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On the singular graph and the Weyr characteristic of an M-matrix

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Abstract. Let A be an M-matrix in standard lower block triangular form, with diagonal blocks A_{ii} irreducible. Let $\mathbf{S}$ be the set of indices α such that the diagonal block $A_{\alpha\alpha}$ is singular. We define the singular graph of A to be the set $\mathbf{S}$ with partial order defined by $\alpha > \beta$ if there exists a chain of non-zero blocks $A_{\alpha i}, A_{ij}, \ldots, A_{l\beta}$.

Let Λ_1 be the set of maximal elements of $\mathbf{S}$, and define the p-th level Λ_p, $p = 2, 3, \ldots$, inductively as the set of maximal elements of $\mathbf{S}\backslash(\Lambda_1 \cup \cdots \cup \Lambda_{p-1})$. Denote by λ_p the number of elements in Λ_p. The Weyr characteristic (associated with 0) of A is defined to be $\omega(A) = (\omega_1, \omega_2, \ldots, \omega_h)$, where $\omega_1 + \cdots + \omega_p = \dim \operatorname{Ker} A^p$, $p = 1, 2, \ldots$, and $\omega_h > 0$, $\omega_{h+1} = 0$.

Using a special type of basis, called an $\mathbf{S}$-basis, for the generalized eigenspace $E(A)$ of 0 of A, we associate a matrix D with A. We show that $\omega(A) = (\lambda_1, \ldots, \lambda_h)$ if and only if certain submatrices $D_{p,p+1}$, $p = 1, \ldots, h-1$, of D have full column rank. This condition is also necessary and sufficient for $E(A)$ to have a basis consisting of non-negative vectors, which is a Jordan basis for $-A$.

We also consider a given finite partially ordered set $\mathbf{S}$, and we find a necessary and sufficient condition that all M-matrices A with singular graph $\mathbf{S}$ have $\omega(A) = (\lambda_1, \ldots, \lambda_h)$. This condition is satisfied if $\mathbf{S}$ is a rooted forest.

1. Introduction

In this paper, we study the structure of the elementary divisors associated with the Perron–Frobenius root of a non-negative matrix P (see §2 for definitions). It is clearly equivalent to consider the elementary divisors associated with 0 of a singular M-matrix and for technical reasons we state our results for M-matrices. Let A be a singular M-matrix in standard lower block triangular form with diagonal blocks A_{ii} irreducible (§2), and let $\mathbf{S}$ be the set of indices α for which $A_{\alpha\alpha}$ is singular. Then, there is a natural partial order or graph corresponding to $\mathbf{S}$, which we call the *singular graph* of A. The general problem is this: To what extent is the structure of the elementary divisors of A (associated with 0) determined by the singular graph $\mathbf{S}$?

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Schneider [1956, Theorems 2 and 5] (also cf. Carlson [1963, Theorem 3], Rothblum [1975, Corollary 3.4]) proved two results in this area:

(1.1) the elementary divisors are all linear if and only if **S** is trivially ordered;
(1.2) there is exactly one elementary divisor if and only if **S** is linearly ordered.

Schneider also showed, by an example, that **S** alone does not always determine the elementary divisor structure.

Cooper [1973, Theorem 3] proved

(1.3) the number of linearly independent eigenvectors of A equals the number of maximal elements of **S**, if **S** is a rooted forest.

This also generalized one direction of (1.1) and (1.2).

In the present paper, we introduce level numbers $(\lambda_1, \ldots, \lambda_h)$ for the singular graph of A and we describe necessary and sufficient conditions for the Weyr [1890] characteristic associated with 0 to be $(\lambda_1, \ldots, \lambda_h)$. These conditions are stated (Theorem (4.7)) in terms of the singular graph of A and of the nullity of certain rectangular matrices which are induced by the action of A on a special basis for the generalized eigenspace $E(A)$ (associated with 0). An equivalent condition (Theorem (6.5)) is that $E(A)$ has a basis consisting of non-negative vectors which is a Jordan basis for $-A$. We deduce necessary and sufficient conditions (Theorem (5.6)) on a graph **S** that every M-matrix A with singular graph **S** should have Weyr characteristic equal to $(\lambda_1, \ldots, \lambda_h)$. In particular (Corollary (5.7)), this condition is satisfied when **S** is a rooted forest, and hence the results quoted from Schneider [1956] and Cooper [1973] are special cases.

Our chief tool is the construction of a sequence of generalized eigenvectors with special zero properties (see §3). A special case of this construction was stated in Theorem 6 of Schneider [1956]. Rothblum [1975] has independently carried out the construction of the general case (and this came to our attention after we had obtained the main results of this paper). We formulate the results explicitly, but the essentials may be found in the proof of Rothblum's main theorem, Theorem (3.1), Parts 1 and 2. We require somewhat finer structure than Rothblum and we do not use all the positivity properties of Rothblum's sequence of generalized eigenvectors until §6. By means of this construction we associate with each singular M-matrix A a strictly lower triangular matrix D which has the same Weyr characteristic and the same singular graph as A. Thus the problems we are considering may be discussed in terms of matrices of a much simpler type.

2. Preliminaries

Let $\mathbb{R}$ be the real field and let $\mathbb{R}^{mn}$ be the set of all $m \times n$ matrices with elements in $\mathbb{R}$. Let $A \in \mathbb{R}^{mn}$. We use the following terminology and notation:

$A \geq 0$ (A is non-negative) if $a_{ij} \geq 0$, $i = 1, \ldots, m$; $j = 1, \ldots, n$;
$A > 0$ (A is semi-positive) if $A \geq 0$ and $A \neq 0$;
$A \gg 0$ (A is strictly positive) if $a_{ij} > 0$, $i = 1, \ldots, m$; $j = 1, \ldots, n$.

Then $A \geq B$ will mean that $A - B \geq 0$, etc.

Remark. Unfortunately, the terminology and notation in the subject of non-negative matrices is not standardized, cf. Gantmacher [1959], Schneider [1956], Carlson [1963], Cooper [1973], Rothblum [1975]. We have followed Barker [1973] (who discusses a more general situation), except that we have sacrificed economy for the sake of clarity by avoiding the variously used term "positive" without any prefix.

Let $A \in \mathbb{R}^{nn}$. We will make continual use of a standard form for A developed by Frobenius [1912]. Since we require some details not found explicitly in the principal references, we sketch in the next few paragraphs a construction of this form (apparently first found in Doeblin [1938, pp. 81–82]). On $\{1, \ldots, n\}$ we define a relation by $i \to j$ if $a_{ij} \neq 0$, and then a transitive relationship by $i \succ j$ if there exists a sequence $i = i_1 \to i_2 \to \cdots \to i_r = j$, where $1 \leq i_q \leq n$, $q = 1, \ldots, r$. We also define an equivalence relation on $\{1, \ldots, n\}$ by $i \sim j$ if either $i = j$ or both $i \succ j$ and $j \succ i$. We define a partial order on the set $\mathbf{G}$ of equivalence classes for $\sim$ by $X \succ Y$ if $X \neq Y$ and there exists $i \in X$, $j \in Y$ such that $i \succ j$. The matrices $A[X] = (a_{ij})$, $i, j \in X$, are irreducible (e.g., Varga [1962, p. 19]).

Let $\mathbf{S} = \{X \in \mathbf{G} : A[X]$ is singular$\}$. Then $\mathbf{S}$ is partially ordered by the order induced from $\mathbf{G}$. We define recursively the p-th level Λ_p of $\mathbf{S}$, $p = 1, \ldots$ by $\Lambda_1 = \{X \in \mathbf{S} : X$ is maximal in $\mathbf{S}\}$ and, if Λ_{p-1} has been defined, $\Lambda_p = \{X \in \mathbf{S} : X$ is maximal in $\mathbf{S} \backslash (\Lambda_1 \cup \cdots \cup \Lambda_{p-1})\}$. We let h be the largest integer such that $\Lambda_h \neq \phi$, and call h the number of levels of $\mathbf{S}$.

It is easy to prove that we may label the equivalence classes $X_1, \ldots, X_g$ of $\mathbf{G}$ to satisfy the following conditions:

(2.1) If $X_i \succ X_j$, then $i > j$;

(2.2) If $X_i \in \Lambda_r$, $X_j \in \Lambda_q$, $r < q$, then $i > j$.

We may cite Harary [1969, Theorem 16.3] to prove (2.1), and the proof of (2.2) is essentially the same as that of Schneider [1956, Theorem 1]. We shall assume that $\mathbf{G}$ has been labelled to satisfy the above conditions.

Let $n_i = |X_i|$, the cardinality of X_i, $i = 1, \ldots, g$. We may apply a permutation π to $\{1, \ldots, n\}$ so that $\pi(X_1 \cup \cdots \cup X_r) = \{1, \ldots, n_1 + \cdots + n_r\}$, $r = 1, \ldots, g$. If P

is the corresponding permutation matrix and $B=PAP^T$, then

$$B=\begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ \cdot & & & 0 \\ B_{g1} & B_{g2} & \cdots & B_{gg} \end{bmatrix},$$

where $B_{ii}=A[X_i]$, $i=1,\ldots,g$.

Thus B is in lower block triangular form with its diagonal blocks irreducible. Such a form is called *standard form*. We may assume that A is given in standard form since the properties we are investigating are invariant under $A\to PAP^T$, where P is a permutation matrix. Further we shall now denote by i the equivalence class $X_i\in \mathbf{G}$. Thus $\mathbf{S}=\{i\in\mathbf{G}: B_{ii}$ is singular$\}$ and (2.1) and (2.2) become

(2.1)′ If $i>j$, then for $i>j$, $1\le i, j\le g$;

(2.2)′ If $i\in \Lambda_r$, $j\in\Lambda_q$, $r<q$, then $i>j$, for $1\le i$, $j\le g$.

Remark. The existence of the standard form and its uniqueness up to permutations of equivalence classes is due to Frobenius [1912] (cf. also Gantmacher [1959, Vol. II, p. 75] and Varga [1962, p. 46] who use the name "normal form") The partial order on $\mathbf{G}$ has also been defined in Schneider [1956], Carlson [1963], Cooper [1973], and Rothblum [1975]. Schneider and Carlson write $R_{ij}=1$ for $i\ge j$, Rothblum following Karlin [1966, p. 41] says "i has access to j," and Cooper uses $i\le j$ for our $i>j$.

