, it is not hard to deduce the second part of the main inertia theorem from this result. His report, however, was not published till 1973, for a short-sighted (if not blind) referee rejected his paper. Thus the work in this area by Krein and Wielandt was hardly known in the decisive decade of the 1960s and Olga Taussky's influence was paramount.

The debt that Ostrowski and I owe her is acknowledged in our (1962) paper, which was written before we knew of her overlapping (1961b):

> The equivalence of Lyapunov's theorem with some of the results of Arrow and McManus (1958) was noticed by Olga Taussky (1961a), who let us see her unpublished manuscript and thereby sparked this (1962) investigation.

When a "theorem closes in from various sides" (cf. 3.1), it is always possible to ask what might have happened "if": if Wielandt (1951) had been freely available in the 1950s, would someone have realized the link between Lyapunov and Sylvester inertia, and so would the main inertia theorem have been found several years earlier?[27] But there is no doubt about what actually happened. It was Olga Taussky who was aware of the role of Lyapunov's theorem in differential equations in the 1950s, it was she who pulled together the various strands (Lyapunov, S and D-stability), it was she who took the initial step beyond Lyapunov's theorem. In the initial stages of the revival of inertia theory, Olga Taussky's impact was clearly greater than that of any other single mathematician, and probably as great as that of all others combined.

4. THE TAUSSKY UNIFICATION PROBLEM

4.1 Throughout her mathematical career, Olga Taussky has posed and published many research problems, some of which have become well-known. On page 124 of the *Bulletin of the American Mathematical Society* (1958) there appeared three problems by Taussky, and we quote part of one of them: (Her examples 2 and 4 are omitted, but we have retained her numbering)

> A number of similar theorems are known for matrices with positive elements (positive matrices) and for positive definite symmetric matrices, but for which the available proofs are different. Can a unified treatment be given for both cases? Four examples of such theorems are:
>
> 1. The dominant eigenvalue exceeds the diagonal elements.
> 2. . . .
> 3. The inequality
>
> $$\det_{i,k=1,\ldots,n}(a_{ik}) \leqslant \det_{i,k=1,\ldots,p}(a_{ik}) \cdot \det_{i,k=p+1,\ldots,n}(a_{ik})$$
>
> for matrices with non-negative minors of all orders and for positive definite symmetric matrices.
> 4. . . .

The problem will be called the *Taussky unification problem* by us. A similar problem appears in Taussky (1962). As far as I know, Taussky has not published any solution to the problem, nor has she written papers which deal specifically with this problem, though she mentions it in a forthcoming (1978) review paper.

Yet, her problem is famous, and has influenced much subsequent research. It should be observed that the problem is not specific, and cannot be solved by a "yes" or "no". It is precisely the vagueness inherent in the problem that has led to its fruitfulness, for it is unlikely that any single theorem will ever be considered *the* solution of the unification problem. Thus there are several possible solutions, depending on which common features of the classes of matrices are considered. Two different solutions are presented in (4.3) and (4.4); there have been others. Perhaps the best solution of the Taussky unification problem is yet to come.

4.2 Before we discuss the unification problem, we shall introduce some definitions, most of which are standard. In these A will always denote a square matrix with complex elements. We write $\langle n\rangle = \{1, \ldots, n\}$ and for $\phi \subseteq \alpha, \beta \subseteq \langle n\rangle$, $A[\alpha|\beta]$ is the submatrix of A indexed by the rows of α and columns of β (in their natural orders), while $A(\alpha|\beta) = A[\alpha'|\beta']$ where $\alpha' = \langle n\rangle/\alpha$, $\beta' = \langle n\rangle/\beta$. Also $A[\alpha] = A[\alpha|\alpha]$ and $A(a) = A(\alpha|\alpha)$, in the case of principal submatrices. The determinant of a submatrix will be called a *minor*. Further we say

A is positive ($A > 0$):	$a_{ij} > 0$,	all i, j
A is nonnegative ($A \geqslant 0$):	$a_{ij} \geqslant 0$,	all i, j
A is a **Z**-matrix:	$a_{ij} \leqslant 0, i \neq j$,	all i, j
A is Hermitian:	$a_{ij} = \overline{a_{ji}}$,	all i, j
A is a **P**-matrix:	all principal minors of A are positive	
A is a (nonsingular) **M**-matrix:	A is a **Z**-matrix and a **P**-matrix	
A is positive definite:	A is Hermitian and a **P**-matrix	
A is totally nonnegative:	all minors of A are nonnegative	
A is totally positive:	all minors of A are positive.	

The above terminology of **P** and **Z**-matrices can be found in Fiedler and Ptak (1962), who, however, denote the class of **M**-matrices by **K**. Of course, $\mathbb{Z}$ is elsewhere used to denote the integers but we have avoided this symbol. Also note that in Ostrowski (1937a) and many subsequent papers, **M**-matrices are allowed to be singular.

Some other classes of matrices will be mentioned later. Observe that for the sake of simplicity our notation does not indicate the order of the matrices involved. When necessary, we write $\mathbb{C}^{nn}$ for the set of all $(n \times n)$ complex matrices.

4.3 For $A \in \mathbb{C}^{nn}$ and $\phi \subset \alpha \subset \langle n \rangle$, the inequality in Part 3 of Taussky's unification problem can now be restated as

$$\det A \leqslant \det A[\alpha] . \det A(\alpha). \tag{HF}$$

For positive definite matrices, this inequality is due to Fischer (1908), and we shall call it the *Hadamard–Fischer* (HF) inequality for it generalizes the inequality

$$\det A \leqslant a_{11} \ldots a_{nn} \tag{H}$$

proved by Hadamard (1893), and which is always called the *Hadamard* (H) inequality. In fact, an easy consequence, for all real matrices A,

$$|\det A| \leqslant s_1 s_2 \ldots s_n,$$

where

$$s_i = \sqrt{\sum_{j=1}^{n} (a_{ij})^2}, \qquad i = 1, \ldots, n,$$

was conjectured in 1885 by Lord Kelvin (Sir William Thompson) and proved by Sir Thomas Muir that year, but Muir's work was not published[28] till (1901/2). By now there are many proofs of (H) or (HF) for positive definite matrices, and we shall not attempt to outline them, but see Ostrowski (1937b) or Bodewig (1953, Ch. I) for a proof based on the inequality between the geometric and arithmetic means.

The inequality (HF) was known to hold for **M**-matrices, a consequence of Ostrowski (1937a, Th 1). In the case of totally nonnegative matrices, see Gantmacher and Krein[29] (1935, 1937) or (1960, Theorem 8, p. 108) for the special case $\alpha = \{1, \ldots, k\}$, $1 \leqslant k \leqslant n$. To deduce the inequality for all α, $\phi \subset \alpha \subseteq \langle n \rangle$ requires a little work, for the totally nonnegative matrices are not closed under simultaneous permutation of rows and columns, see Engel and Schneider (1976) for further details.

