JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 4, 72-84 (1962)

Some Theorems on the Inertia of General Matrices*

ALEXANDER OSTROWSKI

University of Basel, Switzerland
and Mathematics Research Center, Madison, Wisconsin

AND

HANS SCHNEIDER

University of Wisconsin, Madison, Wisconsin
and Mathematics Research Center, Madison, Wisconsin

Submitted by Richard Bellman

I. INTRODUCTION

1.1. Much is known about the distribution of the roots of algebraic equations in half-planes. (Cf. the corresponding parts in the survey [1] by Marden.) In the case of matrix equations, however, there appears to be only one known general result concerning the location of the eigenvalues of a matrix in the left half-plane. This theorem is generally known as *Lyapunov's theorem*:

(L_0) *Let A be an n-th order matrix with complex elements, and let C be an n-th order positive definite Hermitian matrix. Then there exists a negative definite matrix H for which*

$$AH + HA^* = C \tag{1}$$

holds, if and only if all eigenvalues of A have negative real part.

The real case of this theorem is a special case of some theorems proved by Lyapunov [2, p. 276-277], establishing conditions for the stability of solutions of differential equations. Bellman [3, p. 245; cf. also 14] and Gantmacher [4, Vol. II, p. 189] give proofs of this theorem, which make use of differential equations. An algebraic proof has been given by Hahn [5].

1.2. Recently, investigations into the behavior of economic systems depending on a finite number of parameters have led Arrow and McManus [6] to

* Sponsored by the Mathematics Research Center, United States Army under Contract No.: DA-11-022-ORD-2059.

consider the problem from a different point of view, introducing the concept of the *S-stability*. The equivalence of Lyapunov's theorem with some of the results of Arrow and McManus was noticed by Olga Taussky [7] who let us see her then unpublished manuscript, and thereby sparked this investigation.

1.3. To attack these problems on a broader front, we need some new concepts. For an Hermitian matrix H with π positive, ν negative, and δ vanishing eigenvalues, we shall call the ordered triple (π, ν, δ) the *inertia* of H and denote by In H. More generally, for an $(n \times n)$ matrix A which has π eigenvalues with positive real part, ν with negative real part, and δ purely imaginary ones, we shall again call (π, ν, δ) the *inertia* of the matrix A, and write

$$(\pi, \nu, \delta) = \text{In } A. \tag{2}$$

We have, of course, $\pi + \nu + \delta = n$, the order of A. The indices π, ν, δ are then denoted resp. $\pi(A)$, $\nu(A)$, $\delta(A)$.

A *positive definite* Hermitian H $(H > 0)$ is characterized by the inertia triple $(n, 0, 0)$ and a negative definite H $(H < 0)$ by $(0, n, 0)$. If $\nu = 0$, H will be called *positive semidefinite* $(H \geqq 0)$, and if $\pi = 0$ *negative semidefinite* $(H \leqq 0)$. Throughout this paper, "positive" and "negative *semidefinite*" will be used to include the positive and negative *definite* cases.

If A is a general $(n \times n)$ matrix with $\pi = n$ $(\nu = n)$ in (2) we call it *positive (negative) stable*; if we have in (2) $\nu = 0$ $(\pi = 0)$, it will be called *positive (negative) semistable*. A stable matrix is called *positive (negative) H-stable* if the product AH of A with a Hermitian matrix H is positive (negative) stable if, and only if, H is positive definite. A matrix A is called *positive (negative) H-semistable* if the product AH of A with every positive semidefinite Hermitian matrix H is positive (negative) semistable. If this is only assumed for *real* H, A is called *real positive (negative) H-semistable*.

It is clear that if A is positive stable, semistable, H-stable, or H-semistable, $-A$ is correspondingly negative stable, semistable, H-stable, H-semistable. While, in applications, the "negative" matrices of the types mentioned are important, it is slightly simpler to carry out the discussion with the "positive" types. The results are of course completely equivalent.

