Algebraic Perron-Frobenius Theory

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ABSTRACT

Let V be a vector space over a fully ordered field $\mathbf{F}$. In Sec. 2 we characterize cones K with ascending chain condition (ACC) on faces of K. In Sec. 3 we show that if K has ACC on faces, then an operator A is strongly irreducible if and only if A is irreducible. In Sec. 4 we prove theorems of Perron-Frobenius type for a strongly irreducible operator A in the case that $\mathbf{F}=\mathbf{R}$, the real field, and K is a full algebraically closed cone.

1. INTRODUCTION

In the last quarter of a century the celebrated classical theorems of Perron [25, 26] and Frobenius [10, 11, 12] on nonnegative matrices have been studied and extended in various ways. The original theorems for matrices were generalized by Krein and Rutman [19] to operators A on a Banach space partially ordered by a cone K. In the literature of functional analysis (cf. Bonsall [5], Karlin [17], the books by Jameson [16] and Schaefer [20] and their bibliographies) it is common to impose topological conditions on K or analytic conditions on A. We, on the other hand, wish to emphasize the algebraic features of the theory. We do not impose analytic conditions on the operator or topological conditions on the space, but we use only the topology of the real line.

In 1950 Wielandt [34] gave a new proof of the classical matrix theorems. His methods extend to a more general setting, and we shall employ them extensively.

*The research of this author was supported in part by NSF grant GP-37978X1.

In Sec. 2 we define the concept of a face of a cone in a vector space over a fully ordered field. We obtain theorems which are analogous to those for ideals in rings. Since each finitely generated face is in fact generated by a single element, the structure of faces in a cone tends to be simpler than the structure of ideals in a ring. We also employ the concepts of relative algebraic interior and algebraic closure, and relate them to the face structure of a cone.

In Sec. 3 we consider positive operators, i.e., operators which map the cone into itself. We introduce the concepts of irreducibility and strong irreducibility, and show that they are equivalent if the cone has ascending chain condition on faces.

In Sec. 4 we restrict to the real field. We investigate the existence of a positive eigenvalue for a strongly irreducible operator. While we do not obtain an existence theorem by means of conditions on the operator, we prove a necessary and sufficient condition for such an eigenvalue to exist. If the eigenvalue exists, we show in the main theorem of this section that there is associated with it a unique eigenvector which lies in the interior of the cone. Other desired properties of the eigenvalues and eigenvectors also hold.

Further remarks on results related to ours are found in notes at the end of each section.

2. CONES AND FACES

Let $\mathbf{F}$ be a fully ordered field (see [13, p. 105]), and let V be a vector space over $\mathbf{F}$.

DEFINITION 2.1.

(i) A nonempty subset K of V is a *cone* if

$$\begin{aligned} &\text{(a)}\ K+K\subseteq K,\\ &\text{(b)}\ \alpha K\subseteq K \qquad \text{if } \alpha\geqslant 0,\quad \alpha\in F,\\ &\text{(c)}\ K\cap(-K)=\{0\}. \end{aligned}$$

(ii) A cone K is *reproducing* if $\operatorname{span} K = V$.

Let K be a cone and let x, $y\in V$. We write $x\geqslant y$ if $x-y\in K$. This relation is a partial order on V. Further, $x\geqslant y$ implies $x+z\geqslant y+z$ for $z\in V$ and $\alpha x\geqslant \alpha y$ for $\alpha\in\mathbf{F}$, $\alpha\geqslant 0$. We write $x>y$ for $x\geqslant y$ and $x\neq y$.

DEFINITION 2.2. A nonempty subset F of a cone K is called a *face* of K if

(a) F is a cone,
(b) $x\in F$, $y\in K$, and $x-y\in K$ imply that $y\in F$.

It is easily seen that condition (b) is equivalent to

(b′) $0\leqslant y\leqslant x$ and $x\in F$ imply that $y\in F$.

REMARK 2.3. Observe that $\{0\}$ and K are faces of K, called the *trivial* faces of K. If F is a face of K we write $F\trianglelefteq K$, and if the face F is different from K we write $F\triangleleft K$. The set of all faces of K is denoted $\mathcal{F}\ (K)$.

It is easily checked that the intersection of any collection of faces is again a face. Thus our next definitions are unambiguous.

DEFINITION 2.4.

(i) Let $S\subseteq K$. We define $\varphi(S)=\cap\{F:F\in\mathcal{F}(K),\ K\supseteq S\}$, i.e., $\varphi(S)$ is the smallest face of K containing S. We call $\varphi(S)$ the face *generated* by S.

(ii) Let $F\trianglelefteq K$. If there is a finite set $S=\{x_1,\dots,x_n\}\subseteq K$ such that $F=\varphi(S)$, then F is called *finitely generated*. We also write $F=\varphi(x_1,\dots,x_n)$.