As in the arts, in mathematics the past is encapsuled in the present, though in our field this has not been exploited to a sufficient extent. Thus, I shall now indulge in one of my favorite activities without further apology. That is, I will examine some classical proofs to see what results one may see there with hindsight; results which the author probably did not observe. In this case the hindsight is provided by Taussky's unification problem, and the first proof we shall examine is Hadamard's (1893) original proof of his inequality (H). Actually Hadamard proved an inequality for positive definite matrices intermediate between (H) and (HF), which we shall denote by (H′):

$$\det A \leqslant a_{11} \det A(1). \tag{H′}$$

Hadamard's proof of (H′) *for positive definite* ($n \times n$) *matrices:*

First observe that

$$\det A = a_{11} \det A(1) + \det B$$

where the matrix B is obtained from A by replacing a_{11} by 0. So we must show that $\det B \leqslant 0$. The result is clearly true if the order of B is 1. Assume inductively that the result holds for matrices of order less than n. By means of "l'identité bien connue",[30] which turns out to be (a special case of) Sylvester's (1851) identity,

$$\det B.\det B(1,2) = \det B(1)\det B(2) - \det B(1|2)\det B(2|1).$$

Some of the above submatrices are equal to the corresponding submatrices of A. Thus replacing B by A wherever possible, we have

$$\det B.\det A(1,2) = \det A(1)\det B(2) - \det A(1|2)\det A(2|1).$$

But $\det A(1,2) > 0$, $\det A(1) > 0$, $\det B(2) \leqslant 0$, by inductive assumption. Also $\det A(1|2)$ is the conjugate of $\det A(2|1)$ since A is Hermitian, whence $\det A(1|2)\det A(2|1) \geqslant 0$. Hence $\det B \leqslant 0$. Induction completes Hadamard's proof of (H′) for positive definite matrices.

We now observe that in the above proof it is not necessary to assume that A is Hermitian. The following assumptions suffice:

(W_1) A is a **P**-matrix,
(W_2) the product of symmetrically placed almost principal minors of A is nonnegatived,

where a submatrix of $A[\beta|\gamma]$ is *almost principal* if $\beta = \alpha/\{i\}$, $\gamma = \alpha/\{j\}$, and $\phi \subset \alpha \subset \langle n\rangle$, $i, j \in \alpha$, $i \neq j$ (viz. $A[\beta|\gamma]$ is obtained from the principal submatrix $A[\alpha]$ by deleting the ith row and jth column, $i \neq j$). Matrices satisfying (W_2) were called *weakly sign symmetric* (w.s.s.) by Carlson (1967) and we shall adopt this name. The point of the above remarks is that there is no indication that Hadamard had observed that a w.s.s. matrix with positive principal minors satisfied his inequality, but that a *proof* of this result may essentially be found in his paper of (1893) (we define "essentially" to mean that any first year mathematics graduate student who is told the more general theorem could modify the proof in Hadamard's paper to prove the more general result).

It is obvious that the class of w.s.s. **P**-matrices includes the class of totally positive matrices and the class of positive definite matrices. That the M-matrices also belong to this class follows from the result that $A^{-1} \geqslant 0$ if A is an **M**-matrix [Ostrowski (1937a) our of Frobenius (1908)]. Since an **M**-matrix A is also of the form $A = sI\text{-}B$, where $B \geqslant 0$, the Hadamard–Fischer inequality for w.s.s. **P**-matrices constitutes a solution of part of Taussky's unification problem.

The inequality (HF) for w.s.s. **P**-matrices is in fact known, and appears to be due to Kotelyanskii (1953), see also Gantmacher and Krein (1960, Theorem 9, p. 111) (indeed w.s.s. **P**-matrices are called *GKK* matrices by Fan (1967), in honor of these authors who thus proved a unification theorem before the appearance of Olga Taussky's Bulletin problem). Thus the result had to wait 60 years for discovery, presumably because no one had examined Hadamard's proof with the Taussky unification problem in mind. One should bear in mind that Hadamard's inequality was not neglected in the years 1900–20; Muir (1930) contains 11 pages of reviews of papers devoted to this inequality.

It is surprising, but the inequality does *not* hold for all w.s.s. matrices with nonnegative principal minors, see Carlson (1967) for an example. Thus, if we denote by **W** the set of all w.s.s. $(n \times n)$ matrices with nonnegative principal minors, it follows that **W** strictly contains the closure of the w.s.s. **P**-matrices, i.e. there is a matrix in **W** which is not the limit of a sequence of w.s.s. **P**-matrices. This suggests a problem (first mentioned to me by Carlson) which is so far unsolved: Describe the structure of the set **W**.

At this point it should be mentioned that the w.s.s. **P**-matrices can be characterized by an inequality which is more general than Hadamard–Fischer:

THEOREM [Carlson (1967)] *Let A be a* **P**-*matrix. Then A is weakly sign symmetric, if and only if for all α,β, $\phi \subset \alpha,\beta \subseteq \langle n\rangle$,*

$$\det A[\alpha \cap \beta].\det A[\alpha \cup \beta] \leqslant \det A[\alpha].\det A[\beta].$$

One direction of this theorem is based on results by Gantmacher and Krein (1960, p. 111, Satz 9 and Folgerung), and Kotelyanskii (1953). Carlson's work which completes the results of these Russian mathematicians, certainly influenced by Taussky's (1958) research problem, which is mentioned in his bibliography. But beyond that, Carlson recalls that I brought the problem to Wisconsin from Oak Ridge in 1961, where I had been talking to Alston Householder, who certainly was in touch with Olga Taussky. There was also some input from Ky Fan [and indeed some subsequent work (1967) by him as well as by Fiedler and Ptak (1966) we have not discussed], and he knew of Olga Taussky's (1958) announcement, and is, in fact, mentioned in it. Further acknowledgement of the influence of Olga Taussky's research problem can be found in Bauer (1975).

4.4 We now turn to Perron–Frobenius theory. In two papers Perron (1907a,b) published proofs of his famous theorem on positive matrices. Frobenius gave proofs of Perron's theorem in (1908, 1909). In (1912) he extended the theorem to nonnegative matrices in a highly nontrivial manner. We shall not further discuss the nature of these theorems, except that we shall examine Frobenius (1908) proof[31] of part of the theorem.

THEOREM *Let P be a positive $(n \times n)$ matrix. Then P has a positive eigenvalue $\rho(P)$ such that* adj $(rI-P) > 0$ *for $r \geqslant \rho(P)$.*

Here adj $(rI-P)$ is the usual adjugate (adjoint). (Frobenius also proved that $\rho(P)$ is the spectral radius of P; but we ignore this important point.)

Frobenius' proof The proof proceeds by induction. So let us suppose the result holds for $P(1)$. We put $A = -P$, $A_t = tI+A$ and following Frobenius, we expand as he did in (4.3):

$$\det A_t = (t+a_{11}) \det A_t + \det B_t$$

where (again) B_t is obtained from A_t by replacing $t+a_{11}$ by 0. But now Frobenius uses a special case of what has become known as the Schur complement formula:

$$\det B_t = -x' \operatorname{adj} B_t(1)y$$

where

$$x' = (a_{12}, \ldots, a_{1n}), \qquad y' = (a_{21}, \ldots, a_{n1})$$

and y' is the transpose of y.