We have constructed a standard form for an arbitrary $A\in\mathbb{R}^{nn}$. We shall consider only the case where A is a singular M-matrix.

(2.3) DEFINITION. A matrix $A\in\mathbb{R}^{nn}$ is an *M-matrix* if there is a nonnegative $P\in\mathbb{R}^{nn}$ and an $r\ge\rho(P)$ such that $A=rI-P$. ($\rho(P)$ is the Perron–Frobenius root of P, e.g., Varga [1962, p. 46, Theorem 2]).

Clearly an M-matrix A is singular if and only if $\rho(P)=r$ in the above definition.

The equivalence of Definition (2.3) with the original definition of M-matrix in Ostrowski [1937] is well-known, e.g. Schneider [1953], Fiedler–Ptak [1962]. See also Plemmons [1977] for many other equivalent conditions when A is nonsingular.

Two fundamental properties of the standard form of a singular M-matrix A will be used subsequently without further remark. These are:

(a) the diagonal blocks A_{ii}, $1\le i\le g$, are irreducible M-matrices. Several standard results about irreducible M-matrices of importance to us are summarized in Lemma (3.2);

(b) the blocks below the main diagonal, A_{ij}, $1 \leq j < i \leq g$, are non-positive, i.e., $-A_{ij} \geq 0$.

(2.4) *Notations:*

(a) We shall generally denote elements of $\mathbf{S}$ by $\alpha, \beta, \ldots$.

(b) For $\alpha \in \mathbf{S}$, we define

$$\mathbf{S}(\alpha) = \{\beta \in \mathbf{S} : \beta > \alpha\}.$$

(c) Similarly $\mathbf{G}(\alpha) = \{i \in \mathbf{G} : i > \alpha\}$. Also $\mathbf{G}^*(\alpha) = \mathbf{G}(\alpha) \cup \{\alpha\}$.

(d) For $\alpha \in \mathbf{S}$, we denote by $\Delta(\alpha)$ the minimal elements of $\mathbf{S}(\alpha)$. More generally, for $\Lambda \subseteq \Lambda_p$, $1 \leq p \leq h$, we shall write $\Delta(\Lambda) = \bigcup_{\alpha \in \Lambda} \Delta(\alpha)$.

(e) For $\alpha \in \mathbf{S}$, $\mathbf{G}_1(\alpha) = \mathbf{G}^*(\alpha) \backslash (\bigcup_{\beta \in \Delta(\alpha)} \mathbf{G}^*(\beta))$.

(f) $\lambda_p = |\Lambda_p|$, $p = 1, \ldots, h$. These numbers are called the *level numbers* of $\mathbf{S}$.

We may consider $\mathbf{S}$ as a directed graph with arcs given by (α, β) such that $\beta \in \Delta(\alpha)$.

(2.5) DEFINITION. The partially ordered set $(\mathbf{S}, \succeq)$ is called the *singular graph* of the M-matrix A.

Similarly, we may consider $\mathbf{G}$ as a directed graph with arcs given by (i, j) such that j is a minimal element of $\mathbf{G}(i)$.

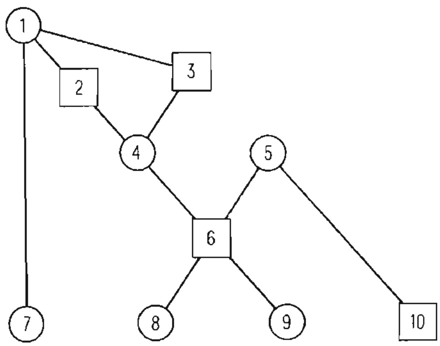
Remark. Although results as early as Frobenius [1912, §11] have natural formulations in terms of $\mathbf{G}$ and $\mathbf{S}$, Cooper [1973] was apparently the first to describe these sets as graphs (cf. his $\gamma(A)$ and $\mu(A)$ resp.).

2.6 DEFINITION. Let $\mathbf{T}$ be a finite partially ordered set. By $\mathfrak{A}(\mathbf{T})$, we shall denote the set of all M-matrices (of all possible orders) whose singular graph $\mathbf{S}$ is isomorphic to $\mathbf{T}$, i.e., $\mathbf{S}$ becomes $\mathbf{T}$ after relabelling.

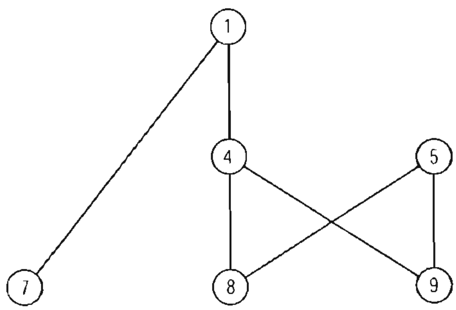
Example. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \cdot & & & \cdot & & & \cdot & 0 \\ -1 & 1 & & & & & & & & \\ -1 & 0 & 1 & & & & & & & \cdot \\ 0 & -1 & -1 & 0 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & -1 & -1 & -1 & -1 & 1 & & & & \cdot \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & & \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & \dot{0} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The graph $\mathbf{G}$ is:

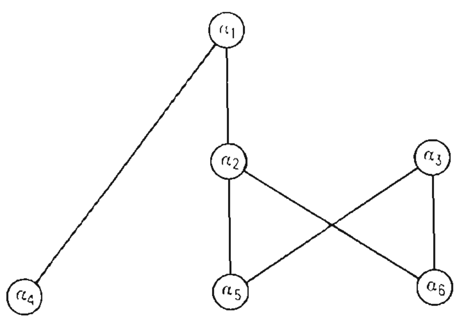


The graph $\mathbf{S}$ is:



$$\mathbf{G}^*(1) = \{1, 2, 3, 4, 6, 7, 8, 9\},$$
$$\mathbf{G}(1) = \{2, 3, 4, 6, 7, 8, 9\},$$
$$\mathbf{G}_1(1) = \{1, 2, 3\},$$
$$\mathbf{S}(1) = \{4, 7, 8, 9\}.$$
$$\Delta(1) = \{4, 7\}$$
$$\Lambda_1 = \{7, 8, 9\} \qquad \Lambda_2 = \{4, 5\} \qquad \Lambda_3 = \{1\}.$$

If $\mathbf{T}$ is



then $A \in \mathfrak{A}(\mathbf{T})$.

3. Construction of an S-basis for $E(A)$

(3.1) *Conventions:*

(a) We shall denote by A an M-matrix in $\mathbb{R}^{nn}$, and by $\mathbf{S}$ the singular graph of A. Also $|\mathbf{S}| = s$.

(b) We shall assume that A is in standard form, with g irreducible diagonal blocks.

(c) The terms eigenvector, eigenspace, etc., will refer to the eigenvalue 0 of A.

(d) If $x \in \mathbb{R}^n$, then x is partitioned conformably with the standard form of A, and we let x_i denote the i-th block component. Thus,

$$(Ax)_i = \sum_{j=1}^{i} A_{ij}x_j, \qquad i = 1, \ldots, g.$$

We begin by stating some standard results for irreducible M-matrices (see Ostrowski [1937], cf. Schneider [1956, Lemmas 5 and 7]). If x is a vector, we use $\sigma(x)$ for the sum of the (actual) 1×1 components of x.

(3.2) LEMMA. *The following hold for an irreducible M-matrix $B \in \mathbb{R}^{nn}$.*

(a) *If B is non-singular, then $B^{-1} \gg 0$.*

(b) *If B is singular, then there exists a unique $u \in \mathbb{R}^n$ such that $Bu = 0$ and $\sigma(u) = 1$. Further, $u \gg 0$.*

(c) *If B is singular and $Bx \geq 0$ or $Bx \leq 0$, then $Bx = 0$ (e.g., Schneider [1956, Lemma 5]).*

(d) *Let y, $z \in \mathbb{R}^n$ and suppose $y > 0$. Then there exists $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$ such that $Bx = cy + z$ (e.g., Schneider [1956, Lemma 7]).*

(3.3) *Definition.* We call a set $\{v^\alpha \in \mathbb{R}^n : \alpha \in \mathbf{S}\}$ an $\mathbf{S}$-set (for A) if

(a) $\sigma(v^\alpha_\alpha) = 1$,

(b) $v^\alpha_i = 0$ if $i \notin \mathbf{G}^*(\alpha)$,

(c) there exist scalars $d^{\beta\alpha} \in \mathbb{R}$, α, $\beta \in \mathbf{S}$, such that

$$d^{\beta\alpha} = 0 \text{ if } \beta \notin \mathbf{S}(\alpha)$$

and

$$Av^\alpha = -\sum_{\beta \in \mathbf{S}} d^{\beta\alpha} v^\beta. \tag{3.3.1}$$

(The empty sum is 0 by convention.)

We shall say that the scalars $\{d^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$ are *associated with the* $\mathbf{S}$*-set* $\{v^\alpha : \alpha \in \mathbf{S}\}$.

Suppose $\mathbf{S} = \{\alpha_1, \ldots, \alpha_s\}$ where $\alpha_1 < \alpha_2 < \cdots < \alpha_s$. This indexing will remain throughout the rest of this paper.

Define $D \in \mathbb{R}^{ss}$ by

$$d_{ij} = d^{\alpha_i \alpha_j}$$

and define for $1 \le p,\ q \le h$, the $\lambda_p \times \lambda_q$ submatrix

$$D_{pq} = (d^{\beta\alpha})_{\beta \in \Lambda_p, \alpha \in \Lambda_q}. \tag{3.3.2}$$

For the sake of notational convenience, we shall index the entries of D by the elements of $\mathbf{S}$. Thus we shall refer to $d^{\beta\alpha}$ as the (β, α) entry of D, and, for $\alpha \in \Lambda_q$, the α-th column of D_{pq} will mean the column with entries $(d^{\beta\alpha})_{\beta \in \Lambda_p}$, etc.

By (2.2)′,

$$D_{pq} = 0 \quad \text{if} \quad p \le q.$$

Observe that

$$D = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ D_{h-1,h} & 0 & & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \\ \cdot & & & \cdot & \cdot \\ D_{1h} & \cdot & \cdot & D_{12} & 0 \end{bmatrix}$$

the unusual order being natural for this problem. We call D the *matrix associated with the* $\mathbf{S}$*-set* $\{v^\alpha : \alpha \in \mathbf{S}\}$.