(iii) Let $F\trianglelefteq K$. If there is an $x\in K$ such that $F=\varphi(x)$, then F is called a *cyclic* face of K.

REMARK 2.5. It is easy to see that φ is a closure operator,[1] i.e., $S\subseteq\varphi(S)$, $\varphi(\varphi(S))=\varphi(S)$ and $S\subseteq T$ implies $\varphi(S)\subseteq\varphi(T)$. Further, $S=\varphi(S)$ if and only if S is a face.

REMARK 2.6. Let $F\trianglelefteq K$. Then F is a cone in V. If $G\trianglelefteq F$ [i.e., $G\in\mathcal{F}(F)$], then it is easy to show that $G\trianglelefteq K$.

EXAMPLE 2.7.

(i) Let $V=\mathbf{F}^n$, $K=\{x:x_i\geqslant 0\}$. Then K is a cone, called the positive orthant. If $I\subseteq\{1,\dots,n\}$ let $F_I=\{x\in K:x_i=0,\ i\not\in I\}$. Then $F_I\trianglelefteq K$. It is easy to see that all faces of K are of this form.

(ii) Let V be a vector space over $\mathbf{F}$ with a norm $\|\cdot\|$ (which takes values in $\mathbf{F}$), and let f be a nonzero linear functional from V into $\mathbf{F}$. Let

$$K=\{x\in V:\|x\|\leqslant f(x)\}.$$

Then K is a cone. We call the norm $\|\cdot\|$ *strictly convex* if $\|x\|=\|y\|=\frac{1}{2}\|x+y\|$ implies that $x=y$. For $x\in V$ let $\operatorname{ray}(x)=\{\alpha x:\alpha\geqslant 0\}$. If $\|\cdot\|$ is strictly

[1]Called hull operation by Bauer [3, Chapter I].

convex and $0 \neq x \in K$ satisfies $\|x\| = f(x)$, then ray(x) is a face of K. For suppose $0 \leq y \leq v$, where $v = \alpha x, \alpha > 0$. If $z = v - y$, then

$$f(v) = \|v\| \leq \|y\| + \|z\| \leq f(y) + f(z) = f(v),$$

whence $\|y + z\| = \|y\| + \|z\|$. Assume that $\|z\| \geq \|y\| > 0$. Then

$$2 \geq \left\| \frac{y}{\|y\|} + \frac{z}{\|z\|} \right\| \geq \left\| \frac{y}{\|y\|} + \frac{z}{\|y\|} \right\| - \left\| \frac{z}{\|y\|} - \frac{z}{\|z\|} \right\|$$

$$= \frac{\|y\| + \|z\|}{\|y\|} - \|z\| \left(\frac{1}{\|y\|} - \frac{1}{\|z\|} \right) = 2.$$

So by the strict convexity of the norm, $y/\|y\| = z/\|z\|$. Hence $y \in \text{ray}(v) = \text{ray}(x)$.

(ii′) As a special case of (ii), let $V = \mathbf{R}^n$, where $\mathbf{R}$ is the real field. Let

$$K = \left\{ x : \left(x_1^2 + \cdots + x_{n-1}^2 \right)^{1/2} \leq x_n \right\}.$$

Here $\|\cdot\|$ is the Euclidean norm $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $f(x) = \sqrt{2}\, x_n$. We call this cone the n-dimensional ice-cream cone.

LEMMA 2.8. *Let $S \subseteq K$. Then $y \in \varphi(S)$ if and only if the following condition holds:*

There are a natural number n and $x^i \in S$, $\beta_i \in \mathbf{F}$, $\beta_i \geq 0$, $i = 1, \dots, n$

$$\text{such that } 0 \leq y \leq \sum_{i=1}^{n} \beta_i x^i. \tag{2.9}$$

Proof. Let F be the set of all y for which Eq. (2.9) holds. We shall show that $F = \varphi(S)$. It is easy to verify that $F \trianglelefteq K$, and clearly $S \subseteq F$. Hence $\varphi(S) \subseteq F$. On the other hand, if $y \in F$, say $0 \leq y \leq \sum_{i=1}^{n} \beta_i x^i = z$, then $z \in \varphi(S) \leq K$, whence $y \in \varphi(S)$. Thus $F = \varphi(S)$, and the result is proved. ∎

COROLLARY 2.10. *Let $x \in K$. Then*

$$\begin{aligned} \varphi(x) &= \{ y \in V : \exists \beta \geq 0,\ 0 \leq y \leq \beta x \} \\ &= \{ y \in V : \exists \beta > 0,\ 0 \leq y \leq \beta x \} \\ &= \{ y \in V : \exists \alpha > 0,\ 0 \leq \alpha y \leq x \}. \end{aligned}$$

∎

LEMMA 2.11. *Every finitely generated face of K is cyclic.*

Proof. Let $F=\varphi(x_1,\dots,x_n)$, and let $z=x_1+\cdots+x_n$. Clearly $z\in F$, whence $\varphi(z)\subseteq F$. But for $i=1,\dots,n$, $0\leqslant x_i\leqslant z$, whence $x_i\in\varphi(z)$, and so by Corollary 2.10, $F\subseteq\varphi(z)$. Thus $F=\varphi(z)$. ∎

DEFINITION 2.12. Let K be a cone in V.

