We conclude with the remark, which is easily verified by considerations precisely similar to those employed in the proofs of Theorems 8 and 9, and which we shall accordingly not take the space to prove, that

THEOREM 11. *The necessary and sufficient condition for the existence of* ${}^{\pi}H_k{}^n$ *on which the general* Q_{k-1}^2 *is non-singular is*

$$n \leqslant \frac{2k}{k-1}(\pi+1).$$

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AN INEQUALITY FOR LATENT ROOTS APPLIED TO DETERMINANTS WITH DOMINANT PRINCIPAL DIAGONAL

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The absolute value of a latent root of a matrix does not exceed the greatest of the sums of absolute values of elements in a row of the matrix. This well-known inequality is due to G. Frobenius [3]. In § 1 this inequality is generalized by the use of compound matrices. In § 2 further inequalities are derived, by means of which bounds for determinants with dominant principal diagonal elements are obtained. These bounds are improvements of bounds due to H. v. Koch [6] and A. Ostrowski [7], which have also been previously improved by A. Ostrowski [7, 8] and G. B. Price [10]. In § 3 conditions are found under which a matrix is similar to a matrix with dominant principal diagonal when transformed by a diagonal matrix. A distinction is made between singular and non-singular matrices, and it is

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pointed out that a similar condition of A. Ostrowski [8] may fail in the case of a singular, reducible matrix*.

§ 1.

Let A be an $n \times n$ matrix with complex elements and C the non-negative matrix $(c_{ij}) = (|a_{ij}|)$. (We use the terms non-negative matrix C, positive vector y, etc., to mean a matrix of non-negative elements, a vector of positive elements, and write $C \geqslant 0$, $y > 0$.)

Let X be the diagonal matrix $X = \mathrm{diag}\,[x_1, x_2, \ldots, x_n]$, $x_i > 0$, $i = 1, 2, \ldots, n$. The vector r of generalized row sums of A is defined as $r = X^{-1} CXe$ $\left[r_i = \left(\sum_{j=1}^{n} |a_{ij}| x_j\right) \Big/ x_i\right]$, where e is the column vector $e = \{1, 1, \ldots, 1\}$. By R_i, $i = 1, 2, \ldots, n$, we denote an arrangement of the r_i for which $R_1 \geqslant R_2 \geqslant \ldots \geqslant R_n$.

The latent roots of A are λ_i, $i = 1, 2, \ldots, n$, supposed arranged so that $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$.

We shall use some results on the λ_i and R_i:

$$|\lambda_1| \leqslant R_1. \tag{1}$$

When A is irreducible, $|\lambda_1| = R_1$ if and only if

$$R_1 = R_2 = \ldots = R_n \tag{2}$$

and

$$A = e^{i\psi} D^{-1} CD, \tag{3}$$

where D is a diagonal matrix, $|d_{ii}| = 1$, and C is the non-negative matrix defined above. The matrix A is irreducible when it cannot be put in the form

$$A = \begin{bmatrix} A_{11} & \cdot \\ A_{21} & A_{22} \end{bmatrix}$$

by a conjugate permutation of rows and columns, where A_{11}, A_{22} are square matrices and the dot represents a null-matrix.

When A is irreducible, $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_k|$ if and only if (2) and (3) hold, and, after a conjugate permutation of rows and columns,

$$C = \begin{bmatrix} \cdot & C_{12} & & & & \\ & \cdot & C_{13} & & & \\ & & \cdot & \times & & \\ & & & \cdot & \times & \\ & & & & \cdot & C_{k-1\,k} \\ C_{k1} & & & & & \cdot \end{bmatrix}, \tag{4}$$

the dots in the diagonal denoting square null-matrices.

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* *Added* 4 July, 1952: Prof. Ostrowski has recently informed me that he is aware of this.

These conditions were proved by G. Frobenius [3, 4], V. Romanovsky [12] in the non-negative case, A. Brauer [1, 2], H. Wielandt [15], for matrices with complex elements, and others. They are usually stated for $X = I$. For general X they follow from the particular case when $B = X^{-1}AX$ is considered, *cf.* A. Brauer [1].

When A is irreducible and $|\lambda| = R_1$, we give a proof of the necessity of (2), which is rather more concise than those we have found in the literature.