1.4. For a general $(n \times n)$ matrix A with complex elements, the *Toeplitz decomposition*

$$A = R + iQ, \qquad R = \frac{A + A^*}{2}, \qquad Q = \frac{A - A^*}{2i} \tag{3}$$

holds, where R and Q are Hermitian. We write

$$R = \frac{A + A^*}{2} = \mathscr{R}A, \qquad Q = \frac{A - A^*}{2i} = \mathscr{J}A. \tag{4}$$

1.5. The main result of our paper is Theorem 1, which asserts that for any Hermitian solution of the equation (1), we have $\mathrm{In}\, H = \mathrm{In}\, A$, and establishes as a necessary and sufficient condition for the existence of such a solution for at least one positive definite C that A has no purely imaginary eigenvalues.[1] This theorem has many important corollaries. Lyapunov's theorem is contained in Corollaries 1 and 2 of Theorem 1. As Corollary 3 of Theorem 1 we prove that if $\mathscr{R}A$ is positive definite, and H is Hermitian, we have $\mathrm{In}\,(AH) = \mathrm{In}\, H$. This rcsult is due to Wielandt [9, p. 4]. In the special case when A is Hermitian a proof may be found in Ostrowski [3]. The general case was pointed out to us in a letter by K. J. Arrow before the start of this investigation. In Corollary 4 we give corresponding results when $\mathscr{R}A$ is semidefinite.

1.6. Our Theorem 3 gives a necessary and sufficient condition for the H-semistability of a matrix; it is that $\mathscr{R}A$ be semidefinite in the same sense. The next question considered is that of necessary and sufficient conditions for the H-stability of a matrix. In this connection Arrow and McManus [6], who introduced the concept of H-stability for real matrices (under the name of S-stability) proved that if $\mathscr{R}A$ is definite, then A is H-stable, with the same sign as $\mathscr{R}A$. This condition, however, is not necessary. In our Theorem 4, necessary and sufficient conditions are obtained for H-stability from the result of Theorem 3, and from Theorem 2 which establishes a connection between the semidefinite character of $\mathscr{R}A$ and the imaginary eigenvalues of A. Finally we give in Theorem 5 a partial result concerning Eq. (1) in the case of a not necessarily definite C.

II. Lemmata about $AX - XB = C$

2.1. From now on all matrices considered will be assumed to be $(n \times n)$ matrices with complex elements, unless otherwise indicated. Instead of the matrix equation (1) we shall first consider the more general equation

$$AX - XB = C \tag{5}$$

where A, B, X and C are nth order matrices. The following lemma is well-known (cf. MacDuffee [11, p. 91], Gantmacher [4, Vol. I., p. 218-220]); however, for the sake of completeness we shall give a proof, which to us seems simpler than the proofs found in the literature.

[1] After this paper was completed, we learned that Olga Taussky [8] has also obtained generalizations of Lyapunov's theorem. One of her results is equivalent to Corollary 1 to our Theorem 1. She discussed her theorems at the Matrix Conference at Gatlinburg, Tenn., in April, 1961.

LEMMA 1. *For each C, there exists a unique X satisfying (5) if and only if A and B have no common eigenvalues, i.e.,*

$$\prod_{\sigma,\tau=1}^{n} (\lambda_\sigma - \mu_\tau) \neq 0 \tag{6}$$

where λ_σ and μ_τ are the complete sets of eigenvalues (counting repetitions) of A and B respectively.

2.2. PROOF. We note first that (5) is a system of n^2 linear equations for the elements of X, and hence, for any C, there exists a unique solution of (5) if and only if

$$AX - XB = 0 \tag{7}$$

has only the trivial solution $X = 0$. Thus we have to prove that (7) has a nontrivial solution if and only if A and B have a common eigenvalue. Indeed, if λ is a common eigenvalue, we have $u'B = \lambda u'$, $Av = \lambda v$, where the $(n \times n)$ matrix $vu' \neq 0$, and it is immediately seen that $X = vu'$ satisfies (7).

To prove the converse we shall suppose that B has elementary divisors $(\lambda - \mu_\rho)^{s_\rho}$, $\rho = 1, \cdots, r$. Then we can find n linearly independent vectors $v_{\rho\sigma}$ $(\sigma = 1, \cdots, s_\rho;\ \rho = 1, \cdots, r)$, such that

$$Bv_{\rho 1} = \mu_\rho v_{\rho 1}$$

and

$$Bv_{\rho,\sigma} = \mu_\rho v_{\rho\sigma} + v_{\rho,\sigma-1}\ (\sigma = 2, \cdots, s_\rho,\ \rho = 1, \cdots, r).$$

If X is now a nonzero solution of (7), there must exist ρ and σ for which $Xv_{\rho\sigma} \neq 0$, and for one such ρ, let σ be the least integer such that $Xv_{\rho\sigma} \neq 0$. Then

$$0 = (AX - XB)v_{\rho\sigma} = A(Xv_{\rho\sigma}) - X(Bv_{\rho\sigma}) = A(Xv_{\rho\sigma}) - \mu_\rho(Xv_{\rho\sigma}),$$

since either $\sigma = 1$ or $Xv_{\rho,\sigma-1} = 0$. Hence μ_ρ is also an eigenvalue of A, and Lemma 1 is proved.