Let V be the matrix

$$V = [v^{\alpha_1}, \ldots, v^{\alpha_s}] \in \mathbb{R}^{ns}. \tag{3.3.3}$$

Then (3.3.1) is equivalent to

$$AV = -VD. \tag{3.3.4}$$

In the example at the end of §2, we may write the elements of $\mathbf{S}$ in the order 1, 4, 5, 7, 8, 9. Then D must have the form

$$
\begin{array}{cc}
 & \begin{array}{cccccc} 1 & 4 & 5 & 7 & 8 & 9 \end{array} \\
\begin{array}{c} 1 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{array} &
\left[\begin{array}{c:cc:ccc}
0 & \multicolumn{2}{c:}{0} & \multicolumn{3}{c}{0} \\ \hdashline
+ & & & & & \\
\cdot & \multicolumn{2}{c:}{0} & \multicolumn{3}{c}{0} \\ \hdashline
+ & \cdot & \cdot & & & \\
\times & + & + & & 0 & \\
\times & + & + & & &
\end{array}\right]
\end{array}
$$

where $\cdot$ indicates a zero element, $+$ a positive element, and $\times$ a real element. (In Lemma (3.8), we will show that $d^{\beta\alpha} > 0$ if $\beta \in \Delta(\alpha)$).

The *generalized eigenspace* $E(A)$ (associated with 0) of A consists of all $x \in \mathbb{R}^n$ such that $A^n x = 0$. It is clear that $\dim E(A) = s$.

(3.4) LEMMA. *Every* $\mathbf{S}$*-set for* A *is a basis for* $E(A)$.

Proof. Let $\{v^\alpha : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}$-set for A. We first show linear independence. Suppose there exist $c_\alpha \in \mathbb{R}$ such that $\sum_{\alpha \in \mathbf{S}} c_\alpha v^\alpha = 0$ and some $c_\alpha \neq 0$. Let $\beta = \min\{\alpha : c_\alpha \neq 0\}$. If $\alpha \in \mathbf{S}$ and $\alpha > \beta$, then $\beta \notin \mathbf{G}^*(\alpha)$, whence by Definition (3.3b), $v^\alpha_\beta = 0$. Hence $0 = \sum_{\alpha \geq \beta} c_\alpha v^\alpha_\beta = c_\beta v^\beta_\beta$ which contradicts $\sigma(v^\beta_\beta) = 1$. We have proved linear independence.

To show that $v^\alpha \in E(A)$ for $\alpha \in \mathbf{S}$, observe that if D is the matrix associated with $\{v^\alpha : \alpha \in \mathbf{S}\}$, then since D is in strictly lower block triangular form, $D^h = 0$ where h is the number of levels of $\mathbf{S}$. Hence $(-A)^h V = VD^h = 0$, where V is given by (3.3.3), and so $A^h v^\alpha = 0$. Since $\dim E(A) = s$, the result follows.

In view of this Lemma, an $\mathbf{S}$-set for A will be called an $\mathbf{S}$-*basis for* $E(A)$. Before proving the existence of $\mathbf{S}$-bases, we will exhibit some of their properties.

Let $\{v^\alpha : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}$-basis for $E(A)$. For $k \in \mathbf{G}$, (3.3.1) is equivalent to

$$A_{kk} v^\alpha_k + (d^{k\alpha} v^k_k) = y_k - \sum_{\beta \in \mathbf{S}(\alpha), \beta < k} d^{\beta\alpha} v^\beta_k, \tag{3.5}$$

where the term in parenthesis occurs only if $k \in \mathbf{S}$ and where

$$y_k = -\sum_{j=\alpha}^{k-1} A_{kj} v^\alpha_j, \tag{3.6}$$

with the summation starting at $j = \alpha$ since $v^\alpha_j = 0$ for $j < \alpha$.

The following is also a consequence of Theorem 2 [Schneider, 1956], and we include a proof for the sake of completeness. The notation used here was introduced in (2.4).

(3.7) LEMMA. *Let* $\{v^\alpha : \alpha \in \mathbf{S}\}$ *and* $\{\tilde{v}^\alpha : \alpha \in \mathbf{S}\}$ *be* $\mathbf{S}$*-bases for* $E(A)$. *Then, for each* $\alpha \in \mathbf{S}$,

(a) $v_k^\alpha \gg 0$, *for* $k \in \mathbf{G}_1(\alpha)$,

(b) $\tilde{v}_k^\alpha = v_k^\alpha$, *for* $k \in \mathbf{G}_1(\alpha)$.

Proof. Let $\alpha \in \mathbf{S}$. The proof is by induction on k. Observe that α is the smallest element in the natural order of $\mathbf{G}_1(\alpha)$. Hence, for $k = \alpha$, (3.5) reduces to $A_{\alpha\alpha} v_\alpha^\alpha = 0$, and (a) and (b) follow from Lemma (3.2b) and Definition (3.3a).

So suppose that $k > \alpha$ and that (a) and (b) hold for $\alpha, \ldots, k-1$. If $k \notin \mathbf{G}_1(\alpha)$, the assertion is vacuously satisfied. Assume $k \in \mathbf{G}_1(\alpha)$. Then A_{kk} is non-singular and thus $A_{kk}^{-1} \gg 0$. Let $\alpha \le j \le k-1$. If $j \notin \mathbf{G}(\alpha)$, then $v_j^\alpha = 0$ and so $A_{kj} v_j^\alpha = 0$. If $j \in \mathbf{G}(\alpha)$, then $j \in \mathbf{G}_1(\alpha)$ and so $-A_{kj} v_j^\alpha \ge 0$ by induction. Further, since $k \in \mathbf{G}_1(\alpha)$, there is an i, $\alpha \le i \le k-1$, such that $-A_{ki} > 0$. Hence $y_k \ge -A_{ki} v_i^\alpha > 0$.

Note that $k < \beta$ for $\beta \in \mathbf{S}(\alpha)$. Hence (3.5) reduces to $A_{kk} v_k^\alpha = y_k$. Thus $v_k^\alpha = A_{kk}^{-1} y_k \gg 0$, and we have proved (a).

To prove (b), let $\tilde{y}_k = -\sum_{j=\alpha}^{k-1} A_{kj} \tilde{v}_j^\alpha$. Our inductive assumption and Definition (3.3b) give $\tilde{y}_k = y_k$ and hence $\tilde{v}_k^\alpha = A_{kk}^{-1} \tilde{y}_k = A_{kk}^{-1} y_k = v_k^\alpha$. The inductive step is complete and the Lemma follows.

(3.8) LEMMA. *Let* $\{v^\alpha : \alpha \in \mathbf{S}\}$ *and* $\{\tilde{v}^\alpha : \alpha \in \mathbf{S}\}$ *be* $\mathbf{S}$*-bases for* $E(A)$ *with associated scalars* $\{d^{\beta\alpha} : \beta, \alpha \in \mathbf{S}\}$ *and* $\{\tilde{d}^{\beta\alpha} : \beta, \alpha \in \mathbf{S}\}$, *respectively. Then for each* $\alpha \in \mathbf{S}$,

(a) $d^{\gamma\alpha} > 0$, *for* $\gamma \in \Delta(\alpha)$,

(b) $\tilde{d}^{\gamma\alpha} = d^{\gamma\alpha}$, *for* $\gamma \in \Delta(\alpha)$.

Proof. Let $\gamma \in \Delta(\alpha)$ and suppose $\alpha \le j < \gamma$. If $A_{\gamma j} v_j^\alpha \ne 0$, then $\alpha < j < \gamma$, whence $j \in \mathbf{G}_1(\alpha)$ and it follows by Lemma (3.7) that $-A_{\gamma j} v_j^\alpha \ge 0$. Hence for all j, $\alpha \le j < \gamma$, we have $-A_{\gamma j} v_j^\alpha \ge 0$. Further there exists an $i \in \mathbf{G}_1(\alpha)$, $\alpha \le i < \gamma$, such that $-A_{\gamma i} > 0$. Hence $y_\gamma > 0$. Since $\gamma \in \Delta(\alpha)$, $\gamma \notin \mathbf{G}^*(\beta)$ for $\beta \in \mathbf{S}(\alpha) \backslash \{\gamma\}$. Hence $v_\gamma^\beta = 0$ and (3.5) becomes

$$A_{\gamma\gamma} v_\gamma^\alpha + d^{\gamma\alpha} v_\gamma^\gamma = y_\gamma.$$

If $d^{\gamma\alpha} \le 0$, then $A_{\gamma\gamma} v_\gamma^\alpha > 0$, which contradicts Lemma (3.2c). Therefore $d^{\gamma\alpha} > 0$. This proves (a).

Observe that by Lemma (3.7), $y_\gamma = \tilde{y}_\gamma$ since $A_{\gamma j}v_j^\alpha = 0$ unless $j \in \mathbf{G}_1(\alpha)$. Hence

$$A_{\gamma\gamma}v_\gamma^\alpha + d^{\gamma\alpha}v_\gamma^\gamma = y_\gamma = \tilde{y}_\gamma = A_{\gamma\gamma}\tilde{v}_\gamma^\alpha + \tilde{d}^{\gamma\alpha}\tilde{v}_\gamma^\gamma,$$

and so $A_{\gamma\gamma}(v_\gamma^\alpha - \tilde{v}_\gamma^\alpha) = (\tilde{d}^{\gamma\alpha} - d^{\gamma\alpha})v_\gamma^\gamma$ since $\tilde{v}_\gamma^\gamma = v_\gamma^\gamma$ by Lemma (3.7b). Thus Lemma (3.2c) gives $\tilde{d}^{\gamma\alpha} - d^{\gamma\alpha} = 0$, and this completes the proof of (b).

In order to show the existence of an $\mathbf{S}$-basis for $E(A)$, we first prove an extension lemma.