Let u' be a latent row vector associated with λ_1:

$$\lambda_1 u_j = \sum_{i=1}^{n} u_i a_{ij}, \quad j = 1, \ldots, n,$$

$$|\lambda_1||u_j| \leqslant \sum_{i=1}^{n} |u_i||a_{ij}|, \quad j = 1, \ldots, n, \tag{5}$$

$$|\lambda_1| \sum_{j=1}^{n} |u_j| \leqslant \sum_{i,j=1}^{n} |u_i||a_{ij}| = \sum_{i=1}^{n} |u_i| r_i, \quad (X = I), \tag{6}$$

$$|\lambda_1| \sum_{j=1}^{n} |u_j| \leqslant R_1 \sum_{i=1}^{n} |u_i|.$$

This proves (1). If $|\lambda_1| = R_1$, we are now able to assert (2) from (6), provided no u_i is zero. Suppose $u_i \neq 0, i = 1, \ldots, k, u_i = 0, i = k+1, \ldots, n$. The equalities must hold in (5) and (6).

By (5),

$$0 = R_1 |u_j| = \sum_{i=1}^{k} |u_i||a_{ij}|, \quad j = k+1, \ldots, n,$$

and hence $\quad a_{ij} = 0, \quad i = 1, \ldots, k, \quad j = k+1, \ldots, n,$

whence A is reducible. This gives the required condition.

THEOREM 1. $$\prod_{i=1}^{k} |\lambda_i| \leqslant \prod_{i=1}^{k} R_i, \quad 1 \leqslant k \leqslant n. \tag{7}$$

COROLLARY 1. $$|\det A| \leqslant \prod_{i=1}^{n} R_i. \tag{8}$$

COROLLARY 2. *When A is irreducible,*

$$\prod_{i=1}^{k} |\lambda_i| = \prod_{i=1}^{k} R_i \quad \textit{for} \quad k = 1, 2, \ldots, r \leqslant n, \tag{9}$$

if and only if

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = R_1 = R_2 = \ldots = R_n. \tag{10}$$

COROLLARY 3. *When A is irreducible,*

$$|\det A| = \prod_{i=1}^{n} R_i \tag{11}$$

if and only if A is monomial or is the 1×1 null matrix. [We shall call A monomial if it has precisely one non-zero element in each row and column, *cf.* Birkhoff and MacLane, "A survey of modern algebra", The Macmillan Company, New York (1941), 227.]

Proof of Theorem 1. Let S_τ be a subset of the set 1, 2, ..., n containing k members. Let S_τ, $\tau = 1, \ldots, \binom{n}{k}$, form all possible such distinct sets. Let $J_{\mu\tau}$ be an ordered arrangement of S_τ and $j_{i\mu\tau}$, $i = 1, \ldots, k$, the i-th member of $J_{\mu\tau}$. Let the $J_{\mu\tau}$, $\mu = 1, \ldots, k!$, be the possible distinct $J_{\mu\tau}$ for fixed τ. Evidently as μ and τ run through all values, the $J_{\mu\tau}$ represent every manner of selecting k ordered, distinct integers from n.

Let H_ν be an ordered arrangement of k integers (not necessarily all distinct) from the set 1, ..., n, and let h_{i_ν} be the i-th element of H_ν. Let the H_ν, $\nu = 1, \ldots, n^k$, be all possible distinct H_ν. By $A^{(k)}$ we denote the k-th compound of A, and by $a_{\sigma\tau}^{(k)}$ the element of $A^{(k)}$ formed from the minor at the intersection of rows $i \varepsilon S_\sigma$, columns $i \varepsilon S_\tau$ of A. Let $R_i^{(k)}$, $r_i^{(k)}$ be the generalized row sums of $A^{(k)}$.

In the summations below we shall assume that the indices μ, ν, τ run through all the values indicated above. In the product i is taken over the fixed set S_σ which we shall choose as 1, ..., k.

We shall first prove the theorem for $X = I$. We have

$$a_{\sigma\tau}^{(k)} = \sum_\mu \pm \prod_i a_{ij_{i\mu\tau}},$$

$$c_{\sigma\tau}^{(k)} = |a_{\sigma\tau}^{(k)}| \leqslant \sum_\mu \prod_i c_{ij_{i\mu\tau}},$$

$$r_\sigma^{(k)} = \sum_\tau c_{\sigma\tau}^{(k)} \leqslant \sum_\tau \sum_\mu \prod_i c_{ij_{i\mu\tau}} \leqslant \sum_\nu \prod_i c_{ih_{i_\nu}} = \prod_i r_i. \tag{12}$$

We now choose S_σ so that $r_\sigma^{(k)} = R_1^{(k)}$ and put $S_\sigma = S$. We have from (1)

$$\prod_{i=1}^{k} |\lambda_i| \leqslant r_\sigma^{(k)} \leqslant \prod_{i \varepsilon S} r_i \leqslant \prod_{i=1}^{k} R_i, \tag{13}$$

and (13) may be applied to $B = X^{-1}AX$ to yield the theorem.

