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Positive Operators and an Inertia Theorem

By
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1. Introduction

In recent years there has been interest in a theorem on positive definite matrices known as Lyapunov's theorem. Several authors have proved generalizations of this theorem, (WIELANDT [29], TAUSSKY [24], [25], [26], OSTROWSKI-SCHNEIDER [20], GIVENS [10], CARLSON-SCHNEIDER [3], CARLSON [4]). Lyapunov's theorem and its generalizations have become known as *inertia theorems*. In this note we shall use a generalization of the theorem of Perron-Frobenius on matrices with non-negative elements (KREIN-RUTMAN [14]) to prove a new inertia theorem. Our theorem is closely related to known results on M-matrices (OSTROWSKI [19], SCHNEIDER [20], FAN [6], FAN-HOUSEHOLDER [7], FIEDLER-PTÁK [8]). The relation between M-matrix theorems and inertia theorems does not seem to have been observed before.

2. Positive Operators

The theorem of Perron-Frobenius on matrices with non-negative elements (BELLMAN [1], p. 278, GANTMACHER [9], Vol. II, p. 53, VARGA [28], p. 30 and p. 46) has been generalized in several ways to apply to linear transformations mapping a cone into itself. The following result is essentially that stated by KREIN-RUTMAN [14], Theorem I, p. 202, or it may be derived by applying either KREIN-RUTMAN [14], Theorem 4.1, p. 243, or Theorem 6.1, p. 262 to $P+\varepsilon I$, for all $\varepsilon > 0$. Indeed, the proof of the Perron-Frobenius theorem in [1] may easily be adapted to prove:

Theorem 0. (Finite dimensional Krein-Rutman.) *Let $\mathscr{C}$ be a closed cone with interior in a finite dimensional real vector space V. Let P be a linear transformation on V for which $P\mathscr{C} \subseteq \mathscr{C}$. Then the spectral radius ϱ of P is an eigenvalue and there exists a non-zero $x \in \mathscr{C}$ such that $Px = \varrho x$. Similarly there exists a non-zero $y^* \in \mathscr{C}^*$, the cone dual to $\mathscr{C}$, such that $P^* y^* = \varrho y^*$.*

For the relevant definitions see KREIN-RUTMAN [14], pp. 206, 209, and 217. Thus $\mathscr{C}$ is a cone if $\mathscr{C} + \mathscr{C} \subseteq \mathscr{C}$, $\alpha\mathscr{C} \subseteq \mathscr{C}$, for all $\alpha \geqq 0$, and $\mathscr{C} \cap (-\mathscr{C}) = \{0\}$. The cone $\mathscr{C}^*$ in the dual space V^* consists of all linear functionals $y^* \in V^*$ for which $(y^*, x) \geqq 0$ for all $x \in \mathscr{C}$. If $x \in \mathscr{C}$, $x \neq 0$ we shall write $x > 0$, and if $x \in \mathscr{C}^0$ (the interior of $\mathscr{C}$) we write $x \gg 0$. If $P\mathscr{C} \subseteq \mathscr{C}$ and $P \neq 0$, then we write $P > 0$.

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Some additional comments will elucidate theorem 0. The topology on V is, of course, the unique topology which turns V into a topological vector space. We need no hypothesis on $\mathscr{C}^*$, since $\mathscr{C}^*$ has interior whenever $\mathscr{C}$ has. This follows from $\mathscr{C}^{**}=\mathscr{C}$ (EGGLESTON [5], p. 26). Finally note that $V=\mathscr{C}^0-\mathscr{C}^0$, if $\mathscr{C}$ has interior.

For more references to further generalizations of theorem 0 see KARLIN [13], SCHAEFER [21], BONSALL [2], and OSTROWSKI [19].

We now prove a simple consequence of Theorem 0. We denote the spectral radius of P by $\varrho(P)$.