2.3. We shall also use the following remark: In the set of pairs of matrices (A, B) for which (6) holds, the solution X of (5) is *continuous* in the elements of A and B; more precisely: If $A(t)$, $B(t)$, $C(t)$ are continuous matrix functions in the real interval $0 \leq t \leq 1$ and (6) holds for all $A(t)$ and $B(t)$ $(0 \leq t \leq 1)$ and the matrix function $X(t)$ satisfies $A(t)X(t) - X(t)B(t) = C(t)$, then $X(t)$ is also continuous in $0 \leq t \leq 1$. As proof we need merely remark again that we obtain X as the solution of linear equations, and that such a solution is continuous in the coefficients in any *closed* domain where the solution is unique.

2.4. We shall now specialize some of the preceding remarks to the case of Eq. (1) and note that for a Hermitian C, if $AX + XA^* = C$ then also $X^*A^* + AX^* = C$, whence $H = \frac{1}{2}(X + X^*)$ also satisfies (1). Thus, if for given A and C, there exists an X satisfying (1), then there exists a *Hermitian* H satisfying that equation. We next note that the condition (6) gives for the unique existence of a solution of (1) the condition

$$\Delta(A) \equiv \prod_{\sigma,\tau=1}^{n} (\lambda_\sigma + \bar{\lambda}_\tau) \neq 0, \tag{8}$$

where, as before, the λ_σ are the eigenvalues of A. In this case, our previous remarks show that the unique solution of (1) is Hermitian.

We shall reserve the symbol $\Delta(A)$ for the product in (8).

LEMMA 2. *Let A be an $(n \times n)$ matrix and let H be Hermitian. If $\mathscr{R}(AH)$ is positive definite, then H is nonsingular.*[2]

PROOF. Let κ be an eigenvalue of H and suppose $Hu = \kappa u$, $u \neq 0$, so that $u^*H = \kappa u^*$. If

$$AH + HA^* = 2\mathscr{R}(AH) = C$$

then

$$u^*Cu = u^*(AH + HA^*)u = \kappa(u^*Au + uA^*u).$$

But $u^*Cu > 0$ if C is positive definite, whence $\kappa \neq 0$.

III. THE MAIN THEOREM

3.1. We have remarked that if there exists a matrix X such that $AX + XA^* = C$, where C is Hermitian, then there exists a *Hermitian* X satisfying this equation, and for a Hermitian H, $C = AH + HA^* = 2\mathscr{R}(AH)$. Thus there is no loss of generality if in discussing Eq. (1) we suppose the solution matrix to be Hermitian.

THEOREM 1. *Let A be $(n \times n)$ matrix. Necessary and sufficient for the existence of a Hermitian matrix H with $\mathscr{R}(AH)$ positive definite is that A has no purely imaginary eigenvalue* [*that is*, $\delta(A) = 0$]; *and then we have* In H = In A.

---

[2] The assumption that H is Hermitian is not essential. This lemma is due to Picone [12, p. 715]. Indeed $\mathscr{R}(AB) > 0$ is easily seen to imply that AB is nonsingular.

3.2. PROOF. We shall first prove in (a) that if $\delta(A) = 0$, then there exists one Hermitian H_0 for which $\mathscr{R}(AH_0) > 0$ and $\operatorname{In} H_0 = \operatorname{In} A$. In part (b) we show for *each* Hermitian H_1 with $\mathscr{R}(AH_1) > 0$ we have $\operatorname{In} H_1 = \operatorname{In} A$. In part (c) of the proof we show, finally, that if there exists a Hermitian H with $\mathscr{R}(AH) > 0$, then $\mathit{2}(A) = 0$.

3.3. (a) Assuming that $\delta(A) = 0$, we use the reduction to the Jordan canonical form:

$$S_1^{-1}AS_1 = \sum \oplus (\lambda_\kappa I_\kappa + U_\kappa)$$

where U_κ is the first superdiagonal matrix to I_κ, that is

$$U_\kappa = (\delta_{\mu+1\nu}) = \begin{Vmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ & \cdot & & \cdot & & \\ & & \cdot & & \cdot & \\ & & & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & 0 & 1 \\ \cdot & \cdot & \cdot & & 0 & 0 \end{Vmatrix}$$