Corollary 1 is the case $k = n$, and may be proved more simply by applying (1) to $\Phi^{-1}A$, where $\Phi = \mathrm{diag}\,[r_1, r_2, \ldots, r_n]$, provided no r_i is zero.

Proof of Corollary 2. From (9) we have in particular $|\lambda_1| = R_1$. We may now assert (2), and (10) follows. The converse is trivial. In this case A must satisfy (3) and, after a conjugate permutation of rows and columns, also (4) with $k = r$.

Proof of Corollary 3. The matrix $A^{(n)}$ consists of the single element $a_{11}^{(n)}$. When $X = I$ we inspect (12), where the equality must hold under the conditions of the corollary. Hence

$$\sum_\mu \prod_{i=1}^{n} c_{ij_{i\mu\tau}} = \sum_\nu \prod_{i=1}^{n} c_{ih_{i_\nu}} \quad (\tau = 1).$$

As A is irreducible this is satisfied only by a monomial matrix or the 1×1 null-matrix. The converse is again evident. For general X we note that $\det B = \det A$, and that if B is monomial (or null) so is A. This completes the proof.

We deduce that when (11) is satisfied by an irreducible A,

$$|\lambda_1| = |\lambda_2| = \ldots = |\lambda_n| = \left(\prod_{i=1}^{n} R_i\right)^{1/n}.$$

Let us now suppose that (9) is satisfied with $r = n$, for an irreducible A. Then

$$|\det A| = \prod_{i=1}^{n} R_i.$$

By Corollary 3 it follows that A is a monomial matrix or the 1×1 null-matrix. If A is monomial, $R_i = |a_{ij}|\, x_j/x_i$, where a_{ij} is the non-zero element in the i-th row of A.

By Corollary 2,

$$R_1 = R_2 = \ldots = R_n = c, \quad \text{say},$$

and hence C: (c_{ij}), $(c_{ij}) = (|a_{ij}|)$, satisfies $C = cXPX^{-1}$, where P is a permutation matrix.

By Corollary 2, (3) applies. Thus

$$A = a\, D^{-1} XPX^{-1} D,$$

where D is a diagonal matrix, $|d_{ii}| = 1$.

If A is reducible we may suppose it decomposed as

$$A = \begin{bmatrix} A_{11} & \cdot & & & \cdot \\ A_{21} & A_{22} & & & \cdot \\ \times & \times & & & \\ \times & & \times & & \\ \times & & & \times & \cdot \\ A_{r1} & A_{r2} & \times & \times & A_{rr} \end{bmatrix}, \tag{14}$$

where the A_{ii} are square and irreducible, and the dots represent null-matrices. The matrix A' obtained from A by putting $A_{ij} = 0$, $i > j$, has the same latent roots as A.

Let R_i', $i = 1, \ldots, n$, $R_1' \geqslant R_2' \geqslant \ldots \geqslant R_n'$ be the generalized row sums of A'.

Then $$R_i' \leqslant R_i, \quad i = 1, \ldots, n,$$

and by (8), $$\prod_{i=1}^{n} |\lambda_i| = |\det A| \leqslant \prod_{i=1}^{n} R_i' \leqslant \prod_{i=1}^{n} R_i.$$

Hence if (11) holds and no R_i equals zero, then $R_i' = R_i$, and thus $A_{ij} = 0$, $i > j$.

Further restrictions on A are easily found by applying the corollaries to the irreducible A_{ii}.

§ 2.

Let M be a square $n \times n$ matrix whose elements satisfy $m_{ii} \geqslant 0$, $m_{ij} \leqslant 0$, $i \neq j$. Let A be a matrix for which $(|a_{ij}|) = (|m_{ij}|)$. Let h be the vector $h = X^{-1} MXe$, $e = \{1, 1, \ldots, 1\}$ $\left[i.e.\ h_i = 2|a_{ii}| - \left(\sum_{j=1}^{n} |a_{ij}| x_j\right) \Big/ x_i\right]$. When $h > 0$ we shall say that A has a dominant principal diagonal with respect to X (or that A is similar to a matrix with dominant diagonal under transformation X). It was known to Hadamard [**5**] that A is non-singular when $h > 0$. Many bounds for $|\det A|$ have been found. Recently G. B. Price [**10**] has given some that are especially simple. Using a method of H. v. Koch [**6**] we shall deduce bounds which are an improvement on some previously proved.