Observe that $B_t(1) = A_t(1)$. But by the inductive hypothesis $P(1) = -A(1)$ has a positive eigenvalue $\rho(P(1))$ such that for $t \geqslant \rho(P1))$, adj $A_t(1) > 0$. Hence for such t, also $x' \operatorname{adj} A_t(1)y > 0$, whence $\det B_t < 0$. We deduce that for $t = \rho(P(1))$,

$$\det A_t < (t+a_{11}) \det A_t(1).$$

If $t = \rho(P1))$, then $\det A_t(1) = 0$ and so $\det A_t < 0$. But for large t, $\det A_t > 0$. Hence there is a $\rho(P) > \rho(P(1))$, such that for $t = \rho(P)$, $0 = \det A_t = \det(tI-P)$.

This argument does not finish the inductive step, for one still has to prove that adj $A_t > 0$, when $t \geqslant \rho(P)$. This Frobenius does by a similar expansion, but we shall not discuss the argument in detail.

We exclaim that a crucial step in Frobenius' proof is Hadamard's inequality (H′) this time for matrices of form $tI-P$, where $P > 0$ and t is sufficiently large. There is no difficulty in adapting Frobenius' argument to nonnegative matrices or to restate the conclusion for $A = -P$, i.e. for **Z**-matrices. By means of a slight modification of his argument, we may break it up into two steps (a) and (b). Thus

a) Let A be a **Z**-matrix. Then
   (H'_+): for all real t, and all $\alpha, \phi \subset \alpha \subseteq \langle n\rangle$, such that $A_t[\alpha]$ is a **P**-matrix, and all $i \in \alpha$,
   $$\det A_t[\alpha] \leqslant (t+a_{ii}) \det A_t[\alpha\backslash\{i\}].$$
b) If A is a matrix with real principal minors which satisfies (H'_+) then
   (ω_1) Every principal submatrix $A[\alpha]$ of A has a real eigenvalue, and
   (ω_2) If $\lambda(A[\alpha])$ denotes the least real eigenvalue of $A[\alpha]$ and $\phi \subset \beta \subset \alpha \subseteq \langle n\rangle$, then $\lambda(A[\alpha]) \leqslant \lambda(A[\beta])$.

Clearly the Hermitian matrices satisfy (H'_+) by Hadamard's result. Hence we have in (b) a unification of Hermitian and **Z**-matrices of the type desired.

The totally positive matrices also satisfy (ω_1) and (ω_2), Engel–Schneider (1976), but the weakly sign symmetric matrices do not satisfy (ω_1) (they are not closed under addition of positive numbers, thus this unification is somewhat different from the previous one.

It can in fact be shown that the converse of (b) also holds. Indeed one can prove somewhat more. First let us state the required form of the Hadamard–Fischer inequality as

(HF_+) For all real t and all $\alpha, \beta \subset \alpha \subset \langle n \rangle$ such that $A_t[\alpha]$ is a P-matrix, and for all $\beta, \phi \subset \beta \subset \alpha$,

$$\det A_t[\alpha] \leqslant \det A_t[\beta] \det A_t[\alpha \backslash \beta],$$

where, as usual, $A_t = A + tI$. We also denote by (ω) the logical conjunction of (ω_1) and (ω_2).

THEOREM [Engel–Schneider (1976, Theorem 3.12)] *Let A be a complex $(n \times n)$ matrix with real principal minors. Then the three conditions* (H'_+), (HF_+) *and* (ω) *are equivalent.*

4.5 Engel and I thought the class of matrices satisfying the Theorem I have just stated was sufficiently important to justify our naming it. Se we called a matrix A satisfying (ω) (or, of course either of the other two equivalent conditions) an *ω-matrix*. An ω-matrix all of whose principal minors are nonnegative we called a *τ-matrix*. It is gratifying to note that there have already been two further papers on ω and τ-matrices, viz. Engel and Varga (1977) and Varga (1977). Just why we chose those particular names may or may not be a mystery:

$$\omega - - \tau,$$

and will be left unsaid. Instead I would like to describe what led to the invention (discovery, identification; the choice of words is left to the reader) of ω-matrices.

A day or so before the Gatlinburg[23] meeting held at Los Alamos in June 1972, Engel and I were driving through New Mexico, admiring the scenery. We were discussing our paper on improvements of the Hadamard-inequality for M-matrices [viz. Engel and Schneider (1973)] which—if my memory serves me right—was already largely written up. Suddenly Engel said to me "Do you realize that our main result for M-matrices also holds for positive definite matrices?" I had not observed this point, so naturally I replied: "Of course, that's obvious. All we use in our proofs is a property [like the one I have here called (ω)] which is shared by both classes of matrices". In other circumstances I might have dismissed this observation, or considered it as a minor addition to our paper, but instead I added "I think we have a solution to Olga Taussky's unification problem". I should be mentioned that, with the unification problem in mind, for some years previously I had been staring at Frobenius' (1908) proof, convinced that it "really" proved a theorem for a class of matrices much wider than the class of positive matrices. But every attempt to identify the class or the theorem had failed or resulted in triviality. At that moment in the car I was convinced that (ω) provided the clue to the desired unification. It would be going too far to say that I saw clearly how this was to be done in detail but I was sure that it was only a matter of time and energy to formulate the right property and that the link between Hadamard–Fischer and Perron–Frobenius was closer than I had previously suspected. It was the sort of obscure but strong conviction which I have had on a few occasions before and since, which I cannot explain but has never been substantially wrong. Yet the decisive moment would have passed, perhaps leaving no trace, had Engel and I not been strongly sensitized to the possibilities of various kinds of unifications by long familiarity with Taussky's problem. It might seem presumptious of me to report at such length on the origin of one of my papers, for it is just one of about 30,000 in mathematics that appear each year. But I am describing a phenomenon that is not confined to just one or two instances. And that is why this tribute is in order, for to express the point in the pop-psychological terms of the 1970s: Olga Taussky's work has altered the consciousness of several generations of matrix theorists. Long may her influence continue.

5. NOTES

[1] In 1852, the term "algebra" was not strongly differentiated from "analysis". In the 1850s, papers in pure mathematics published in Crelle's journal were classified as analysis, geometry or mechanics. Thus some of Grassman's work appeared as geometry while Cayley (1855) was classified as analysis.

[2] The first occurrence of the word "matrix" appears to be in Sylvester (1850). This paper gives some insight into the immediate reasons that led Sylvester to abandon the then customary practice of employing the term "determinant" without differentiating between a square matrix and its determinant. After taking an intermediate position ("Imagine any determinant set out under the form of a square array of terms"), he considers an "oblong arrangement of terms", is thus forced to make an explicit distinction and so introduces the word "matrix", cf. Kline (1972) for a similar remark. The old practice, however, of using "determinant" where any editor would now insist on "matrix" persisted through Frobenius [cf. Schneider (1977, Note 12)] and can be found as late as Ostrowski (1937a), who spoke of "M-determinant" in place of the now standard "M-matrix".

[3] By convention, a date enclosed in parentheses refers to a publication. Normally the author's name precedes the date, but in order to avoid tedious repetition, occasionally the author's name is omitted.