Applying this form to A/ϵ for $\epsilon \neq 0$, we obtain more generally

$$S^{-1}AS = A_\epsilon = \sum \oplus (\lambda_\kappa I_\kappa + \epsilon U_\kappa) \qquad (\epsilon \neq 0). \tag{9}$$

Taking then

$$H = \sum \oplus (\operatorname{sgn} \mathscr{R}\lambda_\kappa) I_\kappa,$$

we have $\operatorname{In} H = \operatorname{In} A$ and

$$\mathscr{R}(A_\epsilon H) = \sum \oplus \left| \mathscr{R}\lambda_\kappa \right| I_\kappa + \frac{\epsilon}{2} \sum \oplus (U_\kappa + U_\kappa')$$

and the expression on the right is obviously positive definite for small $|\epsilon|$, since $\mathscr{R}\lambda_\kappa \neq 0$. Thus, by Sylvester's law of inertia,

$$0 < S\mathscr{R}(A_\epsilon H)S^* = \mathscr{R}(SA_\epsilon HS^*) = \mathscr{R}(ASHS^*) = \mathscr{R}(AH_0), \ \ H_0 = SHS^*,$$

and here, again by Sylvester's law of inertia $\operatorname{In} H_0 = \operatorname{In} H = \operatorname{In} A$.

3.4. (b) Assume that for a Hermitian H_1, $\mathscr{R}(AH_1) = P_1 > 0$. Put $P_0 = \mathscr{R}(AH_0)$, where H_0 is a Hermitian matrix with $\operatorname{In} H_0 = \operatorname{In} A$, and $\mathscr{R}(AH_0) > 0$; the existence of H_0 was proved in (a). Put $P_t = tP_1 + (1-t)P_0$, $0 \leq t \leq 1$. Since $x^*P_tx = tx^*P_1x + (1-t)\,x^*P_0x > 0$ for all $x \neq 0$, P_t is positive definite. If $\Delta(A) \neq 0$, there exists by Lemma 1 a unique solution H_t of

$$\mathscr{R}(AH_t) = P_t \qquad (0 \leq t \leq 1)$$

and H_t depends *continuously* on t for $0 \leq t \leq 1$, by our remarks after the proof of Lemma 1. Hence also eigenvalues of H_t, which are real, vary continuously with t. Further, by Lemma 2, none of the H_t, $0 \leq t \leq 1$, is singular. Therefore $\operatorname{In} H_1 = \operatorname{In} H_0 = \operatorname{In} A$. This proves the assertion of (b) *in the case* $\Delta(A) \neq 0$.

3.5. Assume now that $\Delta(A) = 0$, and suppose that we have

$$\mathscr{R}(AH_1) = P_1 > 0.$$

Since $\Delta(A + tI) = \Pi(\lambda_\mu + \bar{\lambda}_\nu + 2t)$ is a nonvanishing polynomial in t, it is nonzero for all sufficiently small $|t| \neq 0$. On the other hand, for sufficiently small t, $\mathscr{R}(A_t H) = P_t > 0$, and $\operatorname{In} A_t = \operatorname{In} A$, since $\delta(A) \neq 0$. Thus $\operatorname{In} H_1 = \operatorname{In} A_t = \operatorname{In} A$, by the result of 3.4, and (b) is proved.

3.6. (c) Suppose now that for a Hermitian H we have $\mathscr{R}(AH) > 0$; replacing again A by $A_t = A + tI$ we have for sufficiently small $t > 0$, $\pi(A_t) = \pi(A) + \delta$, $\nu(A_t) = \nu(A)$, and since still $\mathscr{R}(A_t H) > 0$ and $\delta(A_t) = 0$, we have

$$\pi(H) = \pi(A) + \delta, \qquad \nu(H) = \nu(A).$$

Replacing t by $-t$, we obtain in the same way,

$$\pi(H) = \pi(A), \qquad \nu(H) = \nu(A) + \delta.$$

Hence $\delta = 0$, and theorem 1 is proved.

IV. Corollaries of the Main Theorem

4.1. Combining the last assertion of Theorem 1 with what has been said about relation (8) we obtain

Corollary 1. *If* $\Delta(A) = \Pi_{\sigma,\tau}(\lambda_\sigma + \bar{\lambda}_\tau) \neq 0$, *and* P *is a given positive definite matrix, then there exists a unique* H *satisfying* $AH + HA^* = P$. *The matrix* H *is Hermitian and* $\operatorname{In} H = \operatorname{In} A$.