Lemma 1. *Let $\mathscr{C}$ be a closed cone with interior in a finite dimensional real space. Let R and S be linear transformations on V and suppose that $S\geqq 0$ and either $R\mathscr{C}^0\supseteq\mathscr{C}^0$ or $R\mathscr{C}^0\cap\mathscr{C}^0=\emptyset$. If $T=R-S$, then the following are equivalent:*

(1′) *R is non-singular, $R^{-1}>0$ and $\varrho(R^{-1}S)<1$,*

(2′) *T is non-singular, and $T^{-1}\mathscr{C}^0\subseteq\mathscr{C}^0$,*

(3′) *$T\mathscr{C}^0\cap\mathscr{C}^0\neq\emptyset$.*

Proof. (1′) ⇒ (2′). Let $P=R^{-1}S$. Since both $R^{-1}>0$ and $S\geqq 0$, it follows that $P\geqq 0$. But $T=R(1-P)$ and $\varrho(P)<1$, whence $T^{-1}=\left(\sum_{m=0}^{\infty}P^m\right)R^{-1}>0$. Since T^{-1} is non-singular, T^{-1} maps open sets into open sets. Hence $T^{-1}\mathscr{C}_0$ is an open set contained in $\mathscr{C}$, and (2′) follows.

(2′) ⇒ (3′). Trivial.

(3′) ⇒ (1′). Let $x\gg 0$ and $Tx\gg 0$. Then $Rx=Tx+Sx\gg 0$, whence by our assumptions on R, $R\mathscr{C}^0\supseteq\mathscr{C}^0$. By $V=\mathscr{C}^0-\mathscr{C}^0$, R is non-singular and so $R^{-1}\mathscr{C}^0\subseteq\mathscr{C}^0$. We draw two conclusions from this. First, that $R^{-1}\mathscr{C}\subseteq\mathscr{C}$ and $R^{-1}>0$, and, second, that, for $P=R^{-1}S$, $(1-P)x=RTx\gg 0$. Let $\varrho=\varrho(P)$. By Theorem 0, there exists $y^*\in\mathscr{C}^*\subseteq V^*$ such that $P^*y^*=\varrho y^*$. Both $(y^*,(1-P)x)>0$ and $(y^*,x)>0$, and it now follows that $0<(y^*,(1-P)x)=((1-P^*)y^*,x)=(1-\varrho)(y^*,x)$, whence $\varrho(R^{-1}S)<1$.

Remark 1. The generalized Perron-Frobenius Theorem has been used only in the proof that (3′) implies(1′) in Lemma 1. We have found several other direct proofs of this implication but all of these use a result close to Theorem 0 either for P or P^*. On the other hand, it is possible to give a purely elementary proof that (3′) implies (2′). A proof that (3′) implies (1′) may also be based on the following observations: For $Q>0$, the set of all R such that $0\leqq R\leqq Q$ is compact. If $P>0$ and $Q=(1-P)^{-1}>0$ then $\sum_{m=0}^{t}P^m=Q-QP^{t+1}\leqq Q$.

This remark may be expanded to form part of a return journey from the lemma to the Perron-Frobenius theorem.

Remark 2. If $\mathscr{C}$ is the cone of non-negative n-tuples then $R\geqq 0$ if and only if R is (represented by) a matrix whose elements are non-negative. If $R>0$, $S\geqq 0$ and R is diagonal then $R-S$ is a matrix with non-positive elements off the diagonal, and in this case the equivalence of (1′), (2′) and (3′) and many other conditions is known, e.g. FAN [6], Theorem 5′, FIEDLER-PTÁK [8] (4, 3), HOUSEHOLDER [12], Lemma 0, VARGA [28], Theorems 3.11 and 3.13. Each of these conditions characterizes the class of non-singular M-matrices in the sense of OSTROWSKI [19], within the class of matrices with non-positive non-diagonal elements.

3. An Inertia Theorem

We shall apply Theorem 0 to the *cone of positive semi-definite matrices* in the real space of $n\times n$ Hermitian matrices* to obtain an inertia theorem. The interior of the cone consists of all positive definite matrices and in accordance with our general conventions, $H\gg 0$ here means that H is positive definite.