(a) The cone K has *ascending chain condition*, or ACC, on faces if and only if there is no infinite chain of faces

$$F_0\lhd F_1\lhd F_2\lhd\cdots.$$

(b) Let p be a natural number. We say that K has *chain length* p if there is a chain of faces in K of length p (i.e., $F_0\lhd F_1\lhd\cdots\lhd F_p$), and no chain of faces of length $p+1$.

LEMMA 2.13. *Let K be a cone in V. Then the following are equivalent.*

(1) *K has* ACC *on faces.*
(2) *Every face of K is cyclic.*

Proof. The standard proof (e.g., see [35] for ideals in a ring) shows that (1) holds if and only if every face of K is finitely generated. By Lemma 2.11 then, (1) is equivalent to (2). ∎

If $S\subseteq V$, we write $\operatorname{span}S$ for the linear space spanned by S in V. If F is a cone, observe that $\operatorname{span}F=F-F$. In addition we write $\dim F$ for $\dim\operatorname{span}F$.

LEMMA 2.14. *Let $F\trianglelefteq K$. Then $F=K\cap\operatorname{span}F$.*

Proof. Clearly $F\subseteq K\cap\operatorname{span}F$. So let $x\in K\cap\operatorname{span}F$. Then $0\leqslant x=y-z$, where $y,z\in F$. Hence $0\leqslant x\leqslant y$, whence $x\in F$. ∎

COROLLARY 2.15. *Let K be a cone in V, and let* $\dim V=n$. *Then K has chain length less than or equal to n.* ∎

DEFINITION 2.16. Let K be a cone in V.

(1) The *relative algebraic interior* of K is denoted by $\operatorname{rai}K$ and is defined by

$$\operatorname{rai}K=\{x\in K:\forall y\in\operatorname{span}K\ \exists\varepsilon_y>0,\ x+\varepsilon_y y\in K\}.$$

(2) The *algebraic closure* of K is denoted by $\operatorname{acl} K$ and is defined by

$$\operatorname{acl} K = \{\, y \in V : \exists x_y \in V,\ y + \alpha x_y \in K\ \forall \alpha \in (0,1] \,\}.$$

(3) If $K = \operatorname{acl} K$, we call K *algebraically closed*. Note that $\operatorname{acl} K$ need not be algebraically closed [18, p. 177].

(4) The *relative algebraic boundary* of K is denoted by $\operatorname{rab} K$ and is defined by

$$\operatorname{rab} K = \operatorname{acl} K \setminus \operatorname{rai} K.$$

(We use \ for the set theoretic complement.)

(5) If K is reproducing and $\operatorname{rai} K \neq \emptyset$, we call K *full*. In this case we write $\operatorname{int} K$ for $\operatorname{rai} K$.

For properties of $\operatorname{rai} K$ and $\operatorname{acl} K$ see [3] or [18]. If $\operatorname{int} K$ exists, then we write $\operatorname{bdy} K$ for $\operatorname{rab} K$.

LEMMA 2.17. *Let $F \trianglelefteq K$. If $x \in F \cap \operatorname{rai} K$, then $K = F = \varphi(x)$.*

Proof. Let $y \in K$. Then there exists an $\varepsilon > 0$ such that $x - \varepsilon y \in K$. Hence $0 \leqslant \varepsilon y \leqslant x$, so that $y \in \varphi(x)$ by Corollary 2.10. Thus $K \subseteq \varphi(x) \subseteq F \subseteq K$. ■

COROLLARY 2.18. *Let $F \triangleleft K$. Then $F \subseteq \operatorname{rab} K$.* ■

EXAMPLE 2.19. Let V be a linear space over $\mathbf{F}$ with a strictly convex norm, and let K be a cone defined as in Example 2.7(ii). Let $x \in K$. Then $x \in \operatorname{int} K$ if and only if $\|x\| < f(x)$. Hence the only nontrivial faces are of the form ray(x) for $\|x\| = f(x)$. It follows that K has chain length 2 if $\operatorname{int} K \neq \emptyset$ and if $\dim K > 1$.

