Conditions for strict inequality in comparisons of spectral radii of nonnegative matrices

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Oberwolfach 1982 , LAA 1984
Szyld 1985 Zeng (preprint)

Theorem 1. *Let T and F be nonnegative matrices. Suppose that the spectral radius satisfies $\rho(T) < 1$. Assume also that $F \neq 0$ and that $F + T$ is irreducible. Let $Q = F(I-T)^{-1}$. Then, after a permutation similarity,*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & 0 \end{pmatrix}, \tag{1}$$

where Q_{11} is a nontrivial irreducible nonnegative matrix, Q_{12} is a nonnegative matrix every column of which has a positive entry, and the 0 rows of Q correspond to the 0 rows of F, if any. Further $\rho(Q) > 0$.

Q is irreducible $\Longleftrightarrow$
F has positive element in each row

Theorem 2. *Let T and F be nonnegative matrices. Suppose that the spectral radius satisfies $\rho(T) < 1$. Assume also that $F \neq 0$ and that $F + T$ is irreducible. Let $Q = F(I-T)^{-1}$.*
Let $0 \leq T' \lneq T$, and put $Q' = F(I-T')^{-1}$. Then, after the same permutation similarity and partitioning,

$$Q' = \begin{pmatrix} Q'_{11} & Q'_{12} \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where $Q'_{11} \lneq Q_{11}$ and $\rho(Q') < \rho(Q)$.

Proofs?

Relevant $F + T$ *path*:
First an F-arc, then maybe T-arcs.

$Q = F(I - T)^{-1}$

$$q_{ij} > 0 \iff \exists \text{ relevant path from } i \text{ to } j$$

$A = M - N$ a regular splitting :
$M^{-1} \geq 0, N \geq 0$

$A = M - N$ an M-splitting :
M an M-matrix, $N \geq 0$

Theorem 3. *(Theorem 2 reformulated) Let* $A_1 = M_1 - N$ *and* $A_2 = M_2 - N$ *be* M-*splittings of* A_1 *and* A_2 *respectively, where* $M_1 \lneq M_2$. *If* ${A_1}^{-1} > 0$, *then* $\rho(N{M_1}^{-1}) > \rho(N{M_2}^{-1}) > 0$.

Open Question 4. Is Theorem 3 still valid is if "M-splitting" is replaced by "regular splitting"?

Example 5.

$$M = \frac{1}{4}\begin{pmatrix} 4 & -2 & -2 \\ -2 & 3 & 1 \\ -2 & -1 & 3 \end{pmatrix}, \quad N = \frac{1}{10}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$M_2 = M, \quad M_1 = M - N, \quad A = M - 2N$$

$$A^{-1} = \frac{1}{10}\begin{pmatrix} 27 & 10 & 24 \\ 24 & 20 & 28 \\ 10 & 0 & 20 \end{pmatrix}.$$

$$NM_2^{-1} = NM_1^{-1} = \frac{1}{10}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\rho(NM_2^{-1}) = \rho(NM_1^{-1}) = 0$$

Theorem 6. [Varga] *Let* $A = M_1 - N_1 = M_2 - N_2$ *be regular splittings of* A*, where* $A^{-1} > 0$*. If* $N_1 \gneq N_2$ *then* $\rho(N_1 M_1^{-1}) > \rho(N_2 M_2^{-1} > 0$*.*

Example 7.

$$M_1,\ M_2,\ N,\ A \text{ as in Ex. 5}$$

$$N_1 = 2N,\ N_2 = N$$

$$A = M_1 - N_1 = M_2 - N_2$$

$$N_1 M_1^{-1} = 2 N_2 M_2^{-1} = 2NM^{-1}$$

$$\rho(N_1 M_1^{-1}) = \rho(N_2 M_2^{-1}) = 0$$