(3.9) LEMMA. *Let $1 \in \mathbf{S}$ and suppose that for $\beta \in \mathbf{S}(1)$, there is a vector $v^\beta \in \mathbb{R}^n$ such that*

$$v_\beta^\beta \gg 0,$$

and

$$v_k^\beta = 0, \quad \textit{if} \quad k \notin \mathbf{G}^*(\beta).$$

Then there exist $u \in \mathbb{R}^n$ and scalars c^β for $\beta \in \mathbf{S}(1)$ such that

(a) $\sigma(u_1) = 1$,
(b) $u_k = 0$ *for* $k \notin \mathbf{G}^*(1)$,
(c) $Au = -\sum_{\beta \in \mathbf{S}(1)} c^\beta v^\beta$.

Proof. We define the block-component vectors u_k, $k = 1, \ldots, g$ and the scalars c^k, $k \in \mathbf{S}$, by induction on k. Observe that (c) is equivalent to

$$A_{kk}u_k + (c^k v_k^k) = y_k - \sum_{\beta \in \mathbf{S}(1), \beta < k} c^\beta v_k^\beta, \tag{3.9.1}$$

where the parenthesis indicates that this term occurs only if $k \in \mathbf{S}(1)$, and

$$y_k = -\sum_{j=1}^{k-1} A_{kj}u_j. \tag{3.9.2}$$

For $k = 1$, let u_1 be the vector satisfying $A_{11}u_1 = 0$ and $\sigma(u_1) = 1$. Let $k > 1$ and suppose that $u_1, \ldots, u_{k-1}$ and c^β, $\beta \in \mathbf{S}(1)$, $\beta < k$, satisfy (3.9.1) and condition (b). There are two possibilities: either $k \notin \mathbf{G}^*(1)$ or $k \in \mathbf{G}(1)$.

Suppose $k \notin \mathbf{G}^*(1)$. In this case, $k \notin \mathbf{S}(1)$ and we define $u_k = 0$, which satisfies (b). Let $1 \leq j \leq k-1$. If $j \in \mathbf{G}^*(1)$, then $A_{kj} = 0$. If $j \notin \mathbf{G}^*(1)$, then $u_j = 0$. Hence

$y_k = 0$. Further, if $\beta \in \mathbf{S}(1)$, then $k \notin \mathbf{G}(\beta)$, and so $v_k^1 = 0$. It follows that u_k satisfies (3.9.1).

Otherwise, $k \in \mathbf{G}(1)$. If $k \notin \mathbf{S}$, then A_{kk} is non-singular, and we may solve (3.9.1) for u_k. If $k \in \mathbf{S}$, then $k \in \mathbf{S}(1)$, and by Lemma (3.2d), since $v_k^k \gg 0$, there exist u_k and c^k satisfying (3.9.1). This completes the inductive step.

Remarks. (i) Observe that $\mathbf{S}(1)$ may be empty in Lemma (3.9). In that case, $Au = 0$.

(ii) If $\mathbf{S} = \phi$, then A is a non-singular M-matrix, and $E(A) = \{0\}$. We consider the empty set to be an $\mathbf{S}$-basis for $E(A)$.

Our next theorem summarizes results on $\mathbf{S}$-bases. For a list of properties of $\mathbf{S}$-bases, Definition (3.3) should be adjoined to Theorem (3.10, II and III).

(3.10) THEOREM. *Let A be an M-matrix with singular graph $\mathbf{S}$.*

(I) *There exists an $\mathbf{S}$-basis for the generalized eigenspace $E(A)$ (associated with 0).*

(II) *Let $\{v^\alpha : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}$-basis for $E(A)$, with associated scalars $\{d^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$. Then, for $\alpha \in \mathbf{S}$,*

(a) $v_k^\alpha \gg 0$, *for* $k \in \mathbf{G}_1(\alpha)$,

(b) $d^{\beta\alpha} > 0$, *for* $\beta \in \Delta(\alpha)$.

(III) *Let $\{\tilde{v}^\alpha : \alpha \in \mathbf{S}\}$ also be an $\mathbf{S}$-basis for $E(A)$ with associated scalars $\{\tilde{d}^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$. Then, for $\alpha \in \mathbf{S}$,*

$\tilde{v}_k^\alpha = v_k^\alpha$, *for* $k \in \mathbf{G}_1(\alpha)$,
$\tilde{d}^{\beta\alpha} = d^{\beta\alpha}$, *for* $\beta \in \Delta(\alpha)$.

Proof. In view of Lemmas (3.7) and (3.8), we need only prove (I). The proof is by induction on g, the number of elements in $\mathbf{G}$. If $g = 1$, then A is irreducible, and the result follows immediately from Lemma (3.2a). So assume $g > 1$, and let $B = (A_{ij})$, $i, j = 2, \ldots, g$.

Then the singular graph of B is $\mathbf{S}' = \mathbf{S} \backslash \{1\}$, and by induction there exists an $\mathbf{S}'$-basis $\{(v_i^\alpha),\ i = 2, \ldots, g : \alpha \in \mathbf{S}'\}$ for $E(B)$.

We consider two cases. If $1 \notin \mathbf{S}$, put $v_1^\alpha = 0$ for all $\alpha \in \mathbf{S}$. Then $\{v^\alpha : \alpha \in \mathbf{S}\}$ is an $\mathbf{S}$-basis for $E(A)$.

If $1 \in \mathbf{S}$, we put $v_1^\beta = 0$ for $\beta \in \mathbf{S}'$. The vectors v^β, $\beta \in \mathbf{S}(1)$, satisfy the hypothesis of Lemma (3.9). So let u and c^β, $\beta \in \mathbf{S}(1)$, be given by Lemma (3.9). Put $v^1 = u$, $d^{\beta 1} = c^\beta$, for $\beta \in \mathbf{S}(1)$, and $d^{\beta 1} = 0$, for $\beta \in \mathbf{S} \backslash \mathbf{S}(1)$. Then $\{v^\alpha : \alpha \in \mathbf{S}\}$ is an $\mathbf{S}$-basis for $E(A)$ with associated scalars $\{d^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$.

Remark. If $D \in \mathbb{R}^{ss}$ is a matrix associated with some $\mathbf{S}$-basis for $E(A)$, we may refer to D as an **S**-*matrix for* A. Observe that if D is an **S**-matrix for A, then, by Theorem (3.10 III), the submatrices $D_{p,p+1}$, $p=1,\ldots,h-1$, do not depend on the choice of **S**-basis.

(3.11) COROLLARY. *Let D be an* **S**-*matrix for A. Then, for $p=1,\ldots,h-1$, each column of $D_{p,p+1}$ is semi-positive.*

Proof. By Theorem (3.10 II) and Definition (3.3c), $D_{p,p+1}\geq 0$. By the definition of Λ_{p+1}, for each $\alpha\in\Lambda_{p+1}$, $\Delta(\alpha)\cap\Lambda_p\neq\phi$ and so each column of $D_{p,p+1}$ has a positive entry.

4. Applications of S-bases for $E(A)$

(4.1) LEMMA. *Let D be an* **S**-*matrix for A. If $p>0$ and D^p is partitioned in the same manner as D, then:*

(a) $(D^p)_{rq}=0$ *for* $r>q-p$, *and* $q, r=1,\ldots,h$;

(b) $(D^p)_{q-p,q}\geq 0$, *and each column of this matrix is semi-positive, if* $p<q$, $q=1,\ldots,h$.

Proof. The proof is by induction on p. For $p=1$, the result follows by Corollary (3.11).

So suppose the result holds for p. By inductive hypothesis, $D_{rt}=0$ if $t\leq r$, and $(D^p)_{tq}=0$ if $t>q-p$. Hence,

$$(D^{p+1})_{rq}=\sum_{t=1}^{h}D_{rt}(D^p)_{tq}=\sum_{t=r+1}^{q-p}D_{rt}(D^p)_{tq}.$$

It follows that $(D^{p+1})_{rq}=0$ if $r>q-p-1$ and (a) is proved.

Further, if $q-p-1>0$,

$$(D^{p+1})_{q-p-1,q}=D_{q-p-1,q-p}(D^p)_{q-p,q}$$

and so $(D^{p+1})_{q-p-1,q}\geq 0$ by inductive assumption. Let $\alpha\in\Lambda_q$. Observe that the inductive assumption implies that there is a $\beta\in\Lambda_{q-p}$ such that $(D^p)^{\beta\alpha}>0$, where $(D^p)^{\beta\alpha}$ is the (β,α) element of D^p. There exists $\gamma\in\Lambda_{q-p-1}$ such that $d^{\gamma\beta}>0$. Hence, $(D^{p+1})^{\gamma\alpha}\geq d^{\gamma\beta}(D^p)^{\beta\alpha}>0$, and so the α-th column of $(D^{p+1})_{q-p-1,q}$ has a positive element. Thus (b) is proved.

We define the integers $\omega_p(A)$ (usually written as ω_p) by $\omega_1+\cdots+\omega_p=\dim\operatorname{Ker}A^p$, $p=1,2,\ldots$. It is well-known (e.g., Weyr [1885], [1890, pp. 184–186]) that $\omega_p\geq 0$ and $\omega_1\geq\omega_2\geq\omega_3\geq\cdots$. The integer k such that $\omega_k>0$, but

$\omega_{k+1}=0$, we call the *index* of A. Thus k is the smallest integer for which $\operatorname{Ker} A^k = E(A)$. The sequence $(\omega_1, \ldots, \omega_k)$ is called the Weyr characteristic of A (associated with 0), e.g., MacDuffee [1933, p. 73].

The following corollary is also proved in Rothblum [1975, Theorem 3.1, Part 2].

(4.2) COROLLARY. *Let A be an M-matrix with singular graph* $\mathbf{S}$, *and let h be the number of levels of* $\mathbf{S}$. *Then the index of A is h.*

Proof. By (3.3.4), $-AV = VD$, where the columns of V are an $\mathbf{S}$-basis for $E(A)$. By Lemma (4.1), $(-A)^h V = VD^h = 0$, but $(-A)^{h-1}V = VD^{h-1} \neq 0$ since V has full column rank and $D^{h-1} \neq 0$.

In view of this Corollary, the Weyr characteristic of A is $(\omega_1, \ldots, \omega_h)$. We now begin our investigation of the relation of the Weyr characteristic to the sequence $(\lambda_1, \ldots, \lambda_h)$, where $\lambda_i = |\Lambda_i|$.