LEMMA 2.20. *Let $F \trianglelefteq K$ and let $x \in V$. Then $x \in \operatorname{rai} F$ if and only if $F = \varphi(x)$.*

Proof. If $x \in \operatorname{rai} F = F \cap \operatorname{rai} F$, then by Lemma 2.17, $F = \varphi(x)$. Conversely, let $F = \varphi(x)$, and let $z \in \operatorname{span} F$. Suppose $z = u - v$ where u, $v \in F$. There is an $\varepsilon > 0$ such that $0 \leqslant \varepsilon u \leqslant \frac{1}{2}x$, $0 \leqslant \varepsilon v \leqslant \frac{1}{2}x$. Hence

$$0 \leqslant \tfrac{1}{2}x - \varepsilon v \leqslant \tfrac{1}{2}x + \varepsilon(u - v) \leqslant \tfrac{1}{2}x + \varepsilon v \leqslant x.$$

Thus $\frac{1}{2}x - \varepsilon z \in \varphi(x)$, and so $x \in \operatorname{rai} F$. ■

COROLLARY 2.21. *Let $F \trianglelefteq K$. Then $\operatorname{rai} F \neq \emptyset$ if and only if F is cyclic.* ■

The preceding lemmas combine to give the next result.

THEOREM 2.22. *Let V be a vector space over the fully ordered field* $\mathbf{F}$, *and let K be a cone in V. Then the following are equivalent*:

(i) K has ACC on faces.
(ii) Each face of K is cyclic.
(iii) For each $F \trianglelefteq K$, $\operatorname{rai} F \neq \emptyset$. ∎

EXAMPLE 2.23. We give two examples of cones without ACC on faces.

(i) Let $\mathbf{R}$ be the real field, and let V be the set of all sequences $x=(x_1,x_2\ldots)$, $x_i\in\mathbf{R}$ such that $\sum_{i=1}^{\infty}x_i^2<\infty$ (i.e., l_2). Let $K=\{x\in V: x_i\geqslant 0, i=1,2,\ldots\}$. Then $\operatorname{rai}K=\emptyset$.

(ii) Let the field again by $\mathbf{R}$, and let V be the vector space of all real sequences that are constant from some point on. Let $K=\{x\in V: x_i\geqslant 0, i=1, 2,\ldots\}$. Then $(1,1,1,\ldots)\in\operatorname{int}K$. Let $F_j=\{x\in K: x_k=0,\ k\geqslant j\}$. Then $F_1\triangleleft F_2 \triangleleft F_3\triangleleft\cdots$ is an ascending chain of faces that does not terminate. Also $F=\cup_{j=1}^{\infty}F_j$ is a noncyclic face.

REMARK 2.24. If K is a cone in a finite dimensional space V, then by Corollary 2.15 every face of K is cyclic.

Note. Our definition of face is related to the concept of an *ideal* in a partially ordered vector space as used by Bonsall [5] or an *order ideal* as in Ellis [8]. Following Namioka [23], Bonsall and Tomiuk [6] employ the same concept as ours under the term *full cone*. Ellis [9] (cf. also Edwards [7]) defines *face* for compact convex sets (cf. also Rockafellar [28, p. 162]). In $\mathbf{R}^n$ Vandergraft [32] defines a *face* as a subcone of K such that $F\subseteq\operatorname{rab}K$ and F is generated by some subset of extremal vectors in K. This definition is less restrictive than ours, as may be seen from the following example. In $\mathbf{R}^4$ let K be the cone generated by the five vectors $(\pm 1,0,1,0)$, $(0,\pm 1,1,0)$, $(0,0,0,1)$, and let F be the cone generated by $(1,0,1,0)$ and $(-1,0,1,0)$. Then F is a face of K by Vandergraft's definition, but not by ours.

3. POSITIVE OPERATORS

In this section we shall assume K to be a full cone in the vector space V over the fully ordered field $\mathbf{F}$. If $x\in\operatorname{int}K$, we write $x\gg 0$.

DEFINITION 3.1. Let $A\in\operatorname{Hom}(V,V)$.

(i) If $AK\subseteq K$, we call A *nonnegative*, and write $A\geqslant 0$. If $A\geqslant 0$ and $A\neq 0$, we write $A>0$.

(ii) If $A(K\setminus\{0\})\subseteq\operatorname{int}K$, we write $A\gg 0$.

(iii) If $A \geqslant 0$ and A maps no nontrivial face of K into itself, we call A *irreducible*.

(iv) If for each $x > 0$ there is a positive integer $p = p(x)$ such that $(I+A)^p x \gg 0$, then we call A *strongly irreducible*.

Note that since K is reproducing, the order defined by (i) in $\mathrm{Hom}(V, V)$ is a partial order.

EXAMPLE 3.2.