[4] So much so, that the Cayley–Hamilton theorem (a matrix satisfies its own characteristic equation), which is stated in Cayley (1858), was written for a (2×2) matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in the remarkable notation

$$\begin{vmatrix} a-M & b \\ c & d-M \end{vmatrix} = 0.$$

[5] The first part of the quotation is taken from (!) the first sentence of the "apotheosis" which forms the introduction to Sylvester's (1884) lectures on universal algebra (viz. matrix algebra). The first paragraph of the "apotheosis" is perhaps the most wonderful piece of mathematical prose in existence. Some might call it wonderfully obscure because of its Hegelian dialectic and Koestlerian hierarchies. If the three pages of the introduction were abridged to a few sentences like "The definition of addition of matrices is due to A. Cayley (1858)", its historical and mathematical content would be preserved, but its deep insight into the process of mathematical abstraction would be lost.

[6] Desplanques describes himself as "élève à l école préparatoire de Sainte-Barbe, classe de M. André".

[7] For a real matrix A the assumption $a_{ij} \leqslant 0$, $i \neq j$, $i, j = 1, \ldots, n$ is not needed here, see a note by L. L. (presumably Lóvy) in Desplanques (1886), Furtwängler (1936), Parodi (1946) Taussky (1948, Theorem IV), and the related result, Rohrbach (1931, Theorem 2).

[8] The three references in this paragraph were communicated to me by Olga Taussky, and the references to Nekrasov (1892) and Janet (1920) were sent to me by A. Ostrowski.

[9] Other independent discoveries are Janet (1920, pp. 145–151) and Tambs–Lynch (1928) (his proof proceeds via differential equations). It is rather hard to discover which of the theorems we have listed Gershgorin was familiar with; there is a laconic reference in (1931) to "Levy's Theorem". In Figure 1 we interpreted this as Lévy (1881). See also Bodewig (1953) for further information.

[10] Olga Taussky's first published remarks on this topic are to be found in a discussion on scholarship examination questions in the *Mathematical Gazette*, see Ref. 17 in her (1949), and Ref. 15 and 16 for previous remarks by other authors.

[11] Jack Todd, Olga's husband.

[12] Price's review of Massonnet (1945) in *Math. Rev.* 8 (1947), 499 lists eight references to the diagonal dominance theorem, see a forthcoming note by Price in this journal.

[13] "In names a mystic virtue lies
Concealed but clear to loving eyes,
And sounds have influence to control
The inmost workings of the soul."

Before the reader replies ". . . a rose by any other name . . ." and dismisses the sentiments of this anonymous poem, quoted by Sylvester in his *Laws of Verse* (1870, p. 91), he might recall that there are many examples of the importance of names and acts of naming in legend and drama. For example, in Wagner's Walküre (poem completed 1852), there are two scenes of namegiving. In one of them the new name expresses a man's transformed character and power ("Siegmund heiss' ich und Siegmund bin ich"). If, as many think, the roots of creativity in mathematics and the arts lie in the unconscious, then it is not absurd to look for specific instances of similar effects. However that may be, Sylvester's invariant twin, Cayley [the phrase is Bell's (1937, Ch. 21)] made a remark in (1855) which could hardly have been made before matrices were identified as separate objects of study: "Il y aurait bien des choses à dire sur cette théorie de matrices, laquelle doit, il me semble, précéder la théorie des déterminants."

[14] Cayley (1858) follows the sentences we have quoted at the head of this article by a statement of the Cayley–Hamilton theoren, cf. Note 3. This beautiful and famous theorem has both an outward and an inward component. If we ignore the latter we obtain as corollary: The algebra of $(n \times n)$ matrices is an algebraic algebra of degree n. We have stated this corollary not merely to mystify the nonexperts by technical uses of "algebra" and "algebraic" [see Herstein (1968, p. 14 and p. 155) for definitions], but to emphasize that a weak form of the Cayley–Hamilton theorem has a natural formulation in terms of the concepts of the next higher level of the hierarchy. I do not know if this could be said of any major theorem which is now regarded as matrix theoretic and which was proved before Cayley–Hamilton. The point is worth making for recently Cayley has rightly been dethroned from his position of the founder of *all* matrix theory, which he has held in mathematical tradition, cf. Hawkins (1975, 1977) for a detailed discussion. Sylvester's emphasis on Cayley (1858) has been blamed for the

historically shallow view frequently found in the literature such as Schneider (1964, p. ix): "The theory of matrices goes back to Sylvester and Cayley, particularly to Cayley's famous memoir of (1858)." But I believe the mistake is not Sylvester's, but that of his interpreters. For today we tend to use "matrix theory" to denote results with a strong inward component (and thus exclude results on algebras like Wedderburn's theorem), while the British–American school of the late nineteenth century used the term "theory of matrices" in an outward sense. This is most explicitly shown in Taber's (1891) identification of "theory of matrices" with the nascent field of linear associative algebras. To assume that Sylvester's (1884) lectures intended to include both inward and outward matrix theory, is to imply that he considered his determinantal identity of (1851) and his law of inertia of (1852) as of no account, since these are not mentioned in his lectures. This difference in emphasis in meaning of the term should be kept in mind when reading Sylvester and Taber.

[15] We suggest that "multiple quantity" as used by Sylvester has a meaning very close to "mathematical holon", where "holon" is Koestler's name for a Janus-like entity and is defined in his (1967, p. 48) immediately following the quotation in section 0. Further, the "forcible injection of addition" may be regarded as a bisociation, see Koestler (1967, p. 183) for definition.

[16] The first occurrence of a matrix in v.d. Waerden's (1930, Vol. II) chapter on linear algebra is as the representation of a linear transformation, and in v. d. Waerden, Vol. I (groups, rings, fields, etc.) "matrix" is listed just once in the index—the reference is to an example. Contrast this with the chapter on matrices in what may be the last "old algebra" textbook—O. Perron (1927).

[17] Unlike his successors, Wedderburn apparently considered his structure theorem part of matrix theory, for it may be found in Wedderburn's (1934) "Lectures on Matrices", but not in Gantmacher (1953) or any other modern textbook on matrices that I know. On the other hand, it would be surprising to find a textbook on non-commutative rings which does not mention the theorem, see Artin (1950) and the Preface to Faith (1976) (a model of its kind) for the role of this structure theorem in the development of ring theory.

[18] In his interesting survey of the rise of modern algebra, Birkhoff (1976a) writes, from a different point of view, of the "emancipation of algebra (from) proofs that depend on analysis" and lists some topics important in classical algebra that early modern algebra ignored (e.g. Sturm sequences).

[19] It's impossible to reverse history, the fight has been won and lost; inward matrix theory is no longer in the mainstream of algebra, but it is a living branch of the tree of mathematics. For this reason I prefer the term "matrix theory" to its synonym "linear algebra"—which is often not linear, and rarely algebraic. But, in order not to overstate my case, it must be noted that there are traces of inward matrix theory left in graduate algebra courses, for example see Jacobson (1974, p. 336–343) for a beautiful exposition of Sylvester's inertia theorem. Continuing in the vein of disclaimers, I am well aware that some of the most interesting results on matrices have both inward and outward components. Cayley–Hamilton has already been mentioned; another such class of theorems are the property P theorems with which Olga Taussky has been associated, see Taussky (1957). Our formulation is based on Goldhaber and Whaples (1953): Let T be a subalgebra of k^{nn} where k is an algebraically closed field. Then the following are equivalent: (i) T/rad T is commutative (outward). (ii) Any pair of matrices in T have property P (outward–inward). (iii) The matrices of T may be simultaneously triangulated by a similarity (more inward than outward).