We shall use a lemma due to H. Weyl [**14**] and G. Polya [**9**] to prove further inequalities between the $|\lambda_i|$ and R_i.

LEMMA. *If* $$a_i,\ b_i \geqslant 0, \quad a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n,$$

and $$\prod_{i=1}^{k} a_i \leqslant \prod_{i=1}^{k} b_i, \quad k = 1, 2, \ldots, n,$$

then $$\sum_{i=1}^{n} \rho(a_i) \leqslant \sum_{i=1}^{n} \rho(b_i),$$

provided that $g(\log x) = \rho(x)$ *is a convex non-decreasing function of* $\log x$.

COROLLARY. $$\sum_{i=1}^{r} |\lambda_i|^{\alpha} \leqslant \sum_{i=1}^{r} R_i^{\alpha}, \quad \alpha \geqslant 0, \quad 1 \leqslant r \leqslant n. \tag{15}$$

For $|\lambda_i|$, R_i satisfy the conditions of the lemma by Theorem 1, and x^{α} is a convex non-decreasing function of $\log x$.

Let $h > 0$, $s_i = (1 - h_i/|a_{ii}|)$, $S_1 = \max s_i$, and $S = \sum_{i=1}^{n} s_i$. Then $s_i < 1$ as $h > 0$.

For $X = I$ Ostrowski [**7**] proved

$$|\det A| \geqslant \prod_{i=1}^{n} h_i = \prod_{i=1}^{n} |a_{ii}|(1 - s_i); \tag{16}$$

and Koch [**6**] proved

$$|\det A| \geqslant e^{S}(1 - S_1)^{S/S_1} \prod_{i=1}^{n} |a_{ii}|. \tag{17}$$

THEOREM 2. *If* $h > 0$

$$e^{S} \prod_{i=1}^{n} |a_{ii}|(1 - s_i) \leqslant |\det A| \leqslant \frac{e^{-S} \prod_{i=1}^{n} |a_{ii}|}{\prod_{i=1}^{n} (1 - s_i)}, \tag{18}$$

where either inequality holds if, and only if, A is diagonal.

Proof. Let $Q = \operatorname{diag}[a_{11}, \ldots, a_{nn}]$. Then Q is non-singular as $|a_{ii}| \geqslant h_i \geqslant 0$. Let $P = I - Q^{-1}A$, and let F be the non-negative matrix $(f_{ij}) = (|p_{ij}|)$. The diagonal elements of F are zero and

$$X^{-1} F X e = S = \{s_1, \ldots, s_n\}.$$

Let the latent roots of P be μ_i, $i = 1, \ldots, n$. Since

$$\det A = \det Q \,.\, \det(I - P) = \prod_{i=1}^{n} a_{ii}(1-\mu_i),$$

we have
$$L = \sum_{i=1}^{n} \log(1-\mu_i) = -\sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \frac{\mu_i^{\nu}}{n},$$

where $L = \log\left(\det A \Big/ \prod_{i=1}^{n} a_{ii}\right)$ and the series converges as, by (1), $|\mu_i| \leqslant S_1 < 1$. Now

$$L = -\sum_{\nu=2}^{\infty} \frac{1}{\nu} \sum_{i=1}^{n} \mu_i^{\nu}, \tag{19}$$

as
$$\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} p_{ii} = 0.$$

Hence
$$|L| \leqslant \sum_{\nu=2}^{\infty} \frac{1}{\nu} \sum_{i=1}^{n} |\mu_i|^{\nu}. \tag{20}$$

We apply (15) to P and obtain from (20)

$$|L| \leqslant \sum_{\nu=2}^{\infty} \frac{1}{\nu} \sum_{i=1}^{n} s_i^{\nu} = -S + \sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \frac{s_i^{\nu}}{\nu},$$

whence, as $s_i < 1$,

$$|L| \leqslant -S - \sum_{i=1}^{n} \log(1-s_i). \tag{21}$$

This is equivalent to (18).