Specializing the assertion of Theorem 1 for $\operatorname{In} H = (0, n, 0)$ we obtain the following analog of Lyapunov's theorem which is partly weaker and partly stronger than L_0:

4.2. Corollary 2. (L_1) *Necessary and sufficient for an* $(n \times n)$ *matrix* A *to have* $\operatorname{In} A = (0, n, 0)$ *is that there exists a negative definite matrix* H *with* $\mathscr{R}(AH) > 0$.

We combine L_1 with Corollary 1 and note that if $\operatorname{In} A = (0, n, 0)$ then $\Delta(A) \neq 0$, and we obtain Lyapunov's theorem L_0 quoted in the introduction.

4.3. Corollary 3. *If for an $(n \times n)$ matrix A, we have $\mathscr{R}A > 0$ and H is Hermitian, then* $\operatorname{In}(AH) = \operatorname{In} H$.

Proof. If H is nonsingular, then $\mathscr{R}(AHH^{-1}) = \mathscr{R}(A)$ is positive definite, whence, by Theorem 1, $\operatorname{In}(AH) = \operatorname{In}(H^{-1})$. But $\operatorname{In}(H^{-1}) = \operatorname{In}(H)$, and so the result is proved in this case.

If H is singular, we proceed as did Wielandt in [9]. We choose a unitary U so that $D = U^*HU$ is a real diagonal matrix and partition D as follows

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where D_1 is a nonsingular matrix. Partition $B = U^*AU$ similarly:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

and note that

$$BD = U^*AHU = \begin{pmatrix} B_{11}D_1 & 0 \\ B_{21}D_1 & 0 \end{pmatrix}.$$

We have to prove $\operatorname{In}(BD) = \operatorname{In} D$. Since the eigenvalues of BD are those of $B_{11}D_1$ together with zeros, it is enough to prove that $\operatorname{In}(B_{11}D_1) = \operatorname{In} D_1$. But, $\mathscr{R}(B_{11})$ being a principal minor of the positive definite matrix, $\mathscr{R}(B)$ is positive definite. Hence the last assertion follows from the nonsingular case.

4.4. Corollary 4. *Let A be a matrix such that $\mathscr{R}(A)$ is positive semi-definite, and let H be Hermitian. If*

$$\operatorname{In}(AH) = (\pi_1, \nu_1, \delta_1), \qquad \operatorname{In} H = (\pi, \nu, \delta),$$

then

$$\pi_1 \leq \pi, \qquad \nu_1 \leq \nu.$$

Proof. Put $\hat{A} = A + \epsilon I$, where $\epsilon > 0$. Then by Corollary 3, we have

$$\operatorname{In}(\hat{A}H) = \operatorname{In} H = (\pi, \nu, \delta).$$

If we now let ϵ go to 0, neither the number of positive nor the number of negative real parts of eigenvalues can *increase* in the limit, and the assertion follows immediately.

V. A Theorem on Purely Imaginary Eigenvalues

5.1. If A is an $(n \times n)$ matrix, we shall write

$$\Lambda(A) = \operatorname{diag}(\lambda_1, \cdots, \lambda_n),$$

where, as usual, the λ_ν are the eigenvalues of A ordered conveniently.

THEOREM 2. *If an* $(n \times n)$ *matrix* A *has exactly* $k > 0$ *purely imaginary eigenvalues* $i\alpha_1, \cdots, i\alpha_k$ *and* $\mathscr{R}A$ *is semidefinite, then the corresponding elementary divisors are linear, while the corresponding eigenvectors are nullvectors for* $\mathscr{R}A$ *and eigenvectors to the eigenvalues* $\alpha_1, \cdots, \alpha_k$ *for* $\mathscr{J}A$. A *is then unitarily similar to a cartesian sum of* diag $(i\alpha_1, \cdots, i\alpha_k)$ *and of a matrix of order* $n - k$.

If beyond these assumptions A *is real, then* $k = 2m$ *is even, the imaginary eigenvalues can be written as* $\pm i\alpha_1, \cdots, \pm i\alpha_m$ *and* A *is orthogonally similar to the cartesian sum*

$$\sum \oplus \begin{pmatrix} 0 & \alpha_\mu \\ -\alpha_\mu & 0 \end{pmatrix} \oplus A_0, \tag{10}$$

where A_0 *is a real matrix of order* $n - k$.