(i) Let $\mathbf{R}$ be the real field and let K be the positive orthant in $V = \mathbf{R}^n$ (cf. Example 2.7). Let $A \in \mathrm{Hom}(V, V)$, and identify A with its matrix relative to the basis $e^1, \ldots, e^n$, where $e^i = \{\delta_{i1}, \ldots, \delta_{in}\}$ and δ_{ij} is the Kronecker delta. Then $A \geqslant 0$ if and only if $a_{ij} \geqslant 0$, $i, j = 1, \ldots, n$. Further, A is irreducible if and only if there does not exist a permutation matrix P such that $P^{-1}AP$ has the form

$$P^{-1}AP = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11}, B_{22} are square. Thus in this case irreducibility has its usual meaning.

(ii) Let K be the ice-cream cone in $\mathbf{R}^3$ [cf. Example 2.7 (ii′)]. Then any rotation A about the line $x_1 = x_2 = 0$ is a nonnegative transformation. If $A \neq I$ (the identity), then A is irreducible.

LEMMA 3.3 *Let K be a full cone in V over $\mathbf{F}$. If A is strongly irreducible, then A is irreducible.*

Proof. Let A be strongly irreducible. Let $F \lhd K$, $F \neq \{0\}$, and suppose that $AF \subseteq F$. Let $0 \neq x \in F$. Then by assumption there exists a p such that $(I+A)^p x \gg 0$. Hence by Lemma 2.17, $\varphi\big((I+A)^p x\big) = K$. But since $AF \subseteq F$, we have $(I+A)^p x \in F$, whence $\varphi\big((I+A)^p x\big) \subseteq F$. It follows that $F = K$, and hence A is irreducible. ∎

THEOREM 3.4 *Let K be a full cone in the vector space V over the fully ordered field $\mathbf{F}$. Let K have ACC on faces. If $A \geqslant 0$, then A is irreducible if and only if A is strongly irreducible.*

Proof. In view of Lemma 3.3 we need only prove that A irreducible implies A strongly irreducible. So let A be irreducible. If $x \gg 0$, then $(I+A)^p x \geqslant x \gg 0$ for all positive p. If $0 \neq x \in \mathrm{bdy}\, K$, then put $F_{k-1} = \varphi\big((I+A)^k x\big)$. Then $F_0 \lhd F_1 \lhd \cdots$, so by ACC this chain terminates with say $F_m = \varphi\big((I+A)^{m+1}x\big)$. Let $y \in F_m$. Then $Ay \leqslant (I+A)\,y \in F_{m+1} = F_m$. Thus $AF_m \subseteq F_m$, and so $F_m = K$. Hence by Lemma 2.20, $(I+A)^{m+1}x \gg 0$. ■

Note. If K has chain length N, then the integer p in Definition 3.1 (iv) can be chosen independently of x. In particular, $p = N-1$ will suffice. By essentially the same proof we obtain

THEOREM 3.5. *Let K be a full cone in V over $\mathbf{F}$. If K has chain length N and $A \geqslant 0$, then A is irreducible if and only if $(I+A)^{N-1} \gg 0$.* ■

Note. The notion of irreducibility for nonnegative matrices is due to Frobenius [12]. The result that an irreducible matrix is (in our terminology) strongly irreducible is implicitly contained in [12, IV], and is explicitly stated and proved by Wielandt [34]. For operators, Bonsall and Tomiuk [6] use a definition close to ours. Schaefer [30, p. 269] uses a definition of irreducibility which involves an infinite series and is in the same spirit as our concept of strong irreducibility. In $\mathbf{R}^n$ our definition of irreducibility is equivalent to Vandergraft's [33], even though the two definitions of face do not coincide (see the note at the end of Sec. 2). Other properties of a similar nature have been studied: see Marek [21], Sawashima [29]; and for a comparison of these properties when $\dim V$ is finite, see Barker [1].

4. PERRON-FROBENIUS THEORY

Henceforth we shall assume that K is full and algebraically closed. In addition, we take $\mathbf{F} = \mathbf{R}$, the real field.

LEMMA 4.1. *If $A > 0$ and $x \gg 0$, then $Ax > 0$.*

Proof. Suppose $A \geqslant 0$, $x \gg 0$, and $Ax = 0$. By Lemma 2.17, $K = \varphi(x)$, so for any $y \in K$, there is an $\alpha > 0$ such that $0 \leqslant \alpha y \leqslant x$. But then $0 \leqslant \alpha Ay \leqslant Ax = 0$, whence $Ay = 0$ and $AK = \{0\}$. Since $K - K = V$, we have $A = 0$. ■

DEFINITION 4.2. Let $A \geqslant 0$. We define

$$\Omega_1 = \{\omega \geqslant 0 : \exists y \gg 0,\ \omega y \leqslant Ay\},$$

$$\Omega = \{\omega \geqslant 0 : \exists y > 0,\ \omega y \leqslant Ay\},$$

$$\Sigma_1 = \{\sigma \geqslant 0 : \exists x \gg 0,\ \sigma x \geqslant Ax\},$$

and

$$\Sigma = \{\sigma \geqslant 0 : \exists x > 0,\ \sigma x \geqslant Ax\}.$$

REMARK 4.3. Let $x \gg 0$. Then for σ sufficiently large, $\sigma x \geqslant Ax$. Hence $\Sigma_1 \neq \emptyset$. Clearly $0 \in \Omega_1$, so $\Omega_1 \neq \emptyset$ either.