Remark. Let V be the matrix of an $\mathbf{S}$-basis for $E(A)$, and let

$$f = \begin{bmatrix} f_h \\ f_{h-1} \\ \cdot \\ \cdot \\ \cdot \\ f_1 \end{bmatrix} \in \mathbb{R}^s$$

be partitioned conformably with an $\mathbf{S}$-matrix D for A. By Lemma (4.1a), $(D^p)_{rq} = 0$ for $q = 1, \ldots, p$, and $r = 1, \ldots, h$. If $f_{p+1} = 0, \ldots, f_h = 0$, then $(-A)^p Vf = VD^p f = 0$ since $(D^p f)_r = \sum_{q=1}^{h} (D^p)_{rq} f_q = 0$, and so the vector $Vf \in \operatorname{Ker} A^p$.

Let $E_p = \{Vf : f \in \mathbb{R}^s \text{ and } f_{p+1} = 0, \ldots, f_h = 0\} = \operatorname{span}\{v^\alpha : \alpha \in \Lambda_1 \cup \cdots \cup \Lambda_p\}$. Then observe that

$$E_p \subseteq \operatorname{Ker} A^p; \tag{4.3}$$

$$\dim E_p = \lambda_1 + \cdots \lambda_p. \tag{4.4}$$

Also note that the space E_p is independent of the particular choice of $\mathbf{S}$-basis, since we can define B as the matrix obtained from A by replacing $A_{\alpha\alpha}$ by the identity matrix of the same order, for $\alpha \in \Lambda_{p+1} \cup \cdots \cup \Lambda_h$. Then $\{v^\alpha : \alpha \in \Lambda_1 \cup \cdots \cup \Lambda_p\}$ is a $(\Lambda_1 \cup \cdots \cup \Lambda_p)$-basis for $E(B)$, whence $E_p = E(B)$.

(4.5) COROLLARY. *Let A be an M-matrix with singular graph* $\mathbf{S}$ *and with Weyr characteristic* $(\omega_1, \ldots, \omega_h)$. *Then:*

(a) $\omega_1 + \cdots + \omega_h = s = \lambda_1 + \cdots + \lambda_h$, *i.e.* $E_h = \operatorname{Ker} A^h$;
(b) $\omega_1 + \cdots + \omega_p \leq s - h + p$, *for* $p = 1, \ldots, h$;
(c) $\omega_1 + \cdots + \omega_p \geq \lambda_1 + \cdots + \lambda_p$, *for* $p = 1, \ldots, h$.

Proof.

(a) Clear, since both sums equal dim $E(A)$.
(b) Follows from $\omega_q \geq \omega_h \geq 1$, for $q = p-1, \ldots, h$.
(c) Immediate from (4.3).

It is interesting to investigate when equality occurs in Lemma (4.5c).

(4.6) LEMMA. *Let A be an M-matrix with singular graph* $\mathbf{S}$ *and with Weyr characteristic* $(\omega_1, \ldots, \omega_h)$, *and let D be an* $\mathbf{S}$*-matrix for A. Let* $1 \leq p < h$. *If* $\lambda_1 + \cdots + \lambda_p = \omega_1 + \cdots + \omega_p$, *then* $D_{p,p+1}$ *has full column rank.*

Proof. Suppose $D_{p,p+1}$ does not have full column rank. Then there exists $f^T = [0, \ldots, 0, f_{p+1}^T, 0, \ldots, 0]$ where $f_{p+1} \neq 0$ and $D_{p,p+1} f_{p+1} = 0$. Thus $(Df)_q = 0$, $q = p, \ldots, h$. Hence $VDf \in E_{p-1}$ and so by (4.3), $VDf \in \operatorname{Ker} A^{p-1}$, whereby $Vf \in \operatorname{Ker} A^p$. But $Vf \notin E_p$. Thus by (4.4), $\omega_1 + \cdots + \omega_p > \lambda_1 + \cdots + \lambda_p$.

(4.7) MAIN THEOREM. *Let A be an M-matrix with singular graph* $\mathbf{S}$, *and let the Weyr characteristic of A (associated with 0) be* $(\omega_1, \ldots, \omega_h)$. *Further let* $(\lambda_1, \ldots, \lambda_h)$ *be the level numbers of* $\mathbf{S}$, *and let D be an* $\mathbf{S}$*-matrix for A. Then the following are equivalent:*

(i) $\omega_q = \lambda_q$, $q = 1, \ldots, h$;
(ii) $D_{p,p+1}$ *has full column rank*, $p = 1, \ldots, h-1$.

Proof. (i) $\Rightarrow$ (ii) Immediate by Lemma (4.6).

(ii) $\Rightarrow$ (i) Assume (ii) does not hold. By Lemma (4.5c), there is a smallest integer p for which $\omega_1 + \cdots + \omega_p > \lambda_1 + \cdots + \lambda_p$. Hence, by (4.3) and (4.4), $E_q = \operatorname{Ker} A^q$, for $q = 1, \ldots, p-1$, and $E_p \neq \operatorname{Ker} A^p$. Let $v \in \operatorname{Ker} A^p \backslash E_p$. Let $\{v^\alpha : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}$-basis for $E(A)$ and let V be the matrix as in (3.3.3). Then by Corollary (4.5a), $v = Vf$, where for some r, $p < r \leq h$, $f_{r+1} = 0, \ldots, f_h = 0$, but $f_r \neq 0$. Thus $(Df)_{r-1} = D_{r-1,r} f_r$. But $-VDf = Av \in \operatorname{Ker} A^{p-1} = E_{p-1}$, whence $(Df)_{r-1} = 0$. Hence $D_{r-1,r} f_r = 0$. Since $f_r \neq 0$, $D_{r-1,r}$ does not have full column rank.

Remark. Result (1.1) in the Introduction corresponds to the case $h = 1$. Result

(1.2) corresponds to the case $\lambda_p = 1$, $p = 1, \ldots, h$, in which case $D_{p,p+1}$ is a positive scalar for $p = 1, \ldots, h-1$.

5. The class of M-matrices with singular graph S

(5.1) DEFINITIONS.

(a) Let $a \in \mathbb{R}^m$ and let $a \geq 0$. We define the (0, 1) vector $\pi(a) \in \mathbb{R}^m$ by

$$\pi(a)_j = 1, \quad \text{if} \quad a_j > 0,$$
$$\pi(a)_j = 0, \quad \text{if} \quad a_j = 0.$$

(b) Let $a^i \in \mathbb{R}^m$, $a^i \geq 0$, $i = 1, \ldots, l$. We call $a^1, \ldots, a^l$ *combinatorially dependent* if either $a^i = 0$, for some i, $1 \leq i \leq l$, or there exist non-empty disjoint subsets Φ_1, Φ_2 of $\{1, \ldots, l\}$ such that $\pi(\sum_{i \in \Phi_1} a^i) = \pi(\sum_{i \in \Phi_2} a^i)$. Otherwise, we call $a^1, \ldots, a^l$ *combinatorially independent.*

LEMMA. *Let $a^1, \ldots, a^l$ be non-negative vectors in $\mathbb{R}^m$. The following are equivalent:*

(i) *$a^1, \ldots, a^l$ are combinatorially dependent;*
(ii) *There exist linearly dependent non-negative $b^1, \ldots, b^l$ in $\mathbb{R}^m$ such that $\pi(b^i) = \pi(a^i)$, $i = 1, \ldots, l$.*

Proof. (ii) $\Rightarrow$ (i). Suppose that (ii) holds for $b^1, \ldots, b^l$.

If $b^i = 0$, for some i, then $a^i = 0$ and the result follows. So assume that $b^1, \ldots, b^l$ are semi-positive, and that

$$\sum_{i=1}^{l} \mu_i b^i = 0, \qquad \mu_i \in \mathbb{R} \quad \text{and some} \quad \mu_i \neq 0.$$

Let

$$\Phi_1 = \{i \in \{1, \ldots, l\} : \mu_i > 0\},$$
$$\Phi_2 = \{i \in \{1, \ldots, l\} : \mu_i < 0\}.$$

Since $b^i > 0$, $1 \leq i \leq l$, Φ_1 and Φ_2 are non-empty. Then

$$\sum_{i \in \Phi_1} \mu_i b^i = \sum_{i \in \Phi_2} (-\mu_i) b^i,$$

whence

$$\pi\Big(\sum_{i\in\Phi_1} b^i\Big)=\pi\Big(\sum_{i\in\Phi_1} \mu_i b^i\Big)=\pi\Big(\sum_{i\in\Phi_2} (-\mu_i b^i)\Big)=\pi\Big(\sum_{i\in\Phi_2} b^i\Big),$$

and so

$$\pi\Big(\sum_{i\in\Phi_1} a^i\Big)=\pi\Big(\sum_{i\in\Phi_2} a^i\Big).$$

(i) $\Rightarrow$ (ii). Suppose (i) holds. If $a^i=0$ for some i, then $a^1,\ldots,a^l$ are themselves linearly dependent. So suppose there exist non-empty disjoint subsets Φ_1, Φ_2 of $\{1,\ldots,l\}$ such that

$$\pi\Big(\sum_{i\in\Phi_1} a^i\Big)=\pi\Big(\sum_{i\in\Phi_2} a^i\Big).$$

Let $d=\sum_{i\in\Phi_1} a^i-\sum_{i\in\Phi_2} a^i$. We now define $b^1,\ldots,b^l$ in $\mathbb{R}^m$. Let $1\le j\le m$. There are three possible cases:

CASE I. If $d_j=0$, then we put $b^i_j=a^i_j$, $i=1,\ldots,l$.

CASE II. If $d_j>0$, then there exists $r\in\Phi_1$ depending on j such that $a^r_j>0$. Put $b^r_j=a^r_j+d_j$ and $b^i_j=a^i_j$ for $i\neq r$, $i=1,\ldots,l$.

CASE III. If $d_j<0$, then there exists $q\in\Phi_2$ depending on j such that $a^q_j>0$. Put $b^q_j=a^q_j-d_j$ and $b^i_j=a^i_j$ for $i\neq q$, $i=1,\ldots,l$.

It is easy to check that $b^i\ge 0$ and $\pi(b^i)=\pi(a^i)$, $i=1,\ldots,l$. Clearly $\sum_{i\in\Phi_1} b^i=\sum_{i\in\Phi_2} b^i$, so $b^1,\ldots,b^l$ are linearly dependent.