5.2. PROOF. Let $A = R + iQ$ be the Toeplitz decomposition of A, with Hermitian R and Q, where R is positive semidefinite. Let x be an eigenvector of A corresponding to the eigenvalue $i\alpha$, $xx^* = 1$,

$$Ax = i\alpha x. \tag{11}$$

Multiplying this from the left by x^* we obtain

$$i\alpha = x^*Ax = x^*Rx + ix^*Qx. \tag{12}$$

As the expression on the left is purely imaginary, we have $x^*Rx = 0$, and hence, as R is semidefinite, $Rx = 0$. Indeed both $\mathscr{R}x$ and $\mathscr{J}x$ give an extremum of x^*Rx. Using this result in (11) we obtain $Qx = \alpha x$, and we see that x is a eigenvector of R corresponding to the root 0, and of Q corresponding to the eigenvalue α. Now let U be a unitary having in its first column the vector x. If we then form

$$B \equiv U^*AU = U^*RU + iU^*QU, \tag{13}$$

we have, denoting by $B_{\sigma\tau}$ the element of B in the σ-th row and the τ-th column and by U_σ the σ-th column of U:

$$B_{\sigma\tau} = (U_\sigma)^*RU_\tau + i(U_\sigma)^*QU_\tau. \tag{14}$$

This vanishes if $\sigma \neq \tau$ and either σ or τ is 1, since

$$Rx = 0, \qquad x^*R = 0, \qquad (U_\sigma)^*Qx = (U_\sigma)^*\alpha x = 0 \ (\sigma > 1),$$
$$x^*QU_\tau = \alpha U_1 U_\tau = 0.$$

As to B_{11}, we have $B_{11} = x^*Rx + ix^*Qx = i\alpha$, by (12). Therefore (13) is totally reducible to the form

$$\begin{pmatrix} i\alpha & 0 \\ 0 & B_1 \end{pmatrix}$$

5.3. We have further

$$\mathscr{R}B = U^*(\mathscr{R}A)U = \begin{pmatrix} 0 & 0 \\ 0 & \mathscr{R}B_1 \end{pmatrix}, \qquad \mathscr{J}B = U^*(\mathscr{J}A)U,$$

so that $\mathscr{R}B_1$ is semidefinite and all $i\alpha_2, \cdots, i\alpha_k$ are eigenvalues of B_1. Applying the procedure to B_1 and so on, the first part of Theorem 2 is proved.

Assume now that A is real. Let $x = \xi + i\eta$, $x^*x = 1$, again be an eigenvector of A to the eigenvalue $i\alpha$. Then for a constant $a + ib$, $a^2 + b^2 = 1$,

$$(a + ib)x = (a\xi - b\eta) + i(a\eta + b\xi) = \xi_1 + i\eta_1$$

is an eigenvector. We show first that a and b can be chosen so that $\xi_1'\eta_1 = 0$. This is almost evident from geometrical considerations. Algebraically the orthogonality condition becomes $(a^2 - b^2)\eta'\xi + ab(|\xi|^2 - |\eta|^2) = 0$, and this is, if $\eta'\xi \neq 0$, certainly satisfied by some positive a and b with $a^2 + b^2 = 1$. We shall therefore assume that already $\eta'\xi = 0$. As in the complex case we see that $Rx = 0$ and therefore

$$R\xi = R\eta = 0. \tag{15}$$

Further, as above, $Qx = \alpha x$, and as Q now is purely imaginary,

$$Q\xi = i\alpha\eta, \qquad Q\eta = -i\alpha\xi. \tag{16}$$

We have therefore in ξ and η two orthogonal vectors satisfying (15) and (16). We normalize these and obtain two orthogonal vectors U_1, U_2 of unit norm satisfying (15) and (16). Let

$$U = [U_1, U_2, \cdots, U_n]$$

be a real orthogonal matrix, having U_1 and U_2 in its first two columns. If we now form the matrix B in (13) with this U, we have again (14). But here $B_{1\tau}$ $(\tau \neq 2)$, $B_{2\tau}$ $(\tau \neq 1)$, $B_{\tau 1}$ $(\tau > 2)$, $B_{\tau 2}$ $(\tau > 2)$ vanish by (15) and (16). And from the same relations it follows that $B_{12} = \alpha$, $B_{21} = -\alpha$. We see that B is totally reducible and has as one component the skew symmetric matrix $\left(\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right)$. Repeating this argument m times, the second part of Theorem 2 is proved.

VI. Conditions for H-Semistability

6.1. Theorem 3. *An $(n \times n)$ matrix A is positive H-semistable if and only if $\mathscr{R}A$ is positive semidefinite. If A is real, A is real positive H-semistable if and only if $\mathscr{R}A$ is positive semidefinite.*

PROOF. Assume that $\mathscr{R}A$ is positive semidefinite. Denote the inertia of a positive semidefinite Hermitian matrix H by $(\pi, 0, \delta)$ and the inertia of AH by (π_1, ν_1, δ_1). Then it follows from the Corollary 4 to Theorem 1 that $\nu_1 \leq 0$, $\nu_1 = 0$, that is that AH is positive semistable.