LEMMA 4.4. *Let* $u \gg 0$ *and let* $-v \not\in K$. *Put* $\varepsilon = \sup\{\alpha : u - \alpha v \geqslant 0\}$. *Then* $0 < \varepsilon < \infty$ *and* $u - \varepsilon v \in \operatorname{bdy} K$.

Proof. Since $-v \not\in K$ and K is algebraically closed, it follows that for α sufficiently large $u - \alpha v \not\in K$. But for $\alpha > 0$ sufficiently small $u - \alpha v \geqslant 0$. Thus $0 < \varepsilon < \infty$. Finally, $u - \varepsilon v \geqslant 0$, since $K = \operatorname{acl} K$, and $u - \varepsilon v \in \operatorname{bdy} K$ since $u - \eta v \not\in K$ for $\eta > \varepsilon$. ■

LEMMA 4.5. *Let* $A > 0$. *Then*

$$\sup \Omega \leqslant \inf \Sigma_1.$$

Proof. Let $\omega \in \Omega$ and $\sigma \in \Sigma_1$. Let $v > 0$ and $u \gg 0$ correspond to ω and σ, respectively. Thus we have $v > 0$, $Av \geqslant \omega v$ or $-Av \leqslant -\omega v$, $u \gg 0$, and $Au \leqslant \sigma u$. Since $-v \not\in K$, for the ε of Lemma 4.4 we conclude that $u - \varepsilon v \in \operatorname{bdy} K$. But then

$$0 \leqslant A(u - \varepsilon v) = Au + \in(-Av) \leqslant \sigma u + \varepsilon(-\omega v) = \sigma\left(u - \varepsilon \frac{\omega}{\sigma} v\right),$$

since $\sigma > 0$ by Lemma 4.1. Hence from the definition of ε it follows that $\varepsilon(\omega/\sigma) \leqslant \varepsilon$. But $\varepsilon > 0$, so $\omega/\sigma \leqslant 1$. ■

LEMMA 4.6. *If A is strongly irreducible, then* $\Sigma = \Sigma_1$ *and* $\Omega = \Omega_1$.

Proof. Clearly $\Sigma \supseteq \Sigma_1$. Conversely, let $\sigma \in \Sigma$, and let $x > 0$ satisfy $Ax \leqslant \sigma x$. Then there is a p such that $(I + A)^p x \gg 0$. Thus $A(I + A)^p x \leqslant \sigma(I + A)^p x$, whence $\sigma \in \Sigma_1$. Therefore $\Sigma = \Sigma_1$. A similar proof shows that $\Omega = \Omega_1$. ■

LEMMA 4.7. *Let A be strongly irreducible, and let* $\rho = \inf \Sigma$. *If A has an eigenvector* $u \geqslant 0$, *then* $\{\rho\} = \Sigma \cap \Omega$ *and ρ is the eigenvalue belonging to u.*

Proof. Let $Au = \tau u$ for $u > 0$. Clearly $\tau \in \Sigma \cap \Omega$. But by Lemmas 4.5 and 4.6, $\tau = \rho$ and $\Sigma \cap \Omega = \{\tau\} = \{\rho\}$. ■

The next lemma is a converse to Lemma 4.7. For the finite dimensional nonnegative orthant, the lemma is due to H. Wielandt.

LEMMA 4.8. *Let A be strongly irreducible. Let* $\rho = \inf \Sigma$. *If* $\rho \in \Sigma$, *then ρ is a positive eigenvalue and any* $u > 0$ *satisfying* $Au \leqslant \rho u$ *is an eigenvector belonging to ρ. Further* $u \gg 0$.

