The Concepts of Irreducibility and Full Indecomposability of a Matrix in the Works of Frobenius, König and Markov

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ABSTRACT

Frobenius published two proofs of a theorem which characterizes irreducible and fully indecomposable matrices in an algebraic manner. It is shown that the second proof, which depends on the Frobenius-König theorem, yields a stronger form of the result than the first. Some curious features in Frobenius's last paper are examined; these include his criticisms of a result due to D. König and the latter's application of graph theory to matrices. A condition on matrices formulated by Markov is examined in detail to show that it may coincide with Frobenius's concept of irreducibility, and several theorems on stochastic matrices of Perron-Frobenius type proved by Markov are exhibited. In a research part of the paper, a theorem is proved which characterizes irreducible matrices and which contains Frobenius's theorem and was motivated by Markov's condition.

0. *Introduction and Motivation*

This article combines detailed examination of parts of classical and influential papers by G. F. Frobenius (1849–1917), A. A. Markov (1856–1922), and D. König (1884–1944) ("mathematical history"[1])† with a new result and proof ("research"). Such a combination is appropriate in this instance, since our Theorem (7.1) clarifies some results in these papers and was conjectured after studying them. In Part I of this article we discuss the following topics listed in (0.1) through (0.4):

(0.1) There is a theorem of Frobenius which characterizes irreducible matrices and fully indecomposable matrices; see (1.1) and (1.2) for definitions. This theorem is stated in identical words in Frobenius [1912] and

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†Superscripts refer to notes collected in Sec. 9.

Frobenius [1917], his last paper. Yet in [1912] an essential ingredient of the proof given in the [1917] paper is missing. How is this possible? I contend[2] that Frobenius in fact proved a somewhat stronger result in [1917] than in [1912]; see Sec. 1.

The theorem in question is[3] F#92-I and F#102-I, and in a slightly different form F#92-XVI, misprinted as XII; see also König [1915], [1933], [1936, p. 241], Mirsky [1971, p. 212] and Ryser [1973], [1975]. The missing ingredient in the [1912] proof is Lemma 102-II. This Lemma generalizes König [1916, Theorem D]—see Sec. 1—and has become famous (and more familiar than 102-I itself) under the name of the Frobenius-König theorem (e.g., König [1933], [1936, p. 240], Marcus-Minc [1964, p. 97], Mirsky [1971, p. 189, Corollary 11.2.6]) and, in a slightly more general form, as P. Hall's theorem on systems of distinct representatives (P. Hall [1935], Ryser [1963, p. 48], Mirsky [1971, p. 27, Theorem 2.2.1]).

(0.2) At the end of Frobenius [1917] there is criticism of a theorem in König [1916] and, more generally, the use of graph theory in matrix theoretic proofs. In Sec. 2 we consider various kinds of points related to the resultant controversy.[4] The unusual features in Frobenius [1917] have led me to the very speculative hypothesis that the final version of this paper may not have been prepared by Frobenius himself; see the end of Sec. 2.

(0.3) Frobenius [1912] is generally credited with the introduction of the concept of irreducibility[5] of a matrix and its exploitation in the theory of non-negative matrices. Yet an examination of Markov [1908] shows that Markov was aware of the need of some such concept. The passage in which Markov states his "important condition" (3.2) is unclear. Did Markov introduce the same concept of irreducibility as Frobenius (and even the concept of aperiodicity[6])? This question is discussed in Sec. 3. We argue that in 1908 Markov proved a substantial part, but by no means all, of what is usually called the Perron-Frobenius theorem[7] for an irreducible non-negative matrix, and which may be found in Frobenius [1912].

(0.4) In Sec. 4 we remark that it is a matter of judgment which arguments in the past are or are not graph theoretic.

(0.5) While comparing Markov's condition (3.2) with Frobenius's definition of irreducibility (1.1), it occurred to me to investigate whether (3.2) with the last seven words omitted was equivalent to irreducibility. To prove this is the purpose of the self-contained Part II. Our Theorem (7.1) clearly contains Frobenius's Theorem 92-XVI. Whether Theorem (7.1) is "new" or "essentially due to Markov" is for the reader to judge. I believe however that no proof has hitherto been published.

PART I. HISTORY

1. *Frobenius's Theorem:* 92-I *and* 102-I

In his great paper on non-negative matrices Frobenius [1912] defined the concept of irreducibility for a (square) matrix. This definition is fundamental[8] in the theory of non-negative matrices and has the character of Columbus egg: it is so simple that anyone could have given it—Frobenius did. We quote[9] Frobenius [1912b, p. 548]:

> I call a matrix or a determinant of order $p+q$ reducible [*zerfallend, zerlegbar*] if in it there vanish all elements which p rows have in common with the q columns whose indices are complementary to the p rows (complete them to 1, $2,\ldots,p+q$).

In other words, Frobenius made the

(1.1) *Definition.* An $(n\times n)$ matrix A is *reducible* if we may partition $\{1,\ldots,n\}$ into two non-empty subsets E, F such that $a_{ij}=0$ if $i\in E$, $j\in F$.

To show the uniqueness of the decomposition of a reducible matrix into irreducible components, Frobenius gave two proofs. The second proof involves Theorem 92-XVI, which characterizes irreducible matrices algebraically, and below we state this theorem in essentially the same form as our Corollary (7.2). In his introduction to this paper, Frobenius singles out this theorem[10] from a collection of results that were to become famous and calls it remarkable or surprising[11] (*merkwürdig*).

But as 92-I in the introduction to F#92 the theorem is stated in a slightly different form. We quote:

> *Let the elements of a determinant*[12] *of order n be n^2 independent variables. Let some of these be put equal to zero, however in such a manner that the determinant does not vanish identically. Then the determinant remains*[13] *an irreducible function, except when for some value $m<n$ all elements vanish which m rows have in common with $n-m$ columns.*

Following[14] Marcus-Minc [1963], [1964, p. 123], we make the following:

(1.2) *Definition.* An $(n\times n)$ matrix A is *partly decomposable* (*not fully indecomposable*) if for some m, $0<m<n$, there exist subsets E,F of $\{1,\ldots,n\}$

with m and $n-m$ elements, respectively, such that $a_{ij}=0$ if $i\in E$, $j\in F$.