One of the equalities in (18) implies the equality in (20). Comparing this equation with (19) we see that μ_i^{ν}, $\nu = 2, 3, \ldots$, have equal arguments. Hence $\mu_i \geqslant 0$; but $\sum_{i=1}^{n} \mu_i = 0$ and so $\mu_i = 0$, $i = 1, \ldots, n$. This gives $|\det A| = \prod_{i=1}^{n} |a_{ii}|$. We have assumed one of the equalities and thus we must have $s_i = 0$, $i = 1, \ldots, n$. Hence A is diagonal. The converse is obvious. This completes the proof of the theorem.

The lower bound of (18) is evidently as sharp as, or sharper than (16) for all $\det A$ for which $h > 0$. That a similar relation holds between (18) and (17) is easily proved.

The inequality (17) is equivalent to

$$|L| \leqslant -S - (S/S_1) \log(1-S_1),$$

and

$$-S-S/S_1 \log(1-S_1) = -S+(S/S_1) \sum_{\nu=1}^{\infty} (S_1{}^{\nu}/\nu)$$

$$\geqslant -S+\sum_{i=1}^{n} \sum_{\nu=1}^{\infty} \frac{s_i{}^{\nu}}{\nu} = -S-\sum_{i=1}^{n} \log(1-s_i).$$

The result follows by the equivalence of (21) and (18).

It should be pointed out that other authors have found improvements of (16) and (17) for determinants with dominant diagonal.

Ostrowski [7] has proved

$$|\det A| \geqslant \prod_{i=1}^{n} |a_{ii}| e^{\sigma^2/S_1}(1-S_1)^{\sigma^2/S_1{}^2}, \tag{22}$$

where
$$\sigma^2 = \sum_{i,\, j=1}^{n} |p_{ij}|^2 \leqslant (\max|p_{ij}|)\, S; \tag{23}$$

and [8]

$$\left(\prod_{i=1}^{n} |a_{ii}|\right) \prod_{i=1}^{[n/2]} (1-S_{2i-1} S_{2i}) \leqslant |\det A| \leqslant \left(\prod_{i=1}^{n} |a_{ii}|\right) \prod_{i=1}^{[n/2]} (1+S_{2i-1} S_{2i}), \tag{24}$$

where the S_i, $i = 1, \ldots, n$ are an arrangement of the s_i such that

$$S_1 \geqslant S_2 \geqslant \ldots \geqslant S_n$$

and $[n/2]$ is the integral part of $n/2$. Price [10] has proved

$$\prod_{i=1}^{n} (|a_{ii}|-t_i) \leqslant |\det A| \leqslant \prod_{i=1}^{n} (|a_{ii}|+t_i), \tag{25}$$

where
$$t_i = \sum_{j=i+1}^{n} |a_{ij}|.$$

For all $\det A$ for which $h = X^{-1} M X e > 0$, it is easily seen that (24) and (25), like (18), yield bounds as sharp as, or sharper than (16), while (22) [because of (23)] like (18) yields bounds as sharp as, or sharper than (17). There does not appear to be a relation of this sort between (18), (22), (24), and (25), *e.g.* (18) may appear sharper or less sharp than (22), depending on the particular A considered. One may construct examples to suit one bound or another. Generally speaking, however, Ostrowski's (24) gives the best bounds. When the super-diagonal elements of A are small compared to the sub-diagonal elements, Price's (25) is generally best*.

§ 3.

If M is a square matrix for which $m_{ii} \geqslant 0$, $m_{ij} \leqslant 0$, $i \neq j$, and $h = X^{-1} M X e > 0$, it is known that all latent roots of M have positive real

* The following papers have appeared since this summary of results was written: A. M. Ostrowsky, "Note on bounds for determinants with dominant principal diagonal", *Proc. American Math. Soc.*, 3 (1952), 26–30; Y. K. Wong, "Some inequalities of determinants of Minkowski type", *Duke Math. Journal*, 19 (1952), 231–242.

parts, while if $h \geqslant 0$, the latent roots of M are zero or have positive real parts (Rohrbach [11], Taussky [13]). We shall prove these results with some converses which are apparently new.

THEOREM 3. *There is an* $X = \operatorname{diag}[x_1, \ldots, x_n]$, $x_i > 0$, $i = 1, \ldots, n$, *such that* $h = X^{-1}MXe > 0$, *if and only if the latent roots of* $M : (m_{ij})$, $m_{ii} \geqslant 0$, $m_{ij} \leqslant 0$, $i \neq j$, *have positive real parts.*

Among the latent roots of largest modulus of a non-negative matrix G there is one, γ, that is non-negative. If $\alpha > \gamma$, then $(\alpha I - G)$ is non-singular and $(\alpha I - G)^{-1} \geqslant 0$ (Frobenius [4], Wielandt [15]).