[20] Here we are not asserting that matrix theory is disjoint from analysis or algebra. Rather we mean that there are matrix theoretic techniques and results which can be applied to problems in these fields and others [cf. Schneider (1967, p. ix) for a similar remark], and that this body of techniques and results constitutes a discipline because of their interrelations. We have a second meaning in mind, which we illustrate by an example. To regard the matrix version of Perron–Frobenius *merely* as a baby theorem in partially ordered topological vector spaces [for adult theorems see Schaeffer (1974)] is to miss much that was latent in Frobenius (1908), see section 4 of this article.

[21] For example, most of the 40 conditions equivalent to "A is a non-singular M-matrix" listed in Plemmons' (1977) survey have been found after 1950. Of course, M-matrix theory is just one branch of inward matrix theory. Also, U. G. Rothblum has counted about 100 papers referring to Perron–Frobenius in the 1960s. Many of these were in mathematical journals, but some are in journals devoted to econometrics, biometrics, ecology, operations research, computer science, genetics, nuclear engineering, statistics and physics.

[22] Ledermann [1950] and Price [1951] also deserve mention, we do not include Gantmacher and Krein (1937) which was little known in the west before similar results appeared in the German translation of their book (1960). In recent years Karlin and others have heavily used the ideas of Gantmacher and Krein, e.g. Karlin (1968).

[23] "Gatlinburg" refers to a sequence of meetings in numerical linear algebra, which take their name from Gatlinburg, Tenn., where they were held until A. S. Householder's retirement from Oak Ridge National Laboratory. For further information see Householder (1974).

[24] See Remark 3 in Taussky (1961a).

[25] In order to state the theorems discussed in this section, we first need a definition:

DEFINITION *Let $A \in \mathbb{C}^{nn}$, and suppose that A has π eigenvalues in the open right half-plane, ν eigenvalues in the open left half-plane and δ eigenvalues on the imaginary axis. Then the* inertia *of A is the triple*

$$\operatorname{In} A = (\pi,\nu,\delta).$$

SYLVESTER'S (1852) INERTIA THEOREM (in part) *Let H be Hermitian, and let S be non-singular. Then* $\operatorname{In}(S^*HS) = \operatorname{In} H$.

LYAPUNOV'S (1892) THEOREM (matrix version) *Let $A \in \mathbb{C}^{nn}$. Then there exists a positive definite H such that $AH+HA^*$ is positive definite if and only if* $\operatorname{In} A = (n,0,0)$ *(i.e. the spectrum of A lies in the open right half-plane).*

THE MAIN INERTIA THEOREM [Taussky (1961b), Ostrowski–Schneider (1962)] (i) *Let $A \in \mathbb{C}^{nn}$. There exists a Hermitian H such that $AH+HA^*$ is positive definite if and only if* $\operatorname{In} A = (\pi,\nu,0)$ *(i.e. A has no eigenvalues on the imaginary axis).*

(ii) *If $AH+HA^*$ is positive definite, then* $\operatorname{In} A = \operatorname{In} H$.

[26] This containment is not explicitly mentioned before Carlson and Schneider (1963), though Sylvester's inertia theorem is used in the proof of the main inertia theorem in Ostrowski and Schneider (1962). The omission is curious, for the importance of the main inertia theorem is surely partly due to the fact that it contains two nineteenth century theorems which previously appeared merely rather similar in spirit. But in the early 1960s the impetus came from Lyapunov, not Sylvester.

[27] Wielandt (1951) is referred to in Ostrowski and Schneider (1962), but the reference was inserted at a very late stage.

[28] The story is told in Muir (1909), where it is stated that a letter dated 1885 from Kelvin to Muir concerning the publication of the proof was shown to the President of the Royal Society of South Africa, but no cause for the delay in publication is given [see also Muir (1923, p. 32)]. Could it be that the inequality was so unrelated to the body of results on determinants that it seemed quite minor?

[29] Contrary to our usual practice of giving references to first editions of books, in the date of Gantmacher and Krein (1960) we refer to the appearance of the German translation of the second edition. The first edition was published in the U.S.S.R. in 1941, and I have never seen a copy. The second Russian edition appeared in 1950, and a typewritten English translation exists prepared by the U.S. Atomic Energy Authority. Weakly sign symmetric matrices do not occur in it. As noted in the preface of the German edition, there are some changes precisely in the section we are referring to; cf. also Kotelyanskii (1953) and Carlson (1967).

[30] It seems unlikely that anyone would have called Sylvester's identity well-known in 1950, or used it without reference or explanation. Fortunately, thanks to Gantmacher and Krein (1960) and Karlin (1968), its importance is again being recognized, see also Gragg (1972) and Householder (1972). The identity is one of a class known as

extensional identities can and be derived easily by means of Muir's (1883) metatheorem; cf. Aitken (1939, pp. 103–107).

[31] This proof is very little known today, since most modern books follow Wielandt's (1950) proof, which is simpler, gives great insight into aspects of the theory and also lends itself to generalization, e.g. Barker and Schneider (1975), among others. Wielandt did us great service with his proof, but we have done ourselves disservice by ignoring Frobenius' proof, as, I hope, is shown by section 4.4. Frobenius' proof can be found in Gantmacher and Krein (1960), but as far as I know in no other post-1950 book which discusses Perron–Frobenius.

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References

Aitken, A. C. (1939), *Determinants and Matrices*, University Math. Texts, Oliver and Boyd, Edinburgh.

Arrow, K. J. and McManus, M. (1958). A note on dynamic stability, *Econometrica*, **26,** 448–454.

Artin, E. (1932). Über Einheiten relativ galoisscher Zahlkörper, *J. reine angew. Math.* **167,** 153–156.

E. Artin (1950). The influence of J. H. M. Wedderburn on the development of modern algebra, *Bull. Amer. Math. Soc.* **56,** 65–72.

Bankwitz, C. (1930), Über die Torsionszahlen der alternierenden Knoten, *Math. Ann.* **103,** 145–161.

Barker, G. P. and Schneider, H. (1975), Algebraic Perron–Frobenius theory, *Lin. Alg. Appl.* **11,** 219–233.

Barnett, S. (1971), *Matrices in Control Theory*, Van Nostrand Reinhold Co., London.

Bauer, F. L. (1975), Positivity and norms, *Comm. ACM,* **18,** 9–13.

Bell, E. T. (1937), *Mem of Mathematics*, Simon and Schuster, also Dover Publ. Co. and Penguin Books (1953).

Bellman, R. (1960), *Introduction to Matrix Analysis*, McGraw-Hill Co., New York.

Birkhoff, G. (1976a), The rise of modern algebra to 1937, Men and Institutions in American Universities, Grad. Stud., Texas Tech. Univ. No. 13, 45–63.