So here the theorem characterizes fully indecomposable matrices.

In Frobenius [1917] the theorem is restated as 102-I in the same words as in 92-I, and it is followed by a sentence to which we shall again refer in Sec. 2:

> The proof which I gave there [in F#92] for this theorem is an incidental product which flows from hidden [*verborgen*] properties of determinants with non-negative elements.

Frobenius then explains that he will now give an elementary proof.

We shall call the result stated in 92-I and 102-I (and quoted above) *Frobenius's theorem*, and we shall now discuss the proofs of this theorem as found in F#92 and F#102. First, we remark that strictly speaking there is no proof of Frobenius' theorem in F#92; rather there is a proof of 92-XVI. Presumably, Frobenius takes the view that it is clear how to derive his theorem from 92-XVI. Second, the proof of Frobenius's theorem in F#102 rests on Lemma 102-II, usually called the *Frobenius-König theorem*[15] and mentioned in (0.1):

> *If all terms of a determinant of order n vanish, then all elements vanish which p rows have in common with $n-p+1$ columns, for $p=1$ or $2,\dots,$ or n.*

A curious point arises: As a consequence of this lemma the qualifying phrase "however... identically", which was needed for the proof of 92-XVI, is now superfluous. The omission of the qualifying phrase results in a stronger form of the theorem; and it is strange that Frobenius included the phrase in the statement of his theorem in F#102.

As it may be hard to distinguish between the two versions of the theorem at first sight, we shall explain in detail, though the mathematical point involved may be minor. For a matrix A whose entries are independent indeterminates or 0, consider the following three propositions:

P: $\det A\neq 0$,

Q: A is fully indecomposable,

R: $\det A$ is an irreducible polynomial.

The weaker form of Frobenius's theorem, as stated in 92-I and 102-I, is

$$P\Rightarrow(Q\Leftrightarrow R).$$

The stronger form (as virtually shown by the proof in F#102) is

$$Q\Leftrightarrow R.$$

Since $R \Rightarrow P$ is trivial, the stronger form yields $Q \Rightarrow P$. But (in my opinion) the last implication cannot be derived from the proof of 92-XVI, as there it is assumed at the outset that the main diagonal of A has no zero entry.

In his paper, König [1915] gives a proof of Frobenius's theorem using graph theory. He there states the theorem in essentially the same form as 92-I (or 102-I), but *without* the qualifying phrase. However, in his proof, König appears to assume that the determinant does not vanish. Thus Frobenius [1917] proved more than he claimed; König [1915] claimed a correct theorem, but more than he proved. The theorem is proved in the stronger form without the qualifying phrase in König [1933] and [1936, p. 238]. The theorem is also stated in the stronger form in G. Szegö's review, König [1915b], where the result is attributed to Frobenius [1912].

A final observation in this section: It is not surprising that slightly different versions of the same theorem (i.e., 92-I and 92-XVI) characterize fully indecomposable matrices on the one hand and irreducible matrices on the other. For by a lemma in Brualdi-Parter-Schneider [1966] (whose proof uses Frobenius-König), a matrix A is fully indecomposable if and only if, for some permutation matrix P, PA is irreducible and has non-zero entries on the main diagonal.

2. *König's Theorem D*

If in a determinant of non-negative elements the quantities in each row and each column have the same sum, different from zero, then not all terms of the determinant can vanish.

This is König [1916, Theorem D], as completed by the two sentences following the statement of the theorem in his paper.[16] Here it is stated in the words of Frobenius in F#102. Frobenius then makes the following remark, which forms the last paragraph of his last paper, F#102:

> The theory of graphs, by means of which Mr. König deduced the above theorem, is in my opinion a tool [*Hilfsmittel*] little suited to the development of the theory of determinants. In this case it leads to a quite special[17] theorem of little value [*ein ganz spezieller Satz von geringem Werte*]. What is valuable in its content is expressed in Theorem II [viz. Frobenius-König].

This highly critical remark appears to have no parallel in Frobenius's collected works.[18] It deals with the utility of graph theory in general, and König's theorem in particular. We shall take up these two aspects one after the other.

There are by now a vast number of applications of graphs to matrices, and today there can be no doubt of the usefulness of graph theoretic methods in matrix theory. We make no attempt at a survey, and confine ourselves to three comments directly relevant to Frobenius's theorem.

First, the "hidden properties of non-negative matrices" (see Sec. 1) in the 1912 proof of Frobenius's theorem can be simply formulated in graph theoretic terms: In an irreducible matrix every non-zero element lies on a non-zero cycle. Second, most proofs of the uniqueness of the decomposition into irreducible components published in the last few years bypass Theorem 92-XVI by means of simple considerations of graph theory or, equivalently, partial order by use of arguments very close to Doeblin [1938, pp. 95–96], e.g., Cooper [1973], Richman-Schneider [1977]. Third, the graph theory we use in Part II in our improvement of Theorem 92-XVI is trivial, but crucial.

We now turn to the theorem that Frobenius criticizes specifically, which is quoted at the beginning of this section.[19] It would be bold to question the value of this theorem today. For example, König's Theorem D is precisely the condition that Marcus-Minc [1964, pp. 78–79] and Mirsky [1971, p. 192] use to prove a theorem of Birkhoff [1946] that has many applications: The set of $n \times n$ doubly stochastic matrices forms a convex polyhedron with the permutation matrices as vertices.[20]

Thus it is ironical that the last published words of one of the greatest mathematicians alive in this century should have failed the test of time. Perhaps it is not surprising that Frobenius did not appreciate the uses of the infant discipline of graph theory,[21,22] but why did he fail to acknowledge inF#102 König's [1915] proof[23] of Theorem 92-I, and why did he choose to criticize the method which had yielded this alternative proof, when he himself considered his original proof indirect (depending on non-trivial properties of non-negative matrices)?