6.2. Assume now that A is positive H-semistable, that is that for all positive semidefinite Hermitian H, AH is semistable. It is even sufficient to assume that this holds for all *definite* Hermitian H. Now observe, that the H-semistability of A implies the H-semistability of S^*AS for $| S | \neq 0$. To see this note that S^*ASP is similar to $ASPS^*$, and that if P is positive semidefinite so is SPS^*. Hence, under our hypothesis, $A(SPS^*)$ is semistable, and so is also S^*ASP.

We choose now S so that $S^*(\mathscr{R}A)S$ is a real diagonal matrix $D = \text{diag}$ $(d_1, \cdots, d_n)$ and therefore $S^*AS = D + iQ$, where Q is Hermitian. Suppose that one of the d_ν is negative, for instance d_1. Choose then H as diag $(1, 0, \cdots, 0)$. Then S^*ASH has one nonzero eigenvalue with d_1 as real part and is therefore not positive semistable. The first part of theorem 3 is proved.

6.3. Suppose now that A is real and real positive H-semistable. We note that in this case there exists a real S for which $S'(\mathscr{R}A)S$ is a diagonal matrix. We can then repeat the argument of 6.2 to prove that $\mathscr{R}A$ is positive semidefinite. The converse is already contained in the first part of the theorem.

VII. CONDITIONS FOR H-STABILITY

7.1. An immediate consequence of Corollary 3 to Theorem 1, is that if we have $\mathscr{R}A > 0$ then A is positive H-stable, since then AH has the same inertia as H. We need therefore consider only under what conditions a matrix A with $\mathscr{R}A$ positive semidefinite and singular is H-stable. The answer is given by

THEOREM 4. *Assume that for the $(n \times n)$ matrix A the real part $\mathscr{R}A$ is positive semidefinite and singular. Then A is not H-stable if and only if, for a convenient nonsingular T, we have*

$$T^*AT = K \oplus Q, \tag{17}$$

where K is a skew Hermitian matrix and Q is a square matrix of an order $< n$. If in particular A is real and not real H-stable, then T can be chosen in (17) as a real matrix.

7.2. PROOF. Assume that (17) holds. Multiplying (17) on the left by T^{*-1} and on the right by T^* and putting $TT^* = P$, we obtain

$$AP = T^{*-1}(K \oplus Q)T^*,$$

and the matrix P is positive definite Hermitian, while the matrix on the right-hand side has the same eigenvalues as $K \oplus Q$ and has in particular all purely imaginary characteristic roots of K. Therefore AP is unstable.

7.3. Assume now that for a convenient positive definite P, AP has purely imaginary eigenvalues $i\alpha_1, \cdots, i\alpha_k$. It is well known that $P = SS^* = S^2$, where S is again a positive definite matrix. AP is then similar to the matrix $SAS = B$, which also has $i\alpha$ as an eigenvalue. Then, by Theorem 2, we have for a unitary matrix U

$$U^*BU = K \oplus B_0, \qquad K = \operatorname{diag}(i\alpha_1, \cdots, i\alpha_k),$$

and so $A = (S^{-1}U)(K \oplus B_0)(S^{-1}U)^*$ with the shew Hermitian K. If A is real, then by Theorem 2, U can be assumed to be orthogonal, while K has the form of the first sum in (10). Theorem 4 is proved.

VIII. On the Equation (1) with Arbitrary C

8.1. Theorem 5. *Let A be an $(n \times n)$ matrix such that all eigenvalues of A have nonzero real part and let $\omega = (\pi, \nu, 0)$ be a given inertia triple. Then* (i) *we can find a Hermitian matrix H such that* In $\mathscr{R}(AH) = \omega$; (ii) *if in particular,* $\omega =$ In A, *we can choose H in* (i) *to be positive definite.*[3]

8.2. Proof. We first note that we can find a matrix similar to A in an arbitrarily small neighborhood of $\Lambda = \Lambda(A)$ as follows immediately from (9) (cf. Ostrowski [13, p. 109]). Thus for a convenient R, $R^{-1}AR = \Lambda + C$, where C can be made as small as we please.