Theorem 1. *Let A, C_k, $k=1,\dots,s$ be complex $n\times n$ matrices which can be simultaneously triangulated. Suppose the eigenvalues of A, C_k under a natural correspondence are $\alpha_i, \gamma_i^{(k)}$, $i=1,\dots,n$ and $k=1,\dots,s$. For Hermitian H, let*

$$T(H)=AHA^*-\sum_{k=1}^{s} C_k H C_k^*.$$

Then the following are equivalent:

(1) $\varphi_i=|\alpha_i|^2-\sum_{k=1}^{s}|\gamma_i^{(k)}|^2>0, \qquad i=1,\dots,n.$

(2) *For all $K\gg 0$, there exists a unique $H\gg 0$ such that $T(H)=K$.*

(3) *There exists $H\gg 0$ such that $T(H)\gg 0$.*

Proof. Let V be the real space of all $n\times n$ Hermitian matrices, and let $\mathscr{C}$ be the positive semidefinite cone. Define R by $R(H)=AHA^*$, and let $S(H)=\sum_{k=1}^{n} C_k H C_k^*$. If A is non-singular, then R is non-singular and $R\mathscr{C}=\mathscr{C}$, and $R^{-1}>0$. If A is singular, then for all $H\in\mathscr{C}$, $R(H)$ is semi-definite, but not definite, whence $R\mathscr{C}^0\cap\mathscr{C}^0=\emptyset$. Thus the hypotheses of Lemma 1 apply to R and furthermore $R^{-1}>0$ if R is non-singular. We shall show that R is non-singular and $\varrho(R^{-1}S)<1$ if and only if $\varphi_i>0$, $i=1,\dots,n$. Since (2) and (3) are merely restatements of (2′) and (3′), this will prove the theorem.

It is known that the eigenvalues of $R=A\otimes\bar{A}$ are $\alpha_i\bar{\alpha}_j$, and of $C_k\otimes\bar{C}_k$ are $\gamma_i^{(k)}\overline{\gamma_j^{(k)}}$, $i,j=1,\dots,n$, (e.g., BELLMAN [1], p. 227, MCDUFFEE [15], p. 84, but see § 4). It is easily verified that $A\otimes\bar{A}$, $C_k\otimes\bar{C}_k$, $k=1,\dots,s$ are simultaneously triangulable under our assumptions on A and the C_k. If R is non-singular, it then follows that the eigenvalues of $R^{-1}S$ are $\sum_{k=1}^{n}\alpha_i\gamma_i^{(k)}\overline{\alpha_j^{-1}\gamma_j^{(k)}}$, $i,j=1,\dots,n$. If also $\varrho(R^{-1}S)<1$, then putting $i=j$, we immediately obtain $\sum_{k=1}^{s}|\alpha_i^{-1}\gamma_i^{(k)}|^2<1$, whence $\varphi_i=|\alpha_i|^2-\sum_{k=1}^{s}|\gamma_i^{(k)}|^2>0$, $i=1,\dots,n$. Conversely, if all $\varphi_i>0$, then all $\alpha_i\neq 0$. Thus R is non-singular and $\sum_{k=1}^{s}|\alpha_i^{-1}\gamma_i^{(k)}|^2<1$, $i=1,\dots,s$. Applying Cauchy's inequality, we then obtain

$$\Big|\sum_{k=1}^{s}\alpha_i^{-1}\gamma_i^{(k)}\overline{\alpha_j^{-1}\gamma_j^{(k)}}\Big|^2\leqq\Big(\sum_{k=1}^{s}|\alpha_i^{-1}\gamma_i^{(k)}|^2\Big)\Big(\sum_{k=1}^{s}|\alpha_j^{-1}\gamma_j^{(k)}|^2\Big)<1,$$

and $\varrho(R^{-1}S)=\max_i\sum_{k=1}^{s}|\alpha_i^{-1}\gamma_i^{(k)}|^2<1$ follows. The theorem is proved.