I have the following hypothesis to explain some points I have made in this section and in Sec. 1: Frobenius [1917] was prepared from notes by Frobenius, but the final version was not written by him. Owing to circumstances not fully known to me, it was not carefully read and revised by Frobenius. This hypothesis is speculation on my part—and may not be generally acceptable[24]—but in response to an enquiry I have received an interesting letter (dated 4 December 1975) from K. R. Biermann of the Academy of Sciences of the DDR (Berlin) which may lend some credence to some hypothesis of the above form. I quote (the translation is mine):

> Frobenius did not give the talk concerning reducible determinants on 12 April, 1917 in the phys.-math. class of the Berlin Academy himself, rather he was represented by H. A. Schwarz. At that time Frobenius was already very ill, and he participated only one more time (on 26 April) in a session of the class.

We remark that Frobenius [1917a] is headed "Session of the Prussian Academy of Sciences, 12 April 1917" and Frobenius died on 3 August 1917.

3. *Markov's "Important Condition"*

In a basic paper on the chains that were to bear his name, A. A. Markov

[1908] considers matrices which we now call stochastic and which satisfy an "important condition". This condition is stated in two forms intended to be equivalent and is followed by another condition. We quote[9] in full a passage from his paper, Markov [1908]. [The labels (3.1)–(3.3) are ours.] Condition (3.1) refers to a chain $x_1, x_2, \dots$ with a finite number of different states $\alpha, \beta, \gamma, \dots$, where the probability of occurrence of a value of x_{n+1} depends only on the value taken by x_n, (e.g., Feller [1950b, pp. 338 et seq.]).

Before proceeding to further conclusions it is necessary to note that we are considering only those

(3.1) chains

$$x_1, x_2, \dots, x_n, \dots,$$

where the appearance of some of the numbers

$$\alpha, \beta, \gamma, \dots$$

does not exclude definitely[25] [*ne isklyuchayet okonchatel'no*] the possibility of the appearance of the others.

This important condition can be expressed by means of determinants in the following manner:

(3.2) the determinant

$$\begin{vmatrix} u, & p_{\beta,\alpha}, & p_{\gamma,\alpha}, & \cdots \\ p_{\alpha,\beta}, & v, & p_{\gamma,\beta}, & \cdots \\ p_{\alpha,\gamma}, & p_{\beta,\gamma}, & w, & \cdots \end{vmatrix}$$

with arbitrary elements

$$u, v, w, \dots$$

does not reduce to a product of several determinants of the same type.

This condition is not sufficient, however, for our purpose, and thus we must assume that

(3.3) the determinant indicated by us does not reduce in an obvious way to a product of several determinants even for

$$u = p_{\alpha\alpha}, \qquad v = p_{\beta\beta}, \dots .$$

Markov then proves[26]: If A is a stochastic matrix satisfying (3.2) and (3.3), then

(3.4) 1 is a simple zero of the characteristic polynomial of A and any zero λ, $\lambda \neq 1$, satisfies $|\lambda| < 1$,

i.e., the matrix A is primitive.[6] Details of the proof are discussed in (3.5)–(3.8) below.

In Sapagov's (Markov [1908e]) commentary, (3.2) is taken to be equivalent to irreducibility, while (3.3) is taken to be aperiodicity. We shall now discuss to what extent Markov anticipated Frobenius in the use of these two concepts. That Markov's important condition is logically equivalent to irreducibility is clear, for (3.1) is surely our (5.2) below, which is well known to be equivalent to irreducibility (Doeblin [1938, p. 81], Varga [1962, p. 20], Rosenblatt [1957]). That the alternative form (3.2) is intended to have the same meaning as Frobenius's definition of irreducibility (1.1) (rather than a condition which may be proved equivalent) is less certain, but appears to be so by the use Markov made of it in proofs. There are three such proofs (Markov [1908d, pp. 572, 573, 574]), and we shall discuss each of these briefly.

(3.5) On p. 572, Markov proves the following theorem[27]: *Let M be an $n \times n$ matrix satisfying* (3.2), *and suppose that*

$$m_{ii} > 0, \quad m_{ij} \leqslant 0, \qquad i,j = 1,\ldots,n,$$

$$\sum_{j=1}^{n} m_{ij} \geqslant 0, \qquad j = 1,\ldots,n.$$

Then $\det M \geqslant 0$, *and* $\det M = 0$ *only if*

$$\sum_{j=1}^{n} m_{ij} = 0, \qquad i = 1,\ldots,n.$$

It is hard to see what hypothesis other then irreducibility would make this theorem valid. On the other hand, Markov's proof of this theorem is by induction, and Markov appears to assume that if M satisfies (3.2), then some principal $(n-1)\times(n-1)$ submatrix of A satisfies the same condition, and this is false if (3.2) is taken to be irreducibility.

(3.6) In the second argument, Markov asserts that it follows from the theorem in (3.5) that 1 is a simple eigenvalue of a stochastic matrix (satisfying the important condition). The proof proceeds via differentiation, see Wielandt [1950] for a related argument.

(3.7) In the third proof, on p. 574, Markov argues thus: Let A be a stochastic matrix, and suppose that $Ax=\lambda x$, where $|x_1|,\ldots,|x_n|$ are not all equal. In view of the conditions imposed, there exist i and j, $1 \leqslant i,j \leqslant n$ such that $|x_j|<|x_i|=\max\{|x_1|,\ldots,|x_n|\}$ and $a_{ij}\neq 0$. Hence $|\lambda|<1$. Again it is hard to formulate a hypothesis that would lead to this assertion as directly as irreducibility.

That (3.3) is intended to be aperiodicity is quite possible, but uncertain. On p. 574 Markov [1908d] makes a reference to "one of our basic conditions" [presumably (3.3)] and follows this by an immediate consequence of aperiodicity. It is not explained how he obtains this consequence, and thus it is at least possible that he intended a condition stronger than aperiodicity (e.g., some or all diagonal elements should be non-zero.) We have in mind the following argument:

(3.8) Suppose x is a vector such that $|x_1|=|x_2|=\cdots=|x_n|\neq 0$ and (say) $x_1=-x_n$. Then "in view of one of our basic conditions" (observes Markov) it is impossible to partition $\{1,\ldots,n\}$ into two sets E_1, E_2 such that $(Ax)_i=x_1$ for $i\in E_1$, but $(Ax)_i\neq x_1$ for $i\in E_2$.