Remark. Let $A\in\mathbb{R}^{mn}$ and suppose $A\ge 0$. We define ccr A (the combinatorial column rank of A) to be the maximal number of combinatorially independent columns of A. The combinatorial row rank of A (crr A) is similarly defined. (See Čulik [1960] for a somewhat different concept which applies to arbitrary A.)

If $A\in\mathbb{R}^{nn}$, Lemma (5.2) implies that ccr $A=n$ if and only if crr $A=n$. This is false for a related definition of another type of column and row rank (see Plemmons [1971]). However, P. M. Gibson has provided us with the following interesting example of a rectangular matrix A for which ccr $A\neq$ crr A.

EXAMPLE. Let $A\in\mathbb{R}^{10,5}$ be a $(0,1)$-matrix defined thus:

(a) each row of A contains three 1's and two 0's;
(b) no two rows of A are the same.

Thus the rows of A are all combinations of [11100].

Let $\tilde{A}$ be obtained from A by deleting one row. The following properties are of interest:

(i) $\operatorname{ccr} A = 5$;
(ii) For any non-negative $b \in \mathbb{R}^{10}$, $\operatorname{ccr}[A, b] = 5$;
(iii) $\operatorname{ccr} \tilde{A} = 4$;
(iv) $\operatorname{ccr} A = 4$.

Observe that if $a^1, \ldots, a^5$ are the columns of A (in any order), then $\pi(a^1 + a^2)$ has precisely one 0, and $\pi(a^3 + a^4 + a^5)$ has no 0. The proofs of (i)–(iii) rest on this observation. To prove (iv), observe that if B is any 5×5 submatrix of A, then (iii) implies that $\operatorname{ccr} B < 5$, and hence $\operatorname{crr} B < 5$ (see previous Remark). Then (iv) follows by noting that the rows of

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

are combinatorially independent.

(5.3) LEMMA. *Let A be an M-matrix with singular graph $\mathbf{S}$ and let D be an $\mathbf{S}$-matrix for A. Let $1 \le p < h$. Then the following are equivalent:*

(i) *There do not exist non-empty disjoint subsets Φ_1, Φ_2 of Λ_{p+1} such that*

$$\Delta(\Phi_1) \cap \Lambda_p = \Delta(\Phi_2) \cap \Lambda_p;$$

(ii) *The columns of $D_{p,p+1}$ are combinatorially independent.*

Proof. Note that by Lemma (3.8a), for any $\Phi \subseteq \Lambda_{p+1}$ and $\beta \in \Lambda_p$, $\sum_{\alpha \in \Phi} d^{\beta\alpha} > 0$ if and only if $\beta \in \Delta(\Phi)$. We denote by d^α the α-th column of $D_{p,p+1}$. Observe that $d^\alpha > 0$ for $\alpha \in \Lambda_{p+1}$, by Corollary (3.11).

(i) $\Rightarrow$ (ii). Suppose that the columns of $D_{p,p+1}$ are combinatorially dependent. Then there exist non-empty disjoint subsets Φ_1, Φ_2 of Λ_{p+1} such that

$$\pi\Big(\sum_{\alpha \in \Phi_1} d^\alpha\Big) = \pi\Big(\sum_{\alpha \in \Phi_2} d^\alpha\Big).$$

Hence for $\beta \in \Lambda_p$,

$$\sum_{\alpha \in \Phi_1} d^{\beta\alpha} > 0 \quad \text{if and only if} \quad \sum_{\alpha \in \Phi_2} d^{\beta\alpha} > 0,$$

and so $\Delta(\Phi_1)\cap\Lambda_p=\Delta(\Phi_2)\cap\Lambda_p$.

(ii) $\Rightarrow$ (i) follows by reversing the above steps.

Let $\mathbf{S}$ be a non-empty finite partially ordered set and let $s=|\mathbf{S}|$. We determine conditions on $\mathbf{S}$ such that for every M-matrix A with singular graph $\mathbf{S}$, the Weyr characteristic of A is $(\lambda_1,\ldots,\lambda_h)$, where λ_p is the pth-level number of $\mathbf{S}$.

We remark that $\mathfrak{A}(\mathbf{S})$, defined as in §2, is always non-empty, for we may define an M-matrix $A\in\mathbb{R}^{ss}$ as follows:

For $\alpha,\ \beta\in\mathbf{S}$,

$$\left.\begin{aligned} a_{\beta\alpha}&<0, &&\text{if}\quad \beta\in\Delta(\alpha);\\ a_{\beta\alpha}&\le 0, &&\text{if}\quad \beta\in\mathbf{S}(\alpha);\\ a_{\beta\alpha}&=0, &&\text{if}\quad \beta\notin\mathbf{S}(\alpha).\end{aligned}\right\}\tag{5.4}$$

Then $\mathbf{S}$ is the singular graph of A, i.e., $A\in\mathfrak{A}(\mathbf{S})$.

Further, if e^α, $\alpha\in\mathbf{S}$, is the $s\times 1$ unit vector with $e^\alpha_\alpha=1$ and $e^\alpha_\beta=0$, $\beta\neq\alpha$, then $\{e^\alpha:\alpha\in\mathbf{S}\}$ is an $\mathbf{S}$-basis for $E(A)$ with $D=-A$ as the corresponding $\mathbf{S}$-matrix.

We also remark that if A is a matrix in $\mathfrak{A}(\mathbf{S})$ and D is an $\mathbf{S}$-matrix for A, then by Lemma (3.8), for $\alpha\in\Lambda_{p+1}$, $\beta\in\Lambda_p$, $1\le p<h$, $d^{\beta\alpha}>0$ if and only if $\beta\in\Delta(\alpha)$. Hence $\pi(D_{p,p+1})$ depends on $\mathbf{S}$ only, and not on A, where for $0\le C\in\mathbb{R}^{mn}$, $\pi(C)\in\mathbb{R}^{mn}$ is defined by $\pi(C)_{ij}=1$ if $C_{ij}>0$, $\pi(C)_{ij}=0$ if $C_{ij}=0$.

Let $A\in\mathfrak{A}(\mathbf{S})\cap\mathbb{R}^{nn}$. If $B\in\mathbb{R}^{mm}$ is any non-singular M-matrix, then observe that $B\oplus A$ is a matrix in $\mathfrak{A}(\mathbf{S})\cap\mathbb{R}^{m+n,m+n}$.

(5.5) LEMMA. Let $\mathbf{S}$ be a non-empty finite partially ordered set with level numbers $(\lambda_1,\ldots,\lambda_h)$. Let $1\le p<h$. Suppose that for all $A\in\mathfrak{A}(\mathbf{S})$, $\omega_1(A)+\cdots+\omega_p(A)=\lambda_1+\cdots+\lambda_p$. Then there do not exist non-empty disjoint subsets Φ_1, Φ_2 of Λ_{p+1} such that

$$\Delta(\Phi_1)\cap\Lambda_p=\Delta(\Phi_2)\cap\Lambda_p.$$

Proof. Suppose there exist non-empty disjoint subsets Φ_1, Φ_2 of Λ_{p+1} such that $\Delta(\Phi_1)\cap\Lambda_p=\Delta(\Phi_2)\cap\Lambda_p$. Let $\tilde{A}$ be an M-matrix with singular graph $\mathbf{S}$. If $\tilde{D}$ is an $\mathbf{S}$-matrix for $\tilde{A}$, then by Lemma (5.3), the $\lambda_p\times\lambda_{p+1}$ matrix $\tilde{D}_{p,p+1}=(\tilde{d}^{\beta\alpha})_{\beta\in\Lambda_p,\alpha\in\Lambda_{p+1}}$, has combinatorially dependent columns which we denote by $\tilde{d}^\alpha=(\tilde{d}^{\beta\alpha})_{\beta\in\Lambda_p}$. Hence by Lemma (5.2), there exist scalars $d^{\beta\alpha}\ge 0$, $\beta\in\Lambda_p$, $\alpha\in\Lambda_{p+1}$, such that the vectors $\{d^\alpha:\alpha\in\Lambda_{p+1}\}$ are linearly dependent and $\pi(d^\alpha)=\pi(\tilde{d}^\alpha)$, for $\alpha\in\Lambda_{p+1}$, where d^α is the vector $(d^{\beta\alpha})_{\beta\in\Lambda_p}$.

Define $A \in \mathbb{R}^{ss}$ to satisfy (5.4) and $a_{\beta\alpha} = -d^{\beta\alpha}$, for $\beta \in \Lambda_p$, $\alpha \in \Lambda_{p+1}$. Let $D = -A$. Then by the remarks preceding this Lemma, D is an $\mathbf{S}$-matrix for A and $D_{p,p+1} = (d^{\beta\alpha})_{\beta \in \Lambda_p, \alpha \in \Lambda_{p+1}}$. Thus $D_{p,p+1}$ does not have full column rank, and by Lemma (4.6), $\lambda_1 + \cdots + \lambda_p \neq \omega_1(A) + \cdots + \omega_p(A)$.

(5.6) THEOREM. *Let* $\mathbf{S}$ *be a non-empty finite partially ordered set with level numbers* $(\lambda_1, \ldots, \lambda_h)$, *and let* $\mathfrak{A}(\mathbf{S})$ *be the set of all* M*-matrices whose singular graph is* $\mathbf{S}$. *Then the following are equivalent:*

(i) *For all* $A \in \mathfrak{A}(\mathbf{S})$, $\omega_q(A) = \lambda_q$, $q = 1, \ldots, h$.

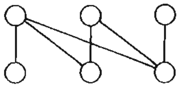
(ii) *For all* p, $p = 1, \ldots, h-1$, *there do not exist non-empty disjoint subsets* Φ_1, Φ_2 *of* Λ_{p+1} *such that*

$$\Delta(\Phi_1) \cap \Lambda_p = \Delta(\Phi_2) \cap \Lambda_p.$$

Proof. (i) $\Rightarrow$ (ii). Immediate by Lemma (5.5).

(ii) $\Rightarrow$ (i). Let $A \in \mathfrak{A}(\mathbf{S})$, and let $1 \leq p < h$. If D is an $\mathbf{S}$-matrix for A, then by Lemma (5.3) the columns of $D_{p,p+1}$ are combinatorially independent, and hence by Lemma (5.2), also linearly independent. Thus (i) follows by Theorem (4.7).