Birkhoff, G. (1976b), The rise of modern algebra, 1936–1950, Men and Institutions in American Universities, Grad. Stud., Texas Tech. Univ., No. 13, 65–85.

Bodewig, E. (1953), *Matrix Calculus* (2nd edn., 1959), North-Holland Publ. Co., Amsterdam.

Brauer, A. (1946), Limits for the characteristic roots of a matrix, *Duke Math. J.*, **13,** 387–395.

Brauer, A. (1947), Limits for the characteristic roots of a matrix II, *Duke Math. J.*, **14,** 21–26.

Brauer, A. (1948), Limits for the characteristic roots of a matrix III, *Duke Math. J.* **15,** 871–877.

Brauer, A. (1973), *Gedenkrede auf Issai Schur*, in I. Schur, *Ges. Abh.*, Vol. 1, v-xiv, Springer, Berlin.

Carlson, D. H. (1967), Weakly sign-symmetric matrices and some determinantal inequalities, *Colloquium Math.* **17,** 123–129.

Carlson, D. H. and Schneider, H. (1963), Inertia theorems for matrices: the semi-definite case, *J. Math. Anal. Appl.* **6,** 430–446.

Cauchy, A. (1829), Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires du mouvement des planètes, *Exerc. de Math.* **4,** and *Oeuvres comp.* (2)**9,** 174–195, Acad. Sci., Paris (1891).

Cayley, A. (1855), Remarques sur la notation des fonctions algébriques, *J. rein. ang. Math, Crelle*, **50**, 282–283 and *Coll. Math. Papers*, No. 134, Vol. II, 185–186, Cambridge (1889).

Cayley, A. (1858), A memoir on the theory of matrices, *Phil. Trans. Roy. Soc. London*, **148**, 17–37 and *Coll. Math. Papers*, No. 152, Vol. II, 475–496, Cambridge (1889).

Collatz, L. (1942), Einschliessungsatz fur die charakteristischen Zahlen von Matrizen, *Math. Zeit.* **48**, 221–226.

Cooper, J. L. B. (1975), Philip Stein, *Bull. London Math. Soc.* **7**, 321–322.

Daleckii, J. L. and Krein, M. G. (1970), *Stability of Solutions of Differential Equations in Banach Space*, *Izd. Nauka Gl. Red. Fis-Mat. Lit., Moscow*, and *Transl. Math. Mon.* No. 43, Amer. Math. Soc., Providence, R.I. (1974).

Desplanques, J. (1886), Théorème d'algèbre, *J. Math. Spec.* (3)**1**, 12–13.

Engel, G. M. and Schneider, H. (1973), Inequalities for determinants and permanents, *Linear and Multilinear Algebra*, **1**, 187–201.

Engel, G. M. and Schneider, H. (1976), The Hadamard–Fischer inequality for a class of matrices defined by eigenvalue monotonicity, *Lin. Multilin. Algebra*, **4**, 155–176.

Engel, G. M. and Varga, R. S. (1977), Minimal Gershgorin sets and ω-matrices, *Lin. Multilin. Alg.* (to appear).

Faith, C. (1976), *Algebra II, Ring Theory*, *Grundlehren math. Wiss.* **191**, Springer, Berlin.

Fan, K. (1967), Subadditive functions on a distributive lattice and an extension of the Szasz inequality, *J. Math. Anal. Appl.* **18**, 262–268.

Fiedler, M. and Ptak, V. (1962), On matrices with non-positive off-diagonal elements and positive principal minors, *Czech. Math. J.* **12**, 382–400.

Fiedler, M. and Ptak, V. (1966), Some generalizations of positive definiteness and monotonicity, *Num. Math.* **9**, 163–172.

Fischer, E. (1908), Über den Hadamardschen Determinanten satz, *Arch. Math. Phys.* **13** (3), 32–40.

Fréchet, M. (1933), Comportement asymptotique des solutions d'un système d'équations linéaires et homogènes aux différences finies du premier ordre à coefficients constants, *Pub. Fac. Sci. Univ. Masaryk*, Brno, No. 178.

Fréchet, M. (1938), *Traité du Calcul des Probabilitiés et de ses Applications*, Ed. E. Borel, Tome I, Fasc. III, Second Livre, *Méthodes des Fonctions Arbitraires, Théorie des événements en chaîne dans le cas d'un nombre fini d'états possibles*, 2nd edn. 1952.

Frobenius, G. F. (1908), Über Matrizen aus positiven Elementen, (i) *Sitzungsber, Kön. Preuss. Akad. Wiss.* Berlin 471–476; (ii) *Ges. Abh.* No. 79, Vol. 3, 404–409, Springer, Berlin (1968).

Frobenius, G. F. (1909), Über Matrizen aus positiven Elementen, II. (i) Stizungsber. *Kön. Preuss. Akad. Wiss.*, Berlin (1908), 474–476; (ii) *Ges. Abh.* No. 80, Vol. 3, 410–414, Springer, Berlin (1968).

Frobenius, G. F. (1912), Über Matrizen aus nicht negativen Elementen (i) Sitzungsber. *Kön. Preuss. Akad. Wiss.*, Berlin, 456–477; (ii) *Ges. Abh.* No. 92, Vol. 3, Springer, Berlin (1968).

Furtwängler, P. (1936), Über einen Determinantensatz, *Sitzungsber. Akad. Wiss. Wien, IIa*, **145**, 527–528.

Gantmacher, F. R. (1953), *Teoriya Matrits, Gos. isd. tech-teoret*, Moscow and *Theory of Matrices*, Chelsea Publ. Co., New York (1959).

Gantmacher, F. R. and Krein, M. G. (1935), Ob odnom spetsial'nom klasse determinantom v svyasi s integral'nymi yadrami Kellog'a. *Mat. Sb.* **42**, 501–508.

Gantmacher, F. R. and Krein, M. G. (1937), Sur les matrices oscillatoires et complètement non-négatives, *Comp. Math.* **4** (1937), 445–476.

Gantmacher, F. R. and Krein, M. G. (1960), *Oszillations matrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme*, Akademie-Verlag, Berlin.

Gershgorin, S. (1931), Über die Abgrenzung der Eigenwerte einer Matrix, *Izv. Akad. Nauk. SSSR*, **7**, 749–754.

Geiringer, Hilde (1949), On the solution of linear equations by certain iteration methods, Reissner Anniv. Vol., Contrib. to *App. Mechanics*, 365–393, J. Edwards (Ed.), Ann Arbor, Mich.

Givens, W. (1961), Elementary Divisors and Some Properties of the Lyapunov Mapping $X \rightarrow AX+XA^*$, Argonne Nat. Lab. Rep. ANL-6546.

Goldhaber, J. K. and Whaples, G. (1953), On some matrix theorems of Frobenius and McCoy, *Canadian J. Math.* **5**, 332–335.

Gragg, W. B. (1972), The Padé table and its relation to certain algorithms of numerical analysis, *SIAM Rev.* **14**, 1–62.

Hadamard, H. (1893), Résolution d'une question relative aux déterminants, *Bull. Sci. Math.* **17** (2) (1893), 240–248, and *Oeuvres*, vol. 1, 239–245, Ed. Can. Nat. Rech. Sci., Paris (1968).