If (3.2) and (3.3) are indeed irreducibility and aperiodicity, then Markov proved that an irreducible aperiodic stochastic matrix A is primitive. Specifically he used the diagonal dominance theorem for irreducible stochastic matrices to obtain information on the multiplicity of the eigenvalue 1. It follows that Markov discovered a considerable part of the Perron-Frobenius theory for stochastic matrices. On the other hand, since it is clear that 1 is an eigenvalue of a stochastic matrix A, there is an important part of the theory for general non-negative matrices that cannot be found in Markov [1908], viz. that the spectral radius of a non-negative matrix is an eigenvalue. Also, I can find no evidence that Markov knew that a stochastic matrix satisfying (3.2) and (3.3) has a positive row eigenvector. This eigenvector, of course, plays a central role in the theory of Markov chains.

This part of Markov's paper and Frobenius's results do not appear to have been noticed by early researchers in discrete Markov chains, who mainly studied positive transition probabilities. In the early 1930's Frobenius's results were applied to Markov chains (in particular by v. Mises [1931, pp. 533–549]), and to some extent there was also independent rediscovery of his results in the special case of stochastic matrices: see the rather fascinating sequence of papers by Romanovsky [1929], [1930] (imprecise hypotheses), [1931] (apparently independent rediscovery), [1933] and [1936] (many references to Frobenius, but none to Markov); and see[28] also [1949].

We observe that it is possible to obtain an unambiguous form of (3.2) by omitting the last seven words, so that (3.2) would then refer to the irreducibility of a polynomial in n indeterminates. I do not claim Markov intended this, but as was indicated in the introduction, it is possible to show that the condition obtained in this way is equivalent to irreducibility, see Theorem (7.1).

4. *Graph Theoretic Methods in Matrix Theory*

Our previous sections raise the question of the date when graph theoretic methods were introduced into matrix theory, particularly in the case of the concepts we have discussed. This question is complicated by the fact that many qualitative statements concerning finite discrete Markov chains have natural and obvious analogues in graph theory.[29] We have an example in Markov's (3.1) with analogue (5.2), (and for a more modern, related example, see Feller [1950b, pp. 349–350].) In Doeblin's [1938, p. 81] version of (5.2) the word path (*chemin*) is used exactly as in Sec. 5 below, but no graph is formally defined. The explicit formulation of results on irreducibility in terms of graphs appears surprisingly late in the literature (Rosenblatt [1957], Varga [1962, Chapter 1]). Thus one might assign any one of 1908, 1938 or 1957 as the date when *irreducibility* was first characterized graph theoretically, dependcing on one's criteria. In the study of *aperiodicity*, cyclic products occur in Frobenius [1912b, p. 558] and are used heavily in Romanovsky [1931], [1933] and [1936]. Cyclic products are surely now regarded as graph theoretic. *Full indecomposability* was associated with graph theoretic concepts by König [1915] and [1916].

PART II. MATHEMATICS

5. *Graphs and Irreducibility*

Let n be a positive integer and put $\langle n\rangle=\{1,\ldots,n\}$. A (directed) *graph* $\mathbf{G}$ on $\langle n\rangle$ is a subset of $\langle n\rangle\times\langle n\rangle$ whose elements will be denoted by $i\to j$ and will be called *edges*. A *path* in $\mathbf{G}$ is a sequence of edges $i_1\to i_2\to\cdots\to i_k$, and a *cycle* γ in $\mathbf{G}$ is a path $i_1\to\cdots\to i_{k+1}$ such that $i_1=i_{k+1}$ and[30] $i_1,\ldots,i_k$ are pairwise distinct. The cycles $i_1\to\cdots\to i_k\to i_1$, $i_2\to\cdots\to i_1\to i_2,\ldots,$ $i_k\to i_1\to\cdots\to i_k$ will be considered as identical.[31] We put $\bar{\gamma}=\{i_1,\ldots,i_k\}$, the *support* of γ. If $\phi\subset\mu\subseteq\langle n\rangle$, and $\bar{\gamma}\cap\mu\neq\phi$, then we shall say that γ *intersects* μ. We shall need the following simple lemma, which is related to Engel-Schneider [1976, Lemma 2.1].

(5.1) LEMMA. *Let $\mathbf{G}$ be a graph on $\langle n\rangle$, and let (μ,ν) be a partition of $\langle n\rangle$. If γ is a cycle in $\mathbf{G}$ which intersects both μ and ν, and γ' is a distinct* *cycle with* $\bar{\gamma}' = \bar{\gamma}$, *then there exists a cycle* δ *which intersects both* μ *and* ν *and has* $\bar{\delta} \subset \bar{\gamma}$. (*We use* $\subset$ *for proper inclusion.*)

Proof. Without loss of generality we may assume that γ is $i_1 \to i_2 \to \cdots \to i_k \to i_1$, where $i_1 \in \mu$ and $i_2 \in \nu$. Then γ' is a cycle of form

$$i'_1 \to i'_2 \to \cdots \to i'_k \to i'_1,$$

where $i'_1 = i_1$ and $(i'_1, \ldots, i'_k)$ is a permutation of $(i_1, \ldots, i_k)$. Thus there exists an integer p, $1 < p < k$, such that

$$i_1 = i'_1, \ldots, i_{p-1} = i'_{p-1},$$

but

$$i_p = i'_q \quad \text{and} \quad q > p.$$

Thus the required cycle is δ:

$$i_1 \to \cdots \to i_p = i'_q \to i'_{q+1} \to \cdots \to i'_k \to i_1.$$

∎

For example, if γ, γ' are respectively

$$1 \to 2 \to 3 \to 4 \to 5 \to 1,$$
$$1 \to 3 \to 5 \to 2 \to 4 \to 1,$$

then δ may be chosen as

$$1 \to 2 \to 4 \to 1.$$

Let D be an integral domain, which is assumed to be commutative and with identity. We write D^{nn} for the set of all $n \times n$ matrices in D. If $A \in D^{nn}$, and $\phi \subset \mu, \nu \subseteq \langle n \rangle$, then $A[\mu|\nu]$ is defined to be the submatrix of A lying in the rows of μ and columns of ν. We shall also put $A[\mu] = A[\mu|\mu]$. Thus by (2.1), $A \in D^{nn}$ is reducible if and only if there exists a partition (μ, ν) of $\langle n \rangle$ (into non-empty subsets) such that $A[\mu|\nu] = 0$.