EXAMPLE. Let $\mathbf{S}$ be



Then $\mathbf{S}$ satisfies condition (5.6ii), and hence any matrix $A \in \mathfrak{A}(\mathbf{S})$ has Weyr characteristic (3, 3).

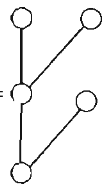
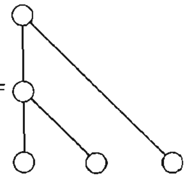
(5.7) DEFINITION. Let $\mathbf{S}$ be a finite partially ordered set. Then $\mathbf{S}$ is called a rooted forest if, for each $\alpha \in \mathbf{S}$, the set $\{\beta \in \mathbf{S} : \beta \preceq \alpha\}$ is linearly ordered.

Cooper [1973, Theorem 3], who used the term inverted tree for our rooted forest, proved that if $\mathbf{S}$ is a rooted forest and $A \in \mathfrak{A}(\mathbf{S})$ then $\omega_1(A) = \lambda_1$. More generally we have:

(5.8) COROLLARY. *Let* $\mathbf{S}$ *be a rooted forest. If* $A \in \mathfrak{A}(\mathbf{S})$, *then* $\omega_p(A) = \lambda_p$ *for* $p = 1, \ldots, h$.

Proof. It is enough to show that $\mathbf{S}$ satisfies condition (5.6ii). So let $1 \leq p < h$ and let Φ_1, Φ_2 be non-empty subsets of Λ_{p+1}. Let $\alpha \in \Delta(\Phi_1) \cap \Delta(\Phi_2)$. Since $\{\beta \in \mathbf{S} : \beta \preceq \alpha\}$ is linearly ordered by hypothesis on $\mathbf{S}$, there exist $\beta_1 \in \Phi_1$, $\beta_2 \in \Phi_2$ with either $\beta_1 \succeq \beta_2$ or $\beta_2 \succeq \beta_1$. But since β_1, $\beta_2 \in \Lambda_{p+1}$, it follows that $\beta_1 = \beta_2$. Hence Φ_1 and Φ_2 are not disjoint.

Remark. Let $\mathbf{S}$ be a non-empty finite partially ordered set, and define $\mathbf{S}^*$ to be the set $\mathbf{S}$ with the reverse partial order, viz., $\alpha \succeq^* \beta$ if $\alpha \preceq \beta$. It is easy to see that $A \in \mathfrak{A}(\mathbf{S})$ if and only if $A^T \in \mathfrak{A}(\mathbf{S}^*)$. Hence, if $\mathbf{S}^*$ satisfies condition (5.6ii), then $(\omega_1(A), \ldots, \omega_h(A)) = (\omega_1(A^T), \ldots, \omega_h(A^T)) = (\lambda_1^*, \ldots, \lambda_h^*)$ where λ_p^* is the p-th level number of $\mathbf{S}^*$, $p = 1, \ldots, h$. For example, if

$\mathbf{S} =$  then $\mathbf{S}^* =$ 

and so the Weyr characteristic of any $A \in \mathfrak{A}(\mathbf{S})$ is (3, 1, 1).

6. Semi-positive Jordan bases

In this section we complete our main theorem by considering positivity properties of eigenvectors and generalized eigenvectors. In the case of eigenvectors, this subject has a long history. By combining some results of Frobenius [1912, p. 563] with Schneider [1956, Lemma 1] one sees that the number of linearly independent semi-positive eigenvectors of the M-matrix A (belonging to 0) is λ_1. This result is clearly implied by Carlson [1963, Lemma 2]. Further, Carlson [1963, Theorem 2] gives a complete description of all non-negative solutions x of $Ax = c$, where $c \geq 0$, in terms of the graphs $\mathbf{G}$ and $\mathbf{S}$. From Carlson's theorem many special results are easily derived. For example, the following are equivalent:

(i) Ker A contains a strictly positive vector;
(ii) Ker A can be spanned by strictly positive vectors;
(iii) $\mathbf{S}$ consists precisely of the minimal elements of $\mathbf{G}$.

(Compare with Gantmacher [1959, Vol. II, p. 77, Theorem 6] and see Cooper [1973, Theorem 2(ii)], where, however, the result is not stated correctly.) Rothblum [1975, Theorem 3, Part 1] has proved that $E(A)$ has a basis consisting of semi-positive vectors. A precise version of Rothblum's result is stated as Theorem (6.2) below after a necessary definition.

(6.1) DEFINITION. Let $\{v^\alpha \in \mathbb{R}^n : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}$-set for A with associated scalars $d^{\beta\alpha}$, $\alpha, \beta \in \mathbf{S}$. We call $\{v^\alpha \in \mathbb{R}^n : \alpha \in \mathbf{S}\}$ an $\mathbf{S}^+$-set for A if:

(a) $v_i^\alpha \gg 0$, if $\quad i \in \mathbf{G}^*(\alpha)$, and

(b) $d^{\beta\alpha} > 0$, if $\beta \in \mathbf{S}(\alpha)$.

It is clear that every $\mathbf{S}^+$-set is an $\mathbf{S}$-basis for $E(A)$, and so we shall refer to an $\mathbf{S}^+$-set as an $\mathbf{S}^+$-basis for $E(A)$.

Observe that if $\{v^\alpha : \alpha \in \mathbf{S}\}$ is an $\mathbf{S}^+$-basis for $E(A)$, then it follows by Lemma (4.1) that

$$(-A)^q V = VD^q > 0 \quad \text{for} \quad 1 \le q < h.$$

(6.2) THEOREM. *Let A be an M-matrix with singular graph $\mathbf{S}$. Then there exists an $\mathbf{S}^+$-basis for the generalized eigenspace $E(A)$.*

Proof. The proof is by induction on g, the number of elements in $\mathbf{G}$. As in the beginning of the proof of Theorem (3.10), the result is clear if $g = 1$ and the inductive step is obvious when $g > 1$ and $1 \notin \mathbf{S}$.

So assume that $g > 1$, $1 \in \mathbf{S}$. As in the proof of Theorem (3.10), we obtain an $\mathbf{S}$-basis $\{w^\alpha : \alpha \in \mathbf{S}\}$ for $E(A)$, with associated scalars $\{c^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$ where by inductive assumption, for $\alpha \in \mathbf{S}\backslash\{1\}$, $w_i^\alpha \gg 0$ if $i \in \mathbf{G}^*(\alpha)$ and $c^{\beta\alpha} > 0$ if $\beta \in \mathbf{S}(\alpha)$.

Let $\gamma \in \mathbf{S}(1)$. We shall now define positive scalars g^γ, h^γ. If $i \in \mathbf{G}^*(\gamma)$, then, by inductive assumption, $w_i^\gamma \gg 0$. Hence we may choose one g^γ to be sufficiently large so that for all

$$i \in \mathbf{G}^*(\gamma), \qquad w_i^1 + g^\gamma w_i^\gamma \gg 0. \tag{6.2.1}$$

If $\beta \in \mathbf{S}(\gamma)$, then $c^{\beta\gamma} > 0$. Hence we may choose h^γ so that for all

$$\beta \in \mathbf{S}(\gamma), \qquad c^{\beta 1} + h^\gamma c^{\beta\gamma} > 0. \tag{6.2.2}$$

Then we put $f^\gamma = \max\{g^\gamma, h^\gamma\}$. We now put

$$v^\gamma = w^\gamma \quad \text{for} \quad \gamma \in \mathbf{S}\backslash\{1\},$$

$$v^1 = w^1 + \sum_{\gamma \in \mathbf{S}(1)} w^\gamma f^\gamma,$$

and for $\alpha, \beta \in \mathbf{S}$,

$$d^{\beta\gamma} = c^{\beta\gamma} \quad \text{for} \quad \gamma \in \mathbf{S}\backslash\{1\},$$

$$d^{\beta 1} = c^{\beta 1} + \sum_{\gamma \in \mathbf{S}(1)} c^{\beta\gamma} f^\gamma.$$

We claim that $\{v^\alpha : \alpha \in \mathbf{S}\}$ is an $\mathbf{S}^+$-basis for $E(A)$ with associated scalars $\{d^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$. For $\alpha \in \mathbf{S}\backslash\{1\}$, the vectors v^α and the scalars $d^{\beta\alpha}$ clearly satisfy the required conditions. It remains to show that v^1 and $d^{\beta 1}$, $\beta \in \mathbf{S}$, satisfy the conditions of an $\mathbf{S}^+$-basis.

(i) $v_i^1 = 0$ for $i \notin \mathbf{G}^*(1)$:

$$v_i^1 = w_i^1 + \sum_{\gamma \in \mathbf{S}(1)} w_i^\gamma f^\gamma = 0 \quad \text{because} \quad \{w^\beta : \beta \in \mathbf{S}\} \text{ is an } \mathbf{S}^+\text{-basis.}$$

(ii) $v_i^1 \gg 0$ for $i \in \mathbf{G}^*(1)$ and $\sigma(v_1^1) = 1$:

There are two cases. If $i \in \mathbf{G}_1(1)$, then $v_i^1 = w_i^1$ and $w_i^1 \gg 0$ by Theorem (3.10.II). Also $\sigma(v_1^1) = \sigma(w_1^1) = 1$. If $i \notin \mathbf{G}_1(1)$, then there is a $\delta \in \mathbf{S}(1)$ such that $i \in \mathbf{G}^*(\delta)$. Since $w_i^\gamma \geq 0$ for $\gamma \in \mathbf{S}(1)$, we have

$$v_i^1 = w_i^1 + \sum_{\gamma \in \mathbf{S}(1)} w_i^\gamma f^\gamma \geq w_i^1 + \sum_{\gamma \in \mathbf{S}(1)} w_i^\gamma g^\gamma \geq w_i^1 + w_i^\delta g^\delta \gg 0, \quad \text{by} \quad (6.2.1)$$

(iii) $d^{\beta 1} = 0$ for $\beta \notin \mathbf{S}(1)$:

$$d^{\beta 1} = c^{\beta 1} + \sum_{\gamma \in \mathbf{S}(1)} c^{\beta\gamma} f^\gamma = 0 \quad \text{because} \quad \{c^{\beta\alpha} : \alpha, \beta \in \mathbf{S}\}$$

are associated scalars of an $\mathbf{S}^+$-basis.