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* In this space a matrix may be multiplied by a real scalar only.

Special cases. We note that for $A=B+I$, $s=2$, $C_1=B$, $C_2=I$, we obtain $T_l(H)=BH+HB^*$, and $\varphi_i=\beta_i+\bar{\beta}_i$, where the β_i are the eigenvalues of B. Thus in this case, Theorem 1 reduces to Lyapunov's theorem (GANTMACHER [9], Vol. II, p. 187, HAHN [11], TAUSSKY [24]). To obtain Stein's theorem (STEIN [23], NEWMAN [17], TAUSSKY [25]), put $A=I$, $s=1$, $C_1=C$. Then $T_s(H)=H-CHC^*$ and $\varphi_i=1-|\gamma_i|^2$.

Remark 3. Let us replace the hypothesis in Theorem 1 that the matrices A, C_k, $k=1,\ldots,s$ are simultaneously triangulable by the stronger assumption that A, C_k, $k=1,\ldots,s$ *commute in pairs*. In this case it is easy to avoid explicit use of Theorem 0. We construct instead positive functionals in $\mathscr{C}^*$ corresponding to y^* in the proof of the lemma. For there exists a common eigenvector z_i: $Az_i=\alpha_i z_i$, $C_k z_i=\gamma_i^{(k)} z_i$, $k=1,\ldots,s$, $i=1,\ldots,n$. If (3) holds in Theorem 1 and $H\gg 0$, $T(H)\gg 0$ then the functional $H\to z_i^* H z_i$ yields that $0<z_i^* T(H) z_i=\varphi_i z_i^* H z_i$, whence $\varphi_i>0$. In the case of the Stein operator $T_s(H)=H-CHC^*$ this proof is known, and I learned of it at the Gatlinburg Matrix Conference in April 1964. It is due to C. G. CULLEN (unpublished).

4. The eigenvalues of $A\otimes\bar{A}$

The operator $X\to AXA^*$ on the *complex* space $\mathscr{M}$ of dimension n^2 of *all* $n\times n$ matrices is usually denoted by $A\otimes\bar{A}$ and its eigenvalues are known to be $\alpha_i\bar{\alpha}_j$, $i,j=1,\ldots,n$. We have used $A\otimes\bar{A}$ to denote the operator $H\to AHA^*$ on the n^2 dimensional *real* space $\mathscr{H}$ of all $n\times n$ *Hermitian* matrices. However, the eigenvalues of the two operators are the same. For suppose T is any linear transformation on $\mathscr{H}$. Define T_e on $\mathscr{M}$ by $T_e(H+iK)=T(H)+iT(K)$. Then if p is any real polynomial, then $p(T)=0$ obviously implies $p(T_e)=0$. Conversely if $r=p+iq$ is a complex polynomial $r(T_e)=0$ implies $p(T_e)=0$ and so $p(T)=0$. More could be said on this subject (for the case of the Lyapunov operator $T(H)=AH+HA^*$, some interesting comments on the relation between T and T_e may be found in TAUSSKY-WIELANDT [27]). Of course, analogous comments would hold for the extension of any linear transformation on a finite dimensional vector space V over a field F to the space $V\otimes_F E$, over E, where E is a finite field extension of F. A similar argument shows that the invariant factors of T and T_e coincide.

5. Transformations mapping the Positive Semi-definite Cone onto itself

Let $\mathscr{H}$ be the real space of all $n\times n$ Hermitian matrices and let $\mathscr{C}$ be the cone of positive semi-definite matrices. If A is non-singular then $R(H)=AHA^*$ and $R'(H)=AH'A^*$ map $\mathscr{C}$ onto itself. We shall show that every transformation for which $R\mathscr{C}=\mathscr{C}$ is of one of these forms.