If $A \in D^{nn}$, we may define the graph $\mathbf{G}(A)$ on $\langle n \rangle$ thus: $i \to j$ if $a_{ij} \neq 0$. Then it is well known that the $A \in D^{nn}$ is irreducible if and only if the following condition holds:

(5.2) For all $i,j \in \langle n \rangle$, there exists a path $i \to \cdots \to j$ in $\mathbf{G}(A)$.

(See Doeblin [1938, p. 81], Rosenblatt [1957], Varga [1962, p. 20]). An easy consequence is the following form of the condition, which will be used later:

(5.3) LEMMA. *Let $A \in D^{nn}$. Then the following are equivalent*:

(i) *A is irreducible.*

(ii) *For every partition (μ, ν) of $\langle n \rangle$, there exists a cycle in $\mathbf{G}(A)$ which intersects both μ and ν.* ■

6. Lemmas on Polynomials

We shall consider polynomials $p(x_1, \ldots, x_n)$ in independent indeterminates $x_1, \ldots, x_n$, with coefficients in D. For ease of notation, we shall put, for any subset $\mu = \{i_1, i_2, \ldots, i_s\}$ of $\langle n \rangle$,

$$f_\mu = x_{i_1} \cdots x_{i_s},$$

where

$$f_\phi = 1.$$

Also we shall use the formal partial derivatives

$$\Delta_\mu p(x_1, \ldots, x_n) = \frac{\partial^s}{\partial x_{i_1} \cdots \partial x_{i_s}} p(x_1, \ldots, x_n),$$

where

$$\Delta_\phi p(x_1, \ldots, x_n) = p(x_1, \ldots, x_n).$$

A polynomial $p(x_1, \ldots, x_n)$ will be called *linear* in $x_1, \ldots, x_n$ (separately) if

$$p(x_1, \ldots, x_n) = \sum_{\phi \subseteq \mu \subseteq \langle n \rangle} p_\mu f_\mu,$$

where $p_\mu \in D$. If

$$p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) q'(x_1, \ldots, x_n)$$

for some polynomials $q(x_1,\ldots,x_n)$, $q'(x_1,\ldots,x_n)$, where $q(x_1,\ldots,x_n)\notin D$, $q'(x_1,\ldots,x_n)\notin D$, then $q(x_1,\ldots,x_n)q'(x_1,\ldots,x_n)$ will be called a *proper factorization* of $p(x_1,\ldots,x_n)$. If there exists a proper factorization for $p(x_1,\ldots,x_n)$, then $p(x_1,\ldots,x_n)$ will be called reducible.[32] (Thus, e.g., $6x_1-3$ is not considered reducible over the integers.) We shall use two simple lemmas on linear polynomials. The first of these is used without proof in Frobenius [1917b, p. 565] and given a short justification by Ryser [1973, p. 152].

(6.1) LEMMA. *Let $p(x_1,\ldots,x_n)$ be a polynomial linear in $x_1,\ldots,x_n$, with coefficients in D, and let*

$$p(x_1,\ldots,x_n)=q(x_1,\ldots,x_n)q'(x_1,\ldots,x_n)$$

be a proper factorization of $p(x_1,\ldots,x_n)$. Then there is a partition $\{i_1,\ldots,i_s\}$, $\{i_{s+1},\ldots,i_n\}$ of $\langle n\rangle$ such that

$$q(x_1,\ldots,x_n)=q(x_{i_1},\ldots,x_{i_s}),$$

$$q'(x_1,\ldots,x_n)=q'(x_{i_{s+1}},\ldots,x_{i_n}),$$

where each polynomial is linear in the indeterminates that occurs in it.

Proof. Let

$$p(x_1,\ldots,x_n)=q(x_1,\ldots,x_n)q'(x_1,\ldots,x_n).$$

Let $1\leqslant i\leqslant n$, and consider p, q, q' as polynomials in x_i with coefficients in $D^*=D[x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n]$. In a factorization of ax_i+b, where $a,b\in D^*$, one factor must be linear, and the other in D^*. The lemma now follows. ■

(6.2) LEMMA. *Let $p(x_1,\ldots,x_n)=\sum_{\phi\subseteq\mu\subseteq\langle n\rangle}p_\mu f_\mu$ be a polynomial linear in $x_1,\ldots,x_n$ with elements in D, and suppose that, for a partition $\mu=\{i_1,\ldots,i_s\}$, $\nu=\{i_{s+1},\ldots,i_n\}$ $(1\leqslant s<n)$ of $\langle n\rangle$,*

$$p(x_1,\ldots,x_n)=q(x_{i_1},\ldots,x_{i_s})q'(x_{i_{s+1}},\ldots,x_{i_n})$$

where each polynomial is linear in the indeterminates that occur in it. Then $p_{\langle n\rangle}p_\phi=p_\mu p_\nu$.

Proof. We have

$$p_{\langle n\rangle} = q_\mu q'_\nu,$$

$$p_\mu = q_\mu q'_\phi,$$

$$p_\nu = q_\phi q'_\nu,$$

$$p_\phi = q_\phi q'_\phi,$$

and the result follows.