(iv) $d^{\beta 1} > 0$ for $\beta \in \mathbf{S}(1)$:

There are two cases. If $\beta \in \Delta(1)$, then $d^{\beta 1} = c^{\beta 1}$ and $c^{\beta 1} > 0$ by Theorem (3.10.II). If $\beta \notin \Delta(1)$, then there is a $\delta \in \mathbf{S}(1)$ such that $\beta \in \mathbf{S}(\delta)$. Since, by inductive assumption, $c^{\beta\gamma} \geq 0$ for $\gamma \in \mathbf{S}(1)$, we have

$$d^{\beta 1} = c^{\beta 1} + \sum_{\gamma \in \mathbf{S}(1)} c^{\beta\gamma} f^\gamma \geq c^{\beta 1} + \sum_{\gamma \in \mathbf{S}(1)} c^{\beta\gamma} h^\gamma \geq c^{\beta 1} + c^{\beta\delta} h^\delta > 0, \text{ by} \quad (6.2.2).$$

(v) $Av^1 = -\sum_{\beta \in \mathbf{S}} d^{\beta 1} v^\beta$:

This follows by a direct calculation.

(6.3) LEMMA. *Let A be an M-matrix with singular graph* $\mathbf{S}$ *and level numbers* $(\lambda_1, \ldots, \lambda_h)$. *The following are equivalent for p, $1 \leq p \leq h$:*

(i) $\omega_1 + \cdots + \omega_p = \lambda_1 + \cdots + \lambda_p$;

(ii) Ker A^p *has a basis of semi-positive vectors.*

Proof. (i) $\Rightarrow$ (ii). By (4.3) and (4.4), (i) implies that $E_p = \operatorname{Ker} A^p$. By Theorem (6.2), there exists an $\mathbf{S}^+$-basis $\{v^\alpha : \alpha \in \mathbf{S}\}$ for $E(A)$. Then $\{v^\alpha : \alpha \in \Lambda_1 \cup \cdots \cup \Lambda_p\}$ is a basis for E_p, consisting of semi-positive vectors.

(ii) $\Rightarrow$ (i). Let V be the matrix of an $\mathbf{S}$-basis for $E(A)$ with associated $\mathbf{S}$-matrix D. Suppose $\omega_1 + \cdots + \omega_p > \lambda_1 + \cdots + \lambda_p$. Then by (4.3) and (4.4), $E_p \subset \operatorname{Ker} A^p$. Let $w \in \operatorname{Ker} A^p \backslash E_p$. We shall show that w is not semi-positive. Using the notation of the Remark preceding Corollary (4.5), $w = Vf \in \operatorname{Ker} A^p$, where $p < r \leq h$, $f_r \neq 0$, and $f_{r+1} = 0, \ldots, f_h = 0$. Then $0 = A^p V f = (-1)^p V D^p f$ and hence $0 = D^p f$. Thus $0 = (D^p f)_{r-p} = (D^p)_{r-p,r} f_r$ by Lemma (4.1a), and so by Lemma (4.1b), f_r can not be semi-positive. Let $\alpha \in \Lambda_r$. Then $w_\alpha = \eta_\alpha v_\alpha^\alpha$, where $\eta_\alpha \in \mathbb{R}$ and we have written $f_r = (\eta_\alpha)_{\alpha \in \Lambda_r}$. Hence, for some $\beta \in \Lambda_r$, $w_\beta \ll 0$. Since $\operatorname{Ker} A^p \backslash E_p$ must contain a basis vector, the result follows.

(6.4) DEFINITION. Let $A \in \mathbb{R}^{nn}$ be a singular matrix and let $\{v^{pi} : p = 1, \ldots, h;\ i = 1, \ldots, \omega_p\}$, where $h > 0$ and $\omega_1 \geq \cdots \geq \omega_h \geq 1$, be a basis for the generalized eigenspace $E(A)$ for the eigenvalue 0 of A. This basis for $E(A)$ is called a *Jordan basis* for A if

$$Av^{pi} = v^{p-1,i}, \qquad p = 2, \ldots, h;\ i = 1, \ldots, \omega_p,$$

and

$$Av^{1i} = 0, \qquad i = 1, \ldots, \omega_1.$$

We remark that $(\omega_1, \ldots, \omega_h)$ in the definition is the Weyr characteristic of 0 for A.

Our final result completes the Main Theorem (4.7).

(6.5) THEOREM. *Let A be an M-matrix with singular graph $\mathbf{S}$ and level numbers $(\lambda_1, \ldots, \lambda_h)$. The following are equivalent:*

(i) *The Weyr characteristic for A (associated with 0) is $(\lambda_1, \ldots, \lambda_h)$;*
(ii) *There exists a Jordan basis for $-A$ consisting of semi-positive vectors;*
(iii) *For $p = 1, \ldots, h$, $\operatorname{Ker} A^p$ has a basis consisting of semi-positive vectors.*

Proof. We observe that (ii) $\Rightarrow$ (iii) is trivial and that (iii) $\Rightarrow$ (i) is given by Lemma (6.3). So we need prove only (i) $\Rightarrow$ (ii).

We assume that (i) holds. Let $\{v^\alpha : \alpha \in \mathbf{S}\}$ be an $\mathbf{S}^+$-basis for $E(A)$, which exists by Theorem (6.2). We shall construct a Jordan basis for $-A$, $\{w^{pi} : p = 1, \ldots, h;\ i = 1, \ldots, \lambda_p\}$ by successively defining sets of vectors $W^h, \ldots, W^1$ satisfying the following conditions for $1 \leq p \leq h$:

(a) $W^p = \{w^{pi} : i = \lambda_{p+1}+1, \ldots, \lambda_p\}$ is a subset of $V^p = \{v^\alpha : \alpha \in \Lambda_p\}$ (where, by convention, $\lambda_{h+1} = 0$).

(b) The set $J^p \cup U^{p-1}$ is a basis for E_p, where we have put

$$U^{p-1} = \{v^\alpha : \alpha \in \Lambda_1 \cup \cdots \cup \Lambda_{p-1}\}$$

and

$$J^p = W^p \cup (-A) W^{p+1} \cup \cdots \cup (-A)^{h-p} W^h = \{(-A)^{q-p} w^{qi} : q = p, \ldots, h; \; i = \lambda_{q+1}+1, \ldots, \lambda_q\}$$

(where by convention $U^0 = \phi$).

We remark that U^{p-1} is a basis for E_{p-1}, and, since (i) holds, $E_{p-1} = \operatorname{Ker} A^{p-1}$. Since $W^q \subseteq V^q \subseteq E_q \backslash E_{q-1}$, $q = p, \ldots, h$, it follows by Lemma (4.1) that $J^p \subseteq E_p \backslash E_{p-1}$.

For $q = p+1, \ldots, h$, and $\lambda_{q+1} < i \leq \lambda_q$, we shall put $w^{pi} = (-A)^{q-p} w^{qi}$. Thus $J^p = \{w^{pi} : i = 1, \ldots, \lambda_p\}$

We first define $W^h = V^h$ and $J^h = W^h$. Clearly W^h satisfies (a) and (b). Now suppose $W^h, \ldots, W^p$ have already been defined satisfying (a) and (b). By (b), we have

$$[J^p \quad U^{p-1}] = [V^p \quad U^{p-1}] \begin{bmatrix} Q & 0 \\ R & I \end{bmatrix}$$

where we have written $[J^p \quad U^{p-1}]$ for the $n \times (\lambda_1 + \cdots + \lambda_p)$ matrix whose first λ_p columns are the vectors of J^p, and whose last $\lambda_1 + \cdots + \lambda_{p-1}$ columns are the vectors of U^{p-1}, and similarly for $[V^p \quad U^{p-1}]$. Further, Q is a $\lambda_p \times \lambda_p$ matrix and I is the $\lambda_1 + \cdots + \lambda_{p-1}$ square identity matrix. Since $J^p \cup U^{p-1}$ is a basis for E_p, it follows that Q is non-singular. Hence, using $-AV = VD$, we obtain

$$(-A)[J^p] = [V^p \quad U^{p-1}] \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ D_{p-1,p} & 0 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ D_{1p} & \cdots & & D_{12} & 0 \end{bmatrix} \begin{bmatrix} Q \\ R \end{bmatrix},$$

where the D_{ij} are submatrices of an $\mathbf{S}$-matrix D associated with the $\mathbf{S}^+$-basis

$\{v^\alpha : \alpha \in \mathbf{S}\}$. Thus,

$$(-A)[J^p] = [V^{p-1} \quad U^{p-2}]\begin{bmatrix} D_{p-1,p}Q \\ Z \end{bmatrix},$$

where Z is a $(\lambda_1 + \cdots + \lambda_{p-1}) \times \lambda_p$ matrix.

But Q is non-singular, and, by Theorem (4.7) and our assumption (i), $D_{p-1,p}$ has full column rank, whence $D_{p-1,p}Q$ has full column rank. Hence there exists a $\lambda_{p-1} \times (\lambda_{p-1} - \lambda_p)$ submatrix E of the $\lambda_{p-1} \times \lambda_{p-1}$ identity matrix such that $[D_{p-1,p}Q \quad E]$ is non-singular. We now put $[W^{p-1}] = [V^{p-1}]E$. Clearly W^{p-1} is a subset of V^{p-1} and so satisfies (a). Also, by definition, $J^{p-1} = -AJ^p \cup W^{p-1}$, and so $[J^{p-1}] = [-AJ^p \quad W^{p-1}]$. Hence

$$[J^{p-1} \quad U^{p-2}] = [V^{p-1} \quad U^{p-2}]\left[\begin{array}{cc|c} D_{p-1,p} & E & 0 \\ \hline Z & 0 & I \end{array}\right].$$

Since the matrix on the right is non-singular, $J^{p-1} \cup U^{p-2}$ is a basis for E_{p-1}. Thus (b) is satisfied and the construction is completed.

Clearly $\{w^{pi} : p = 1, \ldots, h; i = 1, \ldots, \lambda_p\} = J^1 \cup \cdots \cup J^h$ is a Jordan basis for $-A$, and $w^{pi} > 0$ since w^{pi} is a column of $(-A)^{q-p}V$, where $\lambda_{q+1} < i \leq \lambda_q$, and $q \geq p$.

In the case that **S** is linearly ordered, Theorem (6.5) reduces to Theorem 6 of Schneider [1956].

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