Lemma 2. *Let $H>0$ (positive semi-definite of order n) and let* rank $H=r$. *Let the cone $\mathscr{C}(H)$ consist of all $K\geqq 0$ for which there is an $\alpha>0$ such that $\alpha K\leqq H$. Then $\mathscr{C}(H)$ is naturally isomorphic to the cone of all positive semi-definite $r\times r$ Hermitian matrices.*

Proof. Let $\mathscr{R}$ be the range of H, and $\mathscr{N}$ the null space of H. Let $K\geqq 0$. If $K\mathscr{R}\subseteq\mathscr{R}$ then $K\mathscr{N}=\{0\}$, and it is easy to see that $\alpha K\leqq H$, for sufficiently small

$\alpha>0$. Conversely if $0\leqq \alpha K\leqq H$, where $\alpha>0$, and $x\in n$ then $0\leqq \big(x, (H-\alpha K)x\big)= -\alpha(x, Kx)\leqq 0$. Hence $(x, Kx)=0$, and by a familiar result it follows that $Kx=0$. Thus $K\mathcal{N}=0$, and so $K\mathcal{R}\subseteq\mathcal{R}$. The restriction of $K\in\mathcal{C}(H)$ to $\mathcal{R}$ is the required isomorphism.

Lemma 3. *Let R be a linear transformation on the space $\mathcal{H}$ of $n\times n$ Hermitian matrices taking the cone $\mathcal{C}$ of positive semi-definite matrices onto itself. Then R preserves rank.*

Proof. Let $H>0$, and define $\mathcal{C}(H)$ as in Lemma 2. For real α, $\alpha K\leqq H$ if and only if $\alpha R(K)\leqq R(H)$, and hence $\mathcal{C}\big(R(H)\big)=R\mathcal{C}(H)$. Since R preserves the dimension of a subspace, the dimensions of the spaces spanned by $\mathcal{C}(H)$ and $\mathcal{C}\big(R(H)\big)$ are the same. Hence, by Lemma 2, $\operatorname{rank} H=\operatorname{rank} R(H)$. If $H\in\mathcal{H}$, then for suitable $H_1, H_2\in\mathcal{C}$, $H=H_1-H_2$ and $\operatorname{rank} H=\operatorname{rank} H_1+\operatorname{rank} H_2$. Hence $\operatorname{rank} R(H)\leqq\operatorname{rank} R(H_1)+\operatorname{rank} R(H_2)=\operatorname{rank} H$, and by applying this argument to R^{-1}, $\operatorname{rank} R(H)=\operatorname{rank} H$.

Theorem 2. *Let R be a linear transformation on the real space $\mathcal{H}$ of $n\times n$ Hermitian matrices which maps the cone $\mathcal{C}$ of positive semi-definite matrices onto itself. Then there exists a non-singular matrix A such that, either $R(H)=AHA^*$ for all $H\in\mathcal{H}$, or $R(H)=AH'A^*$ for all $H\in\mathcal{H}$.*

Proof. Since $R(I)=P^2$, where $P\gg 0$, we may define $R_1(H)=P^{-1}R(H)P^{-1}$. Clearly $R_1(H-\lambda I)=R_1(H)-\lambda I$, and since R preserves rank so does R_1. Hence R_1 preserves spectrum. It now follows by Marcus-Moyls [*16*], Theorem 3, that there exists a unitary U such that either $R_1(H)=UHU^*$ for all $H\in\mathcal{H}$, or $R_1(H)=UH'U^*$ for all $H\in\mathcal{H}$. We now put $A=PU$, and the desired result follows.

Remark 4. Theorem 3 of Marcus-Moyls [*16*] is a consequence of more general results. We shall sketch a simple computational proof of our Theorem 2. It may be shown that there exists a non-singular B such that for R_2 given by $R_2(H)=BR_1(H)B^*$ all matrices $D=\operatorname{diag}(d_i)\in\mathcal{H}$ satisfy $R_2(D)=D$. For all $H\in\mathcal{H}$, $\det H=\det K$, where $K=R_2(H)$. By comparing coefficients of products of d_i in $\det(D+H)$ and $\det(D+K)$ we obtain $h_{ii}=k_{ii}$, and $|h_{ij}|=|k_{ij}|$, $i,j=1,\ldots,n$. Next transform by a unitary diagonal matrix so that for $L=U^*KU$, $l_{i,i+1}=1$ if $h_{i,i+1}=1$, and show that $L=H$, or $L=H'$.