Proof. Let $\rho \in \Sigma$, and let $Au \leqslant \rho u$ for $u > 0$. Set $z = (\rho I - A)u \geqslant 0$, and suppose $z > 0$. Then there is p such that $z^1 = (I + A)^p z \gg 0$. Thus

$$z^1 = (\rho I - A)(I + A)^p u = (\rho I - A)u^1,$$

where $u^1 = (I + A)^p u > 0$. Since $-Au^1 \leqslant 0 \ll z^1$, we have $\rho > 0$. There is an $\alpha > 0$ such that $\alpha u^1 \leqslant z^1 = (\rho I - A)u^1$. Hence $Au^1 \leqslant (\rho - \alpha)u^1$, which contradicts the choice of ρ. Therefore $0 = z = (\rho I - A)u$. Also, for suitable p we have $0 \ll (I + A)^p u = (1 + \rho)^p u$, whence $u \gg 0$. ∎

COROLLARY 4.9. *Let A be strongly irreducible. Then A has an eigenvector $u \in K$ if and only if* $\inf \Sigma \in \Sigma$. ∎

THEOREM 4.10. *Let K be a full algebraically closed cone. Let A be a strongly irreducible operator. Let Σ be given by Definition 4.2 and $\rho = \inf \Sigma$. If $\rho \in \Sigma$, then there is a vector u for which $Au = \rho u$ and such that*

(1) $\rho > 0$,
(2) $u \gg 0$,
(3) u is (up to scalar multiples) the only eigenvector belonging to ρ,
(4) u is the only eigenvector of A in K,
(5) $(\rho I - A)V \cap K = \{0\}$,
(6) Every eigenvalue λ of A satisfies $|\lambda| \leqslant \rho$.

(Note that at this point only real eigenvalues are considered.)

Proof. The existence of u, (1) and (2) are given by Lemma 4.7.

(3) Suppose $v \neq 0$ satisfies $Av = \rho v$. Either $v \notin K$ or $-v \notin K$, say $-v \notin K$. Then by Lemma 4.4, $u - \varepsilon v \in \operatorname{bdy} K$ for some $\varepsilon > 0$. But then $A(u - \varepsilon v) = \rho(u - \varepsilon v)$, which contradicts Lemma 4.8 if $u - \varepsilon v \neq 0$. Hence $u = \varepsilon v$.

(4) Suppose $v > 0$ and $Av = \tau v$. Then $\tau \geqslant 0$ and $\tau = \rho$ by Lemma 4.7. By (3), v is a positive multiple of u.

(5) Suppose $x \in V$ satisfies $(\rho I - A)x \geqslant 0$. Then for β sufficiently large we have $y = x + \beta u > 0$. We obtain

$$(\rho I - A)y = (\rho I - A)x \geqslant 0,$$

whence by Wielandt's lemma (Lemma 4.8), $(\rho I - A)y = 0$. Thus $(\rho I - A)x = 0$.

(6) Suppose $\lambda \neq \rho$ and $Av = \lambda v$ for $v \neq 0$. By (4), neither v nor $-v$ is in K. So by Lemma 4.4 there are maximal positive ε_1 and ε_2 for which

$$u - \varepsilon_1 v \geqslant 0 \quad \text{and} \quad u + \varepsilon_2 v \geqslant 0.$$

Let $\varepsilon=\max\{\varepsilon_1,\varepsilon_2\}$, say $\varepsilon=\varepsilon_1$. Then

$$0\leqslant A(u-\varepsilon_1 v)=\rho u-\lambda\varepsilon_1 v=\rho\left(u-\frac{\lambda}{\rho}\varepsilon_1 v\right),$$

whence $-\varepsilon_1\leqslant-\varepsilon_2\leqslant(\lambda/\rho)\varepsilon_1\leqslant\varepsilon_1$. Thus $|\lambda|/\rho\leqslant 1$. If $\varepsilon=\varepsilon_2$ we argue similarly on $u+\varepsilon_2 v$. ∎

REMARK 4.11. If V is finite dimensional, then (3) and (5) of Theorem 4.10 together imply that ρ is a simple zero of the characteristic polynomial of A (i.e., ρ has algebraic multiplicity 1). For by virtue of (3) we need only show that the elementary divisor belonging to ρ is linear. This is insured by (5), since there can be no x satisfying $(\rho I-A)x=u\gg 0$.

As in the finite dimensional case, we would like to make a comparison with the moduli of any complex eigenvalues. In order to speak of complex eigenvalues we must regard A as the operator on the space $V+iV=\{x+iy:x,y\in V\}$ given by the formula

$$A(x+iy)=Ax+iAy.$$

The process of extension is explained for finite dimensional V by Halmos [14]. The reader can check that this same procedure works in the present situation. Note further that if K is algebraically closed in V, then $K+iK$ is algebraically closed in $V+iV$. If $\lambda=\mu+i\nu$, $\nu\neq 0$, is a complex eigenvalue with eigenvector $z=x+iy$, then

$$Az=A(x+iy)=(\mu x-\nu y)+i(\nu x+\mu y),$$

and x, y are linearly independent over $\mathbf{R}$. Finally, if λ is an eigenvalue of A with eigenvector $x+iy$, then $\bar{\lambda}$ is an eigenvalue with eigenvector $x-iy$.