7. *A Theorem on Irreducibility*

(7.1) THEOREM. *Let A be an $n\times n$ matrix with elements in an integral domain D. Let $x_1,\ldots,x_n$ be independent indeterminates, and let $X=$*diag *$(x_1,\ldots,x_n)$. Then the following are equivalent:*

(1) *A is reducible,*
(2) det $(X+A)$ *is reducible.*

Proof.
(1)⇒(2) is trivial.
(2)⇒(1): Let $\det(X+A)$ be reducible. Since $\det(X+A)$ is linear in $x_1,\ldots,x_n$, we may assume by Lemma (6.1) that

$$\det(X+A) = q(x_{i_1},\ldots,x_{i_s})q'(x_{i_{s+1}},\cdots x_{i_n}),$$

where, for $\mu=\{i_1,\ldots,i_s\}$, $\nu=\{i_{s+1},\ldots,i_n\}$, (μ,ν) is a partition of $\langle n\rangle$ and each of q, q' is linear in the variables that occur in it. Let $Z=X+A$, and let σ be any subset of $\langle n\rangle$ that intersects both μ and ν. Then we claim that $\det Z[\sigma]$ is reducible. For if τ is the complement of σ in $\langle n\rangle$, then

$$\det Z[\sigma] = \Delta_\tau \det Z = \Delta_{\tau\cap\mu} q(x_{i_1},\ldots,x_{i_s})\Delta_{\tau\cap\nu} q'(x_{i_{s+1}},\ldots,x_{i_n}), \qquad (*)$$

and neither of the last two factors lies in D, since $\tau\cap\mu\neq\mu$ and $\tau\cap\nu\neq\nu$.

Suppose now that A is irreducible. We shall fix attention on a particular subset σ of $\langle n\rangle$ defined below. By Lemma (5.3), there exist a cycle which intersects both μ and ν. So let $\delta = j_1\to j_2\to\cdots\to j_k\to j_1$ be such a cycle of minimal length. Let σ be the support of δ. By Lemma (5.1) there is no cycle distinct from δ whose support is contained in σ and which meets both μ and

ν. Hence, it is easy to see[33] that

$$\det Z[\sigma] = \det Z[\sigma \cap \mu] \det Z[\sigma \cap \nu] + b,$$

where $b = (-1)^{k-1} a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_k j_1} \neq 0$. Let $\det Z[\sigma] = \sum_{\phi \subset \rho \subset \sigma} c_\rho f_\rho$. Then by (*) and Lemma (6.2),

$$c_\phi = c_\sigma c_\phi = c_{\sigma \cap \mu} c_{\sigma \cap \nu}.$$

But $\det Z[\sigma] - b = \det Z[\sigma \cap \mu] \det Z[\sigma \cap \nu]$ is also reducible. Hence also

$$c_\phi - b = c_{\sigma \cap \mu} c_{\sigma \cap \nu}.$$

It follows that $b = 0$, which is a contradiction. ■

(7.2) Corollary (Frobenius, Theorem 92-XVI). *Let A be a matrix whose diagonal elements are indeterminates and whose off-diagonal elements are indeterminates or* 0 (*all indeterminates being independent*). *Then the following are equivalent*:

(1) *A is irreducible*,

(2) $\det A$ *is irreducible* (*considered as a polynomial over an arbitrary field F*).

Proof. Let D be the integral domain obtained by adjoining to F all a_{ij}, $i \neq j$. Let A' be the matrix given by $a'_{ii} = 0$, $a'_{ij} = a_{ij}$, $i \neq j$, $1 \leqslant i, j \leqslant n$. Then apply Theorem (7.1) to the matrix A' considered as an element of D^{nn}. ■

We observe that Theorem (7.1) and Corollary (7.2) hold if we replace the determinant function by the permanent function. Essentially the same proofs apply.

8. *Other Generalizations of Frobenius's Theorem*

We have found very few generalizations of Frobenius's theorem in the literature. Żyliński [1921] (see also MacDuffee [1936, pp. 14–15]) found the following related theorem, apparently independently of Frobenius: Let A be a matrix whose entries are independent indeterminates or 0. Suppose each non-zero element of A lies on a diagonal without zero. Then $\det A$ is irreducible if and only if A is chainable.[34]

König [1936, p. 241] observes that Frobenius's theorem holds with the determinant replaced by a more general function of type $f(A) =$

$\Sigma\epsilon_\sigma a_{1\sigma(1)}\cdots a_{n\sigma(n)}$, where the summation is over all permutations σ of $\{1,\ldots,n\}$, and $\epsilon_\sigma \neq 0$ for all σ; cf. also Ryser [1973]. Recently and independently of us, Ryser [1975] has found a characterization of irreducibility which is in the same spirit as ours but is distinct from it.

9. *Notes*

[1]We shall pay attention to the three questions listed in Judith Grabiner [1975] as typical of the concerns of mathematicians writing history: "When was this concept first defined, and what problems led to its definition?", "Who first proved this theorem, and how did he do it?", "Is the proof correct by modern standards?". Historians may wish to raise much wider issues, but those will not be discussed here, except in Ostrowski's comment (see Note 21).

[2]*Note on personal pronouns:* We use the first person singular when we wish to stress that a personal opinion or account is involved. Thus "in Sec. 4 we consider", but "it occurred to me".

[3]*Note on referencing:* We are attempting to introduce a system of referencing to Frobenius's papers which is invariant under transformations induced by the needs of different authors and their papers. It is based on the numbering of papers in Frobenius's collected works (edited by J. P. Serre), and the numbers might be called *Serre* numbers, in analogy with Koechel numbers for the works of Mozart. Thus, in the case of papers by Frobenius, we use two systems of referencing. For example, F#92 is also Frobenius [1912], and Theorem 92-I is Theorem I in F#92, and F#102 is also Frobenius [1917]. We have referenced various versions and derivatives of the papers by Frobenius, Markov and König under the original paper. Thus Frobenius [1912c] is Jacobsthal's review of F#92.

[4]There is no attempt to survey applications of graphs to matrices, or matrices to graphs.