Let us choose $\alpha \geqslant \max m_{ii}$. Then $\alpha I - M \geqslant 0$. We shall now put $G = \alpha I - M$. If $\mu = \alpha - \gamma$ it follows that μ is a latent root of M. Let λ be any other latent root of M, $\lambda \neq \mu$. Then $\beta = \alpha - \lambda$ is a latent root of G. Hence $|\beta| \leqslant \gamma$, and as $\beta \neq \gamma$, $\Re(\beta) < \gamma$, where $\Re(\beta)$ is the real part of β.

We have
$$\Re(\lambda) = \alpha - \Re(\beta) > \alpha - \gamma = \mu.$$

Hence M has a latent root, μ, which is real, and whose real part is less than that of any other latent root.

By applying (1) to G we obtain
$$\gamma = \alpha - \mu \leqslant \max\,(X^{-1}\{\alpha I - M\}Xe)_i = \alpha - \min h_i,$$
where $(z)_i$ denotes the i-th element of the vector z inside the bracket. Hence
$$\min h_i \leqslant \mu \tag{26}$$
for all X. Thus if $h > 0$ it follows that $\mu > 0$, and by $\Re(\lambda) > \mu$ all latent roots of M have positive real parts.

Suppose that all latent roots have positive real parts. Then M is clearly non-singular and $\mu > 0$. Hence $M^{-1} = (\alpha I - G)^{-1} \geqslant 0$, as
$$\alpha = \gamma + \mu > \gamma.$$

Let k be any column vector, $k > 0$, and let $MXe = k$. We have
$$Xe = M^{-1}k > 0,$$
as, of course, every row of the non-singular matrix M contains at least one positive element. The vector Xe determines the diagonal matrix X uniquely. Putting $X^{-1}k = h$, we have $X^{-1}MXe = h > 0$ as required, and the theorem is proved.

While k, $k > 0$, is arbitrary, (26) shows that h is much more restricted. It is of some interest to construct an X for which $h > 0$. Let μ_p be the real latent root of least real part of M_{pp}, where the M_{ii}, $i = 1, \ldots, r$, are the irreducible matrices in the diagonal of the decomposition of M, *cf.* (14). Let y^p be a latent vector of M_{pp} associated with μ_p. If G is partitioned conformably with M, we have $G_{pp} = \alpha I_p - M_{pp}$, where I_p is a unit matrix.

Then y^p is also a latent column vector of G_{pp} associated with $\gamma_p = \alpha - \mu_p$, where γ_p is a non-negative latent root of maximum modulus of G_{pp}. But G_{pp} is irreducible, as an irreducible matrix remains irreducible when the diagonal elements are altered. Hence γ_p is a single latent root of G_{pp} and has a unique positive latent column vector associated with it (Frobenius [4], Wielandt [15]). It follows that $y^p > 0$, on suitable normalization.

Let $Y_p = \text{diag}\,[y_1{}^p,\ y_2{}^p,\ \ldots]$ and $X = \text{diag}\,[\epsilon_1 Y_1,\ \epsilon_2 Y_2,\ \ldots,\ \epsilon_r Y_r]$, where the ϵ_i are positive constants. We shall write $h = \{h^1, \ldots, h^r\}$, h^p conformable with M_{pp}. Similarly $e = \{e^1, \ldots, e^r\} = \{1, 1, \ldots, 1\}$. We note that $\mu_p \geqslant \mu > 0$ and $Y_p^{-1} M_{pp} Y_p e^p = Y_p^{-1} \mu_p y^p = \mu_p e^p$. Now

$$h = X^{-1} M X e,$$

whence

$$h^p = \epsilon_p^{-1} Y_p^{-1} \sum_{j=1}^{p} \epsilon_j M_{pj} Y_j e^j,$$

or

$$h^p = \mu_p e^p - \epsilon_p^{-1} f^p, \tag{27}$$

where

$$f^p = Y_p^{-1} \sum_{j=1}^{p-1} \epsilon_j M_{pj} Y_j e^j. \tag{28}$$

As $M_{pj} \leqslant 0$, $j < p$, it follows that $f^p \geqslant 0$. But f^p, $p \geqslant 2$, is homogeneous and linear in $\epsilon_1, \epsilon_2, \ldots, \epsilon_{p-1}$ and involves no other ϵ_i, while $f^1 = 0$. Hence from (27), $h^1 > 0$, and by choosing ϵ_p sufficiently large compared to $\epsilon_1, \epsilon_2, \ldots, \epsilon_{p-1}$ we shall obtain $h^p > 0$, $p \geqslant 2$. Choosing successively $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ we may ensure that $h > 0$.