LEMMA 4.12. *Let $A\geqslant 0$, and let A have a complex eigenvalue $\lambda=\mu+i\nu$ with $\nu\neq 0$. If $z=x+iy$ is a corresponding eigenvector, put $\pi=\operatorname{span}\{x,y\}$ in the real space V. Then $\pi\cap K=\{0\}$.*

Proof. Note that π and $\pi\cap K$ are invariant under A. Let A_π be the restriction of A to π. If A_π had a real eigenvalue, then A_π would have three linearly independent eigenvectors in the two dimensional complex space $\pi+i\pi$, which is impossible. Hence A_π has no real eigenvalues. If $\dim(\pi\cap K)\geqslant 1$, then the generalized Perron-Frobenius theorem (cf. Birkhoff [4] or Rheinbolt and Vandergraft [17]) implies that A_π has a real eigenvalue. Hence $\pi\cap K=\{0\}$. ∎

Of course, if we assume A is strongly irreducible, then we could have proved Lemma 4.12 by applying our theorems instead of appealing to the finite dimensional Perron-Frobenius theorem.

Since π (as defined in Lemma 4.12) has dimension 2, we may define a norm χ on π by $\chi(\alpha x+\beta y)=(\alpha^2+\beta^2)^{1/2}$. This endows π with the usual Euclidean topology. In what follows, all topological notions on π will refer to this topology.

LEMMA 4.13. *Let $u \gg 0$ be fixed, and let π be defined as in Lemma 4.12. Define, for all $w \in \pi$,*

$$q(w)=\inf\{k>0: u+k^{-1}w \in K\}.$$

Then

(i) $q(w) \geq 0$ *for all* $w \in \pi$.
(ii) $q(w)=0$ *if and only if* $w=0$.
(iii) $q(w+w') \leq q(w)+q(w')$ *for* $w, w' \in \pi$.
(iv) $q(\alpha w)=\alpha q(w)$ *if* $\alpha \geq 0$ *and* $w \in \pi$.
(v) *Let* $S=\{w \in \pi : q(w) \leq 1\}$. *Then S is a compact convex subset of π with 0 in the interior.*

Proof. Clearly (i) holds, and (ii) holds by Lemma 4.12 and because K is algebraically closed. The proofs of (iii), (iv), and (v) are patterned after the standard proofs that relate norms and symmetric convex bodies in a finite dimensional space (e.g., [15]). ∎

THEOREM 4.14. *Let K be an algebraically closed full cone. Let A be a strongly irreducible operator, define Σ as in Definition 4.2, and put $\rho=\inf\Sigma$. If $\rho \in \Sigma$ and if $Az=\lambda z$ for $z \in V+iV$ and $\lambda \in \mathbf{C}$, then $|\lambda| \leq \rho$.*

Proof. We may assume that $\rho=1$, that $u \gg 0$ satisfies $Au=\rho u$, and that $\lambda \notin \mathbf{R}$, since $\lambda \in \mathbf{R}$ is covered by Theorem 4.10. Let q be the function defined as in Lemma 4.13 and let S be given by Lemma 4.13(v). We shall first show that $AS \subseteq S$. Let $S_1=u+S$. Since K is algebraically closed, it is easy to prove that $S_1=(u+\pi)\cap K$. Hence for $w \in S$, $u+Aw \in u+\pi$, and since $u+Aw=A(u+w)$, we also have that $u+Aw \in K$. Thus $u+Aw \in S_1$, whence $Aw \in S$.

For $\lambda=\mu+i\nu \in \mathbf{C}$ and $w=\alpha x+\beta y \in S$ a direct computation shows that

$$Aw=(\alpha\mu+\beta\nu)x+(-\alpha\nu+\beta\mu)y.$$

It follows that $\chi(Aw)=|\lambda|\chi(w)$. Since χ is a continuous function on the compact set S, χ achieves a positive maximum on S at, say, w_0. Hence, since, $Aw_0 \in S$,

$$|\lambda|\chi(w_0)=\chi(Aw_0)\leqslant\chi(w_0).$$

It follows that $|\lambda|\leqslant 1=\rho$. ■

The rotational invariance of the spectrum (cf. Wielandt [34]) has its analog for strongly irreducible operators. This was shown by Barker and Turner [2] for operators in a finite dimensional space. The notion of an elliptic cross section can be carried over to the present setting and the corresponding result proved, viz. *under the hypotheses of Theorem* 4.14, *if K has no elliptic cross section and if* λ *is an eigenvalue of A of modulus* ρ, *then* λ *is a root of unity times* ρ.

Note. Methods similar to Wielandt's have recently been exploited by Marek [21, 22] and Lee [20]. Also the paper by Karlin [17] employs sets much like those defined in Definition 4.1. There is an immense literature on spectral properties of various kinds of operators leaving invariant a cone in a topological vector space; see the references in this paper and the excellent bibliography by Marek [22].

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Received December 1973; revised October 1974