[5]We are here concerned with the irreducibility of a single matrix (under similarity by permutation matrices). The related, but somewhat different concept of irreducibility for matrix groups (under similarity by non-singular matrices) arose earlier; see, e.g., Burnside [1911], where references may be found on p. 244 and a definition on p. 258. The word "irreducible" may also here have been introduced by Frobenius [1899b, p. 130], and the concept is already used implicitely in his [1897] paper. Observe that in the [1899] paper, F#57, the German words *reducibel* and *zerlegbar* have slightly different meanings.

[6]A matrix A is *aperiodic* (cyclic of order 1) if the g.c.d. of its nonzero cycles is 1, i.e., if it cannot be put in a form illustrated by Frobenius [1912b, p. 560] (see also Seneta [1973, p. 15], Romanovsky [1931], [1933], [1936] and [1949]). Some authors prefer the term "acyclic", but this would suggest that A has no non-zero cycles. A non-negative irreducible matrix A is *primitive* if A has no eigenvalue λ equal in magnitude to its spectral radius ρ, other than $\lambda=\rho$ itself. A non-negative matrix is primitive if and only if it is irreducible and aperiodic; see Seneta [1973, p. 18]. This result was essentially proved by Frobenius [1912b, p. 560], but characteristically he prefers to state an algebraic analog of the result as a theorem, viz. Theorem 92-XIV.

[7]For matrices with positive entries, the theorem may be found in two papers by Perron [1907] and two papers by Frobenius [1908 and 1909].

[8]From the second paragraph of the introduction to F#92: A non-negative matrix, which is irreducible, has almost all properties in common with positive matrices.

[9]*Note on translations:* The translation of Frobenius's German words is mine. The translation of Markov's Russian passages is based on Petelin's translation (see Markov [1908d]), but some changes have been made. In both cases I have tried to produce an English version very close to the original, even at the expense of a phrase or two which may not sound natural. We quote Frobenius's theorem and the Frobenius-König theorem in the original German to show that one has to choose between a very literal translation and considerable rephrasing.

Frobenius's theorem (92-I and 102-I):

Die Elemente einer Determinante n ten Grades seien n^2 *unabhängige Veränderliche. Man setze einige derselben Null, doch so, dass die Determinante nicht identisch verschwindet. Dann bleibt sie eine irreduzible Funktion, ausser wenn für einen Wert* $m<n$ *alle Elemente verschwinden, die m Zeilen mit* $n-m$ *Spalten gemeinsam haben.*

Frobenius-König (102-II):

Wenn in einer Determinante n ten Grades alle Elemente verschwinden, welche p ($\leqslant n$) *Zeilen mit* $n-p+1$ *Spalten gemeinsam haben, so verschwinden alle Glieder der entwickelten Determinante.*

Wenn alle Glieder einer Determinante n ten Grades verschwinden, so verschwinden alle Elemente, welche p Zeilen mit $n-p+1$ *Spalten gemeinsam haben für* $p=1$ *oder* $2,\ldots$ *oder n.*

[10]This theorem is the only result stated as a formal theorem in the introduction to F#92. We have found no reference to the theorem in the literature between 1936 and 1971; cf. (0.1). There are of course very many references in this period to the basic theorems on irreducible non-negative matrices proved in F#92.

[11]Jacobsthal calls the theorem beautiful (*schön*) in his review Frobenius [1917c]; Mirsky [1971, p. 212] describes the result as striking.

[12]There are several places in our quotations where Frobenius refers to the determinant, while today we would refer to the matrix.

[13]It was known earlier that $\det A$ is irreducible if all entries of A are independent indeterminates (and remains irreducible even if $a_{ij}=a_{ji}$). These results may be found with different proofs in Kürschák [1906] and in Bôcher [1907a, pp. 176–178], [1907c, pp. 192–194]. Bôcher adds an exercise to show that $\det A$ is irreducible if A is $(n+p)\times(n+p)$, where $p<n$, has a $p\times p$ zero submatrix, and the other entries are independent indeterminates. It would be most interesting to know what proof he had in mind. The absence of earlier references in Muir's review, Kürschák [1906c], suggests that these results had not been published before.

[14]It is surprising that before 1963 there does not seem to have been a term in the literature describing this property. Of course, every fully indecomposable matrix is irreducible.

[15]In addition to the quoted result, the statement of 102-II contains its converse, which is obvious, and which I quote in the original German in Note 9.

[16]König [1916] in fact proved this theorem for matrices of integers (i.e., Mirsky [1971, Theorem 11.1.3]), but then indicated that the theorem was valid for real matrices. See also König [1936, p. 238] for the same theorem, and Gallai [1964] for more discussion.

[17]In German, *speziell* means "not general", as "special" does in mathematical English. In colloquial American, "special" commonly means "not ordinary", but the German word lacks this connotation.

[18]However, Frobenius could write with virulence in official university documents, which were not intended for publication; see Biermann [1973]. For example, see p. 215 of Biermann's book for Frobenius's attack on S. Lie, and see pp. 122–123 for a sketch of Frobenius's character.

[19]"A particularly beautiful and suggestive result in combinatorial matrix theory" (Mirsky [1971, p. 211]).

[20]The analog of Birkhoff's theorem for matrices of non-negative integers had already been proved by König [1916, Theorem F]; see also König [1936, p. 239] and Mirsky [1971, Theorem 11.1.5]. König was surely aware that some of his results proved for matrices of integers had analogues for real matrices; see Note 16. Also Egerváry [1931, Theorem II] proves a result more general than König's Theorem F, and observes that by considerations of continuity a result can be obtained containing Birkhoff's. It is easy to unify the two theorems: see Schneider [1977] for a theorem we propose to call the Birkhoff-Egerváry-König theorem.

[21]In this connection, we have received the following comment (February 1976) from A. M. Ostrowski (born 1893):

> The last sentence in Frobenius' collected papers makes indeed rather an awkward impression. It expresses however a feeling that was rather general in those days. The argumentation of Frobenius belongs of course to graph theory. But he had obviously the feeling that introducing new names for old arguments does not add anything of substance.
>
> It's of course different to day. If I try to analyse why it is so, the main reason appears to be that with coming of computers very long chains of arguments became accessible and the need in systematic combinatorics became very urgent.
>
> It is interesting to observe that in this case, too, the advance in technology made necessary the development of a new branch in pure mathematics.