Let λ_σ be the eigenvalues of A. We choose $E = \operatorname{diag}(\epsilon_\sigma)$, with $\epsilon_\sigma = \pm 1$, and In $E = \omega$. Let $(\lambda_\sigma + \bar{\lambda}_\sigma) = \epsilon'_\sigma \mid \lambda_\sigma + \bar{\lambda}_\sigma \mid$, and write $E' = \operatorname{diag}(\epsilon'_\sigma)$. We note in passing that In $E' =$ In A. Define the diagonal matrix $E'' = \operatorname{diag}(\epsilon'')$ by $E'' = EE'$ and note that $\epsilon''_\sigma = \pm 1$, and $E'E'' = E$.

8.3. Now put $Q = R^{-1}ARE''$ and observe that $\mathscr{R}(Q) = \mathscr{R}(\Lambda E'') + \mathscr{R}(CE'')$ and that $\mathscr{R}(\Lambda E'') = \operatorname{diag} \epsilon'_\sigma\epsilon''_\sigma \mid \lambda_\sigma + \bar{\lambda}_\sigma \mid$, so that In $(\Lambda E'') =$ In $\mathscr{R}(\Lambda E'') =$ In $E = \omega$. Hence by the continuity of eigenvalues In $\mathscr{R}Q = \omega$, provided C was chosen sufficiently small.

Next put $H = RE''R^*$. Then $AH = RQR^*$, $(AH)^* = RQ^*R^*$ so that $\mathscr{R}(AH) = R(\mathscr{R}Q)R^*$, whence by Sylvester's law of inertia, In $\mathscr{R}(AH) =$ In $\mathscr{R}Q = \omega$. We have proved assertion (i).

8.4. To prove assertion (ii) observe that if $\omega =$ In A, then $E = E'$, provided the (ϵ_σ) are ordered suitably, and therefore $E'' = I$. Thus our $H = RR^*$ and is positive definite.

[3] Assertion (ii) may also be derived from Theorem 1 of Lewis and Taussky [15].

8.5. The assertion (ii) is a special case of a more general result whose proof rests on combinational considerations. We shall state this result without proof:

Let A be an n-th order matrix with $\text{In } A = (\pi', \nu', 0)$. If $\omega = (\pi, \nu, 0)$ *is a given inertia triple then we can find a Hermitian H such that both* $\text{In } \mathscr{R}(AH) = \omega$, *and* $\text{In } H = (\pi'', \nu'', 0)$, *provided that* $|\pi + \pi' - n| \leq \pi'' \leq n - |\pi - \pi'|$ *and* $\pi'' - |\pi + \pi' - n|$ *is even.*

References

1. Marden, M. The geometry of zeros. Math. Surveys 3, American Math. Soc., 1949.
2. Lyapunov, A. "Problème Général de la stabilité du mouvement." *Ann. of Math. Studies*, **17**, Princeton Univ. Press, Princeton, New Jersey, 1947.
3. Bellman, R. "Introduction to Matrix Analysis." McGraw-Hill, New York, 1960.
4. Gantmacher, F. R. "The Theory of matrices," Vols. I, II. Chelsea, New York, 1959.
5. Hahn, W. Eine Bemerkung zur zweiten Methode von Lyapunov. *Math. Nachrbl.* **14**, 349-354 (1955).
6. Arrow, K. J. and McManus, M. A note on dynamic stability. *Econometrica.* **26**, 448-454 (1958).
7. Taussky, O. A remark on a theorem by Lyapunov. *J. Math. Anal. and Appl.* **2**, 105-107 (1961).
8. Taussky, O. A generalization of a theorem by Lyapunov, to be published.
9. Wielandt, H. On the eigenvalues of $A + B$ and AB. *Natl. Bur. Standards Rept.* 1367 (1951).
10. Ostrowski, A. Ueber Eigenwerte von Produkter Hermitescher Matrizen. *Hamburger Math. Abhandl.* **23**, 60-68 (1959).
11. MacDuffee, C. C. "The Theory of Matrices." Chelsea, New York, 1956.
12. Picone, M. L'automazione del calcolo e il progresso dell'analisi matematica. *La ricerca sci.* **28**, 697-717 (1958).
13. Ostrowski, A. "Solution of Equations and Systems of Equations." Academic Press, New York and London, 1960.
14. Bellman, R. Kronecker Products and the second method of Lyapunov. *Math. Nachrbl.* **20**, 17-19 (1959).
15. Lewis, C. L., Jr. and Taussky, O. Some remarks concerning the real and imaginary parts of the characteristic roots of a finite matrix. *J. Math. Phys.* **1**, 234-236 (1960).