Hadamard H. (1903), *Leçons sur la Propagation des Ondes*, Paris, 13–14 and Chelsea Publ. Co.

Hahn, W. (1955), Eine Bemerkung zur zweiten Methode von Ljapunov, *Math. Nachr.* **14**, 349–354.

Hahn, W (1959), Theorie und Anwendung der direkten Methode von Ljapunov, *Ergeb. Math. Grenzgeb.*, N.F. No. 22, Springer-Verlag. Berlin.

Hawkins, T. (1975), The theory of matrices in the 19th century, *Proc. Int. Cong. Math.* (Vancouver 1974), Vol. 2, 561–570.

Hawkins, T. (1977), Another look at Cayley and the theory of matrices, to appear.

Herstein, I. N. (1968), *Non-commutative Rings*, Carus Math. Monographs, No. 15, Math. Assoc. America.

Hirzebruch, U. (1974), Produkte positiver Elemente in formal-reellen Jordan Algebren, *Lin. Alg. Appl.* **8**, 263–269.

Hopf, H. (1929), Über die algebraische Anzahl von Fixpunkten, *Math. Zeit*, **29**, 493–524.

Householder, A. S. (1972), Lectures on Numerical Algebra, Math. Assoc. America.

Householder, A. S. (1974), The Gatlinburgs, *SIAM Rev.* **16**, 340–343.

Jacobson, N. (1974), *Basic Algebra I*, W. H. Freeman & Co., San Francisco.

Jacobson, N. (1976), Structure groups and Lie algebras of symmetric elements of associative algebras with involution, *Advances Math.* **20**, 106–150.

Janet, M. (1920), Sur les systèmes d'equations aux derivées partielle, *J. Math. Pure et App.*, **3** (8), 65–151.

Karlin, S. (1968), *Total Positivity*, Vol. **I**, Stanford U.P., Cal.

Kline, M. (1972), *Mathematical. Thought from Ancient to Modern Times*, Oxford U.P.

Koecher, M. (1971), Jordan algebras and differential geometry, *Actes congrès intern. math.* vol. I (1970), 279–283, Gauthier-Villars, Paris.

Koestler, A. (1967), *The Ghost in the Machine*, Hutchinson & Co., London, H. Regnery Co., Chicago, Illinois.

Kotelyanskii, D. M. (1950), The theory of non-negative and oscillating matrices, *Amer. Math. Soc. Transl.*, Ser. 2, **27** (1963) 1–8, origin in *Ukrain. Mat. Zh.* **2** (1950), 94–101.

Kotelyanskii, D. M. (1953), A property of sign symmetric matrices, *Amer. Math. Soc. Transl.*, Ser. 2, **27** (1963), 19–23, orig. in *Usp. mat. nauk* (N.S.) **8** (1953), 163–167.

Ledermann, W. (1950), Bounds for the greatest latent roots of a matrix, *J. London Math. Soc.* **25**, 265–268.

Lévy, L. (1881), Sur la possibilité de l'équilibre électrique, *Comptes Rendus Acad. Sci. (Paris)*, **93** (3), 706–708.

Lewis, D. C. (1951), Differential equations referred to a variable metric, *Amer. J. Math.* **73**. 48–58.

Lewis, D. C., Jr. and Taussky, O. (1960), Some remarks concerning the real and imaginary parts of the characteristic roots of a finite matrix, *J. Math. Phys.* **1**, 234–236.

Lyapunov (Liapunoff) (1892), Problème général de la stabilité du mouvement, (i) *Comm. Math. Soc. Kharkov*; (ii) *Ann. Fac. Sci. Toulouse* **9** (2), (1907); (iii) *Ann. Math. Studies*, No. 17 (1974), Princeton U.P.

Markov, A. A. (1908), (i) Rasprostranyenie predel'nykh teorem ischisleniya veroyastnostei na summu velichin svyazannikh v tsep', *Zap. (Mem.) Imp. Akad. Nauk*, St. Petersburg, *Fis.-Mat.*, Ser. 8, **25**, No. 3; (ii) *Isbr. Trudy*, Moscow (1951), 365–397; (iii) Ausdehnung der Sätze über die Grenzwerte in der Wahrscheinlichkeitsrechnung auf eine Summe verketteter Grössen, pp. 272–298 in A. A. Markoff, *Wahrscheinlichkeitsrechnung*, B. G. Teubner, Leipzig (1912), transl. H. Liebmann; (iv) Extension of the limit theorems of probability theory to a sum of variables connected in a chain, Trans. E. Petelin, pp. 552–576 in R. A. Howard, *Dynamic Probabilities Systems*, Wiley and Sons, New York (1971).

Massonnet, C. (1945), Sur une condition suffisante pour qu'un déterminant soit positif, *Bull. Soc. Roy. Sci. Liège*, **14**, 313–317.

Mehmke, R. (1892), K sposobu Seidela sluzhashemu dlya resheniya linneinykh uravneniy, *Mat. Sborn.* **16**, 342–345.

Minkowski, H. (1900), Zur Theorie der Einheiten in den algebraischen Zahlkörpern, Gottinger Nachr., pp. 90–93 (*Ges. Abh.* **1**, 316–317).

Minkowski, H. (1907), *Diophantische Approximationen*, Teubner, Leipzig.

v. Mises, R. (1931), *Vorlesungen aus dem Gebiete der angewandten Mathematik*, Vol. I, *Wahrscheinlichkeitsrechnung*, F. Deuticke, Leipzig.

Muir, T. (1883), The law of extensible minors in determinants, *Trans. Roy. Soc. Edinburgh*, **30**, 1–4.

Muir, T. (1901/2), Question 14792, *Educ. Times* **54** (1901), 83 and Math. questions and solutions from the *Educ. Times*, N.S. **1**, (1902), 52–53.

Muir, T. (1909), An upper limit for the value of a determinant, *Trans. R. S. South Africa*, **1** (1908–10), 323–334.

Muir, T. (1923), *The Theory of Determinants in the Historical Order of Development*, vol. IV (1880–1900), MacMillan & Co., London & Dover Publ. Co.

Muir, T. (1930), *Contributions to the History of Determinants*, 1900–1920, Blackie & Sons, London.

Nekrasov, P. A. (1892) K voprosu o reshenyi linneinoi sistemy urqvnenyi, *Zap. Imp. Akad. Nauk*, St. Petersburg, **69**, No. 5.

Ostrowski, A. M. (1937a), Über die Determinanten mit überwiegender Haupt-diagonale, *Comm. Math. Helv.* **10**, 69–96.

Ostrowski, A. M. (1937b), Sur la détermination des bornes inférieures pour une classe des déterminants, *Bull. Sci. Math.* (2)**61**, 1–14.

Ostrowski, A. M. and Schneider, H. (1962), Some theorems on the insertia of general matrices, *J. Math. Anal. Appl.* **4**, 72–84.

Parodi, M. (1946), Sur l'éxistence des résaux électriques, *Comp. Rend. Acad. Sci. Paris*, **223**, 23–25.

Parodi, M. (1959), *La localisation des valeurs caractéristiques des matrices et ses applications*, Gauthier-Villars, Paris.