[22]One would not expect to find in print many expressions of the "feeling that was rather general" mentioned by Ostrowski, but a mild example occurs in Muir [1930, p. 59]. Muir is reviewing a graph theoretic solution by G. Pólya (then of Budapest) of a problem on determinants posed by Schur, and he comments that Pólya's graph theoretic approach "in the present instance does not conduce to brevity". Muir has the advantage of knowing that a more direct algebraic solution may have been found by Schur, as indicated by Pólya. In his proof, Pólya foreshadows many results that are now standard in graph theory, and using these, his proof could now be expressed much more briefly. Schur's problem concerns the independence of terms in a determinant and can be reformulated thus: Find n^2-2n+2 permutation matrices of order n such that every permutation matrix is a linear combination with integral coefficients. The problem and solution are listed by Muir, and hence by us, under Schur [1912].

[23]See König [1933] or [1936, pp. 240–241] for a rather restrained reply to Frobenius's remarks, where it is mentioned that König sent his [1915] proof to Frobenius in German translation. Most papers by Hungarians of that time were published twice, once in Hungarian and once in German or French, but König [1915] appeared in Hungarian only. Perhaps Frobenius's reaction to König's paper has some connection with this fact.

[24]L. Mirsky has voiced disagreement with my hypothesis. His comment runs as follows:

If the final text of F-102 was prepared by someone other than Frobenius himself, surely the writer would not have gone out of his way to introduce gratuitously insulting remarks. Is it not much more likely that, on the contrary, he would suppress any passage that struck a controversial note? It seems to me therefore that the explanation you advance is not convincing. However, I am in any case very far from certain that anything calls for an explanation. Consider the actual situation: we are dealing with a man of choleric and unbridled temperament who, for many years, has been accustomed to riding roughshod over other people's feelings. (The way, for example, in which he defied the powerful Althoff and succeeded in having Schottky appointed to a chair in Berlin is wholly characteristic. Nor ought we forget his disparaging observations about a whole host of distinguished mathematicians such as Lie, Klein, Landau, and even Hilbert.) Consider, as I say, such a man, who is now elderly and in failing health, indeed nearing the end of his life. In 1915, he receives a letter from König together with a German translation of a paper on graph theory, and his temper is roused. The precise reasons for the feeling of annoyance can only be guessed at; it may have been the fear of an old man of ideas that might put his own work in the shade, or it may have been something totally irrational. In any case, nervous irritability is precisely the reaction Frobenius had so often envinced in his life.

[25]Russian dictionaries we have consulted list "definitely" before "ultimately" as a translation of *okonchatel'no*. Liebmann's translation (Markov [1908c]), where "*schliesslich*" is used, opts for the latter meaning.

[26]In (3.4)–(3.8) we have paraphrased Markov's words and modernized the notation and terminology. For example, in (3.8) we partition the index set $\{1,\ldots,n\}$; Markov speaks of dividing sums corresponding to $\sum_{j=1}^{n} a_{ij}x_j$, $i=1,\ldots,n$, into two sets.

[27]By referring to Minkowski [1900], Markov shows that he is aware of the *diagonal dominance theorem*:

If

$$m_{ii}>0,\quad m_{ij}\leqslant 0,\qquad i\neq j,\quad i,j=1,\ldots,n,$$

and

$$\sum_{j=1}^{n} m_{kj}>0,\qquad k=1,\ldots,n,$$

then

$$\det M>0.$$

In his fundamental paper on M-matrices, Ostrowski [1937, p. 73] quotes the theorem in (3.5) (with the hypothesis that M is irreducible) and attributes it to Markov [1908]. A slightly more general version of this theorem is to be found in Olga Taussky [1949, Theorem II], where a reference to Markov [1908] is also given. These are the only references we have found in the literature to Markov in the context of irreducibility. It should be noted that an irreducibility condition for symmetric matrices occurs as early as Stieltjes [1887]. On p. 387, he essentially shows that if A is a positive definite irreducible matrix and $a_{ij} \leqslant 0$, $i \neq j$, $i,j = 1,\ldots,n$, then A^{-1} has positive entries. Stieltjes happens to refer to a paper of Markov's at the end of his article, but there is no evidence that 20 years later Markov recalled Stieltjes's. For a very extensive list of references to the diagonal dominance theorem, see Taussky [1949], and for a few additions, see Schneider [1978].

[28]See also Seneta [1973, pp. 99–100] for some remarks in a similar spirit and additional references to papers that are not well known.

[29]For related remarks see Solow [1952], particularly Sec. X, and a footnote on p. 33.

[33]In Part I we informally used the word "cycle" without the restriction that follows.

[31]Thus, strictly speaking, a cycle is an equivalence class of paths.

[32]Observe that the sense of "reducible" here is the usual one for polynomials over a field and differs from that of (1.1).

[33]See DeSoer [1960] for a general discussion, based on work by C. L. Coates [1959].

[34]Following Sinkhorn-Knopp [1969], we call a matrix A *chainable* if for each pair of non-zero entries $a_{i_1 j_1}$ and $a_{i_k j_k}$ there is a sequence of non-zero entries $a_{i_1 j_1},\ldots,a_{i_k j_k}$ such that for $r = 1,\ldots,k-1$ either $i_r = i_{r+1}$ or $j_r = j_{r+1}$. By means of Perfect-Mirsky [1965, Theorem 1], which is also Mirsky [1971, p. 198, Theorem 11.4.1], one may derive any one of Frobenius's theorem, Zylinski's theorem and Sinkhorn-Knopp [1969, Lemma 1] from the other two. It is interesting to observe that applications of chainability have been found, apparently independently, at least four times: in the papers by Zylinski [1921] and Sinkhorn-Knopp [1969] already quoted, in Dulmadge-Mendelsohn [1962] and in Lallement-Petrich [1964], [1966]; see also Engel-Schneider [1975].

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