I do not know how to characterize linear transformations S for which $S\mathcal{C}\subseteq\mathcal{C}$.

6. A Counter-example

Several generalizations of Lyapunov's theorem are known (see § 1 for references). For example, let H be a Hermitian matrix which may not be positive definite and suppose that $T_l(H)=BH+HB^*\gg 0$. Then it is known that all $\varphi_i=\beta_i+\bar\beta_i\neq 0$, and there are as many positive (negative) φ_i as there are positive (negative) eigenvalues of H. For general T, these generalizations, if they exist, must be more subtle. This is shown by the following example. Let

$$A=\begin{bmatrix}\alpha_1 & \cdot\\ \cdot & \alpha_2\end{bmatrix},\quad C_1=\begin{bmatrix}\gamma_1^{(1)} & \cdot\\ c & \gamma_2^{(1)}\end{bmatrix},\quad C_2=\begin{bmatrix}\gamma_1^{(2)} & \cdot\\ \cdot & \gamma_2^{(2)}\end{bmatrix}.$$

If $H=\begin{bmatrix}-1 & \cdot\\ \cdot & \cdot\end{bmatrix}$, then $T(H)=\begin{bmatrix}-|\alpha_1|^2+|\gamma_1^{(1)}|^2+|\gamma_1^{(2)}|^2 & \bar c\,\gamma_1^{(1)}\\ \overline{\gamma_1^{(1)}}\,c & |c|^2\end{bmatrix}.$

Thus $T(H) \gg 0$ if and only if $|\alpha_1|^2 - |\gamma_1^{(2)}|^2 < 0$. Of course, this condition implies that $\varphi_1 = |\alpha_1|^2 - |\gamma_1^{(1)}|^2 - |\gamma_1^{(2)}|^2 < 0$. Considerations of continuity show that irrespective of whether $\varphi_2 > 0$, $\varphi_2 = 0$ or $\varphi_2 < 0$, we can find an H which is negative definite, negative semi-definite or indefinite such that $T(H) \gg 0$.

Note added after submission. The main point of our paper is the close relation of some inertia theorems to M-matrix theorems, both of which may be derived from the finite dimensional Krein-Rutman theorem. If one is only interested in our inertia theorem (Theorem 1), there exists a much shorter matrix theoretic proof, which was communicated to us by H. WIELANDT, after he saw the rest of this manuscript. We give the most interesting part of the proof.

Wielandt's proof that (3) *implies* (1) *in Theorem* 1. Let

$$T(H) = A H A^* - \sum_k C_k H C_k^* \gg 0.$$

After replacing A, C_k by $X A X^{-1}$, $X C_k X^{-1}$, respectively, and H, $T(H)$ by $X H X^*$, $X T(H) X^*$, respectively, we may suppose that A and C_k, $k = 1, \ldots, s$ are all (lower) triangular. There exists a non-singular triangular* Q for which $H = Q Q^*$. We note that A is non-singular, for $A x = 0$ implies that $x^* T(H) x \leqq 0$, whence $x = 0$. Thus

$$0 \ll (A Q)^{-1} T(H) (A Q)^{*-1} = I - \sum_k D_k D_k^*,$$

where $Q^{-1} A C_k Q = D_k = (d_{ij}^{(k)})$. It follows that $1 - \sum_k \sum_{j=1}^{i} |d_{ij}^{(k)}|^2 > 0$. But, for a suitable ordering of the $\alpha_i, \gamma_i^{(k)}$, we have $d_{ii}^{(k)} = \alpha_i^{-1} \gamma_i^{(k)}$, and we have proved rather more than required.

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