Perron, O. (1907a), Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus, *Math. Ann.* **63**, 1–76.

Perron, O. (1907b), Zur Theorie der Matrices, *Math. Ann.* **63**, 248–263.

Perron, O. (1927), *Algebra*, Vol. I, de Gruyter & Co., Berlin.

Plemmons, R. (1977), *M*-matrix characterizations I: Non-singular matrices, *Lin. Alg. Appl.* **18**, 175–188.

Price, G. B. (1951), Bounds for determinants with dominant principal diagonal, *Proc. Amer. Math. Soc.*, **2**, 497–502.

Reidemeister, K. (1932), *Knotentheorie*, *Erg. d. Math. u. Grenzgeb*, Vol. I, Springer, also Chelsea Publ. (1948).

Rohrbach, H. (1931), Bemerkungen zu einem Determinantensatz von Minkowski, *Jahresbericht d. D.M.V.* **40**, 49–53.

Schaeffer, H. H. (1974), Banach Lattices and Positive Operators, *Gundlehren Math. Wiss.* No. 215, Springer-Verlag, New York.

Schneider, H. ed. (1964), *Recent Advances in Matrix Theory*, Univ. Wisconsin Press, Madison, Wis.

Schneider, H. (1977), The concepts of irreducibility and full indecomposability in the works of Frobenius, König and Markov, *Lin. Alg. Appl.* **18**, 139–162.

Stein, P. and Rosenberg, R. L. (1948), On the solution of linear simultaneous equations by iteration, *J. London Math. Soc.* **23**, 111–118.

Sylvester, J. J. (1850), Additions to the articles: "On a new class of theorems" and "On Pascal's theorem", (i) *Phil. Mag.* **37** (3), 363–370; (ii) *Coll. Math. Papers*, No. 25, Vol. I, 145–151, Cambridge U.P. (1904) and Dover Publ. Co. (1973).

Sylvester, J. J. (1851), On the relation between the minor determinants of linearly equivalent quadratic functions. (i) *Phil. Mag.* **1** (4), 295–305; (ii) *Coll. Math. Papers*, No. 37, Vol. I, 241–250, Cambridge U.P. (1904) and Dover Publ. Co. (1973).

Sylvester, J. J. (1852), A demonstration of the theorem that every homogeneous quadratic polynomial is reducible by real orthogonal substitutions to the form of a sum of positive and negative squares. (i) *Phil. Mag.* (4)**4**, 138–142; (ii) *Coll. Math. Papers*, No. 47, Vol. I, 378–381 Cambridge U.P. (1904) and Dover Publ. Co. (1973).

Sylvester, J. J. (1870), *The Laws of Verse, or Principles of Versification Exemplified in Metrical Translations*, Longmans, Green & Co., London.

Sylvester, J. J. (1884), Lectures on the principles of universal algebra, (i) *Amer. J. Math.* **6** (1884), 270–286; (ii) *Coll. Math. Papers*, No. 31, Vol. IV, 208–224, Cambridge U.P. (1904) and Dover Publ. Co. (1973).

Taber, H. (1890), On the theory of matrices, *Amer. J. Math.* **12**, 337–396.

Tambs–Lyche, R. (1928), Un théorème sur les déterminants. *Det. Kong. Vid. Selskab.*, Forh. I. Nr. 41, 119–120.

Taussky, O. (1947), A method for obtaining bounds for characteristic roots of matrices with applications to flutter calculations, *Aeron. Res. Counc.*, Gt. Britain, Rep. 10, 508.

Taussky, O. (1948), Bounds for characteristic roots of matrices, *Duke Math. J.* **15**, 1043–1044.

Taussky, O. (1949), A recurring theorem on determinants, *Amer. Math. Month.* **56**, 672–676.

Taussky, O. (1957), Commutativity in finite matrices, *Amer. Math. Month.* **64**, 229–235.

Taussky, O. (1958), Research problem, *Bull. Amer. Math. Soc.* **64**, 124.

Taussky, O. (1961a), A remark on a theorem by Lyapunov, *J. Math. Anal. and Appl.* **2**, 105–107.

Taussky, O. (1961b), A generalization of a theorem by Lyapunov, *J. SIAM* **9**, 640–643.

Taussky, O. (1962), Some topics concerning bounds for the eigenvalue of finite matrices, *Survey of Numerical Analysis* (ed. John Todd), McGraw-Hill Co., 279–297.

Taussky, O. (1964a), Matrices C with $C^k \to 0$, *J. Alg.* **1**, 5–10.

Taussky, O. (1964b), On the variation of the characteristic roots of a finite matrix under various changes of its elements, in *Recent Advances in Matrix Theory* (ed. H. Schneider), Univ. Wisconsin Press.

Taussky, O. (1967), Positive definite matrices, in *Inequalities*, Vol. 1 (ed. O. Shisha), Academic Press, New York.

Taussky, O. (1968a), On stable matrices, in *Programmation en Mathématiques Numeriques*, Coll. Int. Cen. Nat. Rech. Sci. No. 165, 75–88.

Taussky, O. (1968b). Positive definite matrices and their role in the study of characteristic roots of general matrices, *Advances Math.* **2**, 175–186, also in *Inequalities*, Vol. 2 (ed. O. Shisha) (1970), Academic Press, N.Y.

Taussky, O. (1978), History of sums of squares, in *History of Algebra*, Texas Tech. Univ.

Taussky, O. and Wielandt, H. (1962), On the matrix function $AX+X'A'$, *Arch. Rat. Math. Anal.* **9**, 93–96.

Todd, J. (1965), On smallest isolated Gerschgorin disks for eigenvalues, *Num. Math.* **7**, 171–175.

Varga, R. S. (1965), Minimal Gerschgorin sets, *Pacific Math. J.* **15**, 719–729.

Varga, R. S. (1976), On recurring theorems on diagonal dominance, *Lin. Alg. Appl.* **13**, 1–9.

Varga, R. S. (1977), A note on an open question on ω- and τ-matrices, *Lin. Alg. Appl.* **18**, 45–52.

v. d. Waerden, B. L. (1930), *Moderne Algebra*, Grundl. Math. Wiss., No. 34, Springer-Verlag, Berlin.

Wedderburn, J. H. M. (1908), On hypercomplex numbers, *Proc. London Math. Soc.* (2)**6**, 77–117.

Wedderburn, J. H. M. (1934), Lectures on Matrices, *Amer. Math. Soc. Colloqu. Pub.* Vol. 17, Providence, R.I., also Dover Pub.

Wielandt, H. (1950), Unzerlegbare, nicht negative Matrizen, *Math. Z.* **52**, 642–648.

Wielandt, H. (1951), On the eigenvalues of $A+B$ and AB. *Nat. Bureau Standards Rep.*, No. 1367 and *J. Res. Nat. Bur. Standards* **77B** (1973), 61–63.

Wittmeyer, E. (1936), Einfluss der Änderung einer Matrix auf die Lösung des zugehörigen Gleichungssystems, sowie auf die charakteristischen Zahlen und die Eigenvektoren, *Zeitschr. Ang. Math. Mech.*, **16**, 287–300.