An implication of the conditions of Theorem 3 is worth noting. Let us suppose that m_{ii} is the k-th element of the principal diagonal of M_{pp}. Then for all M: (m_{ij}), $m_{ii} \geqslant 0$; $m_{ij} \leqslant 0$, $i \neq j$,

$$m_{ii} \geqslant (Y_p^{-1} M_{pp} Y_p e^p)_k = \mu_p,$$

whence, by

$$\mu_p \geqslant \mu, \quad p = 1, \ldots, r,$$

$$m_{ii} \geqslant \mu, \quad i = 1, \ldots, n.$$

It follows that the equivalent assumptions $h = X^{-1} M X e > 0$ or $\mu > 0$ of Theorem 3 imply $m_{ii} > 0$, $i = 1, \ldots, n$.

We shall use the above construction for h to prove a result corresponding to Theorem 3 when M is not restricted to be non-singular. Let us say that an irreducible A_{ii} of the decomposition (14) of a reducible matrix A is isolated when $A_{ij} = 0$, $j < i$.

THEOREM 4. *The matrices M and X satisfy the conditions of Theorem 3. If there is an X for which $h = X^{-1} M X e \geqslant 0$ the latent roots of M are zero or have positive real parts.*

If the latent roots of M have non-negative real parts there is an X for which $h = X^{-1} M X e \geqslant 0$ if, and only if, M is irreducible or the singular M_{ii} of the decomposition of M are isolated.

We shall give all symbols the same meanings as in the proof of Theorem 3. The considerations leading to (26) and (27) hold under the conditions of the theorem. If $\mu < 0$ we have from (26) that some $h_i \leqslant \mu < 0$, for all X. Hence if $h = X^{-1}MXe \geqslant 0$ then also $\mu \geqslant 0$, and as $\mathcal{R}(\lambda) > \mu$, where λ is any other latent root of M, $\lambda \neq \mu$; the first part of the theorem follows.

Suppose that $\mu \geqslant 0$ and consider (27).

If $\mu_p > 0$, we may again prove that $h^1 > 0$, and $h^p > 0$, $p \geqslant 2$, provided that ϵ_p is sufficiently large compared to $\epsilon_1, \epsilon_2, \ldots, \epsilon_{p-1}$. Now suppose $\mu = 0$ and $\mu_p = \mu$. If M_{pp} is isolated, then $M_{pj} = 0$, $j < p$, and by (28), $f^p = 0$; whence $h^p = 0$. Thus if all singular M_{pp} are isolated or if $M = M_{11}$ is irreducible we have a vector $h = \{h^1, \ldots, h^r\}$, where either $h^p > 0$, or $h^p = 0$. Hence $h \geqslant 0$.

We must still consider the case when some singular M_{pp} is not isolated. Partition $X = \operatorname{diag}[x_1, \ldots, x_n]$, $x_i > 0$, and $G = \alpha I - M$, conformably with M. Then $X = \operatorname{diag}[X_1, X_2, \ldots, X_r]$, where the X_i are diagonal matrices with positive diagonal elements, and the G_{ii} are irreducible. We also partition h conformably with M.

As
$$h = X^{-1} MXe,$$
we have
$$h^p = X_p^{-1} \sum_{j=1}^{p} M_{pj} X_j e^j,$$
whence
$$h^p = \alpha e^p - r^p - f^p, \tag{29}$$
where
$$f^p = -X_p^{-1} \sum_{j=1}^{p-1} M_{pj} X_j e^j \tag{30}$$
and
$$\alpha e^p - r^p = X_p^{-1} M_{pp} X_p e^p = X_p^{-1}(\alpha I_p - G_{pp}) X_p e^p.$$
Hence
$$r^p = X_p^{-1} G_{pp} X_p e^p. \tag{31}$$
As $M_{pj} \leqslant 0$, $j = 1, 2, \ldots, p-1$, and some $M_{pj} \neq 0$, $j < p$, it follows from (30) that
$$f^p \geqslant 0, \quad f^p \neq 0. \tag{32}$$

But $G_{pp} \geqslant 0$, and so r^p is the vector of generalized row sums of G_{pp}. The non-negative latent root of maximum modulus of G_{pp} is $\gamma_p = \alpha - \mu_p$. By (1), r_i^p, the largest element of r^p, satisfies $r_i^p \geqslant \alpha$.

If $r_i^p > \alpha$, by (29) and (32)
$$h_i^p = \alpha - r_i^p - f_i^p < 0$$
and h is not non-negative.

Suppose $r_i^p = \alpha$. Since G_{pp} is irreducible (2) holds. Hence $r^p = \alpha e^p$ and from (29) and (32)
$$h^p = -f^p \leqslant 0, \quad h^p \neq 0$$
and h is not non-negative.

As X is an arbitrary diagonal matrix with $x_i > 0$, we have proved that $h = X^{-1}MXe$ is not non-negative when a singular M_{pp} is not isolated. This completes the proof of the theorem.

In Theorems 3 and 4 we may replace simultaneously

$$X = \operatorname{diag}[x_1, x_2, \ldots, x_n], \quad x_i > 0, \tag{33}$$

by

$$x > 0; \tag{34}$$

and

$$h = X^{-1}MXe \geqslant 0 \quad (h = X^{-1}MXe > 0) \tag{35}$$

by

$$k = Mx \geqslant 0 \quad (k = Mx > 0). \tag{36}$$

For putting $Xe = x$, and $Xh = k$, we have (33) if, and only if, (34) holds, and (35) if, and only if, (36) holds.

When $\mu = \alpha - \gamma \geqslant 0$ $(\mu = \alpha - \gamma > 0)$, then $\det M = \det(\alpha I - G)$ and all principal minors of M are non-negative (positive), (Frobenius [3], Ostrowski [8]). The converse is also true. The characteristic equation of M is

$$\rho(\theta) = (-\theta)^n + c_1(-\theta)^{n-1} + \ldots + c_n,$$

where c_r is the sum of the principal minors of M of order r, and $c_n = \det M$. If $\det M$ is non-negative (positive) and all principal minors of M are non-negative it follows that all real latent roots are non-negative (positive). Hence $\mu \geqslant 0$ $(\mu > 0)$. We deduce that in Theorems 3 and 4 we may replace "the latent roots of M are zero or have positive real parts (the latent roots of M have positive real parts)" by "$\det M$ and all principal minors are non-negative (positive)", or by "$\det M$ is non-negative (positive) and all principal minors of M are non-negative".

A theorem of Ostrowski ([8], Theorem 4) may be restated thus: "If for M, $m_{ii} \geqslant 0$, $m_{ij} \leqslant 0$, $i \neq j$, $\det M$ and all principal minors of M are non-negative there are column vectors $x > 0$, $k \geqslant 0$, such that $Mx = k$. If M is non-singular we may restrict k to be positive."

Our preceding two remarks indicate that Theorem 3 may be reduced to the non-singular case of Ostrowski's theorem. If M is reducible and singular, however, a similar reduction applied to Theorem 4 shows that any matrix M with a non-isolated singular M_{pp} in its decomposition will not satisfy Ostrowski's theorem*. An example of such a matrix is:

$$A = \begin{bmatrix} 1 & \cdot \\ -1 & \cdot \end{bmatrix}.$$

Finally we shall prove another analogue of Ostrowski's theorem.

THEOREM 5. *The matrix M: (m_{ij}) has $m_{ii} \geqslant 0$, $m_{ij} \leqslant 0$, $i \neq j$. If there are vectors $x > 0$, $k \geqslant 0$, such that $Mx = k$, the latent roots of M are zero or have positive real parts.*

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* There does not appear to be any justification for the second sentence of [8], § 13, p. 87.

If the latent roots of M have non-negative real parts there are vectors $x \geqslant 0$ $(x \neq 0)$, $k \geqslant 0$, such that $Mx = k$.

The first part of the theorem follows from the first part of Theorem 4 and the equivalence of (33) and (34), (35) and (36).

Suppose $\mu \geqslant 0$. Let r be the number of irreducible M_{ii} in the decomposition of M and let $x = \{O^1, O^2, \ldots, O^{r-1}, y^r\}$, where the O^i are null-vectors, and y^r is the positive latent vector of M_{rr} associated with μ_r, the latent root of least real part of M_{rr}.

We have $x \geqslant 0$, $x \neq 0$, and

$$k = Mx = \{O^1, \ldots, O^{r-1}, M_{rr} y^r\} = \{O^1, \ldots, O^{r-1}, \mu_r y^r\} \geqslant 0,$$

as

$$\mu_r \geqslant \mu \geqslant